Discrete Mathematics 317 (2014) 19–32

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Graphs with maximum degree $\Delta \geq 17$ and maximum average degree less than 3 are list 2-distance $(\Delta + 2)$-colorable[☆]

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ARTICLE INFO

Article history:
Received 7 February 2013
Received in revised form 25 October 2013
Accepted 28 October 2013
Available online 22 November 2013

Keywords:
2-distance coloring
Square coloring
Maximum average degree

ABSTRACT

For graphs of bounded maximum average degree, we consider the problem of 2-*distance coloring*. This is the problem of coloring the vertices while ensuring that two vertices that are adjacent or have a common neighbor receive different colors. It is already known that planar graphs of girth at least 6 and of maximum degree Δ are list 2-distance $(\Delta + 2)$-colorable when $\Delta \geq 24$ (Borodin and Ivanova (2009)) and 2-distance $(\Delta + 2)$-colorable when $\Delta \geq 18$ (Borodin and Ivanova (2009)). We prove here that $\Delta \geq 17$ suffices in both cases. More generally, we show that graphs with maximum average degree less than 3 and $\Delta \geq 17$ are list 2-distance $(\Delta + 2)$-colorable. The proof can be transposed to list injective $(\Delta + 1)$-coloring.

1. Introduction

In this paper, we consider only simple and finite graphs. A 2-*distance k-coloring* of a graph G is a coloring of the vertices of G with k colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k-coloring. This is equivalent to a proper vertex-coloring of the square of G, which is defined as a graph with the same set of vertices as G, where two vertices are adjacent if and only if they are adjacent or have a common neighbor in G. For example, the cycle of length 5 cannot be 2-distance colored with less than 5 colors as any two vertices are either adjacent or have a common neighbor: indeed, its square is the clique of size 5. An extension of the 2-distance k-coloring is the *list* 2-*distance k-coloring*, where instead of having the same set of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. We define $\chi^2_\ell(G)$ as the smallest k such that G admits a list 2-distance k-coloring of G for any list assignment. Obviously, 2-distance coloring is a sub-case of list 2-distance coloring (where the same color list is assigned to every vertex), so for any graph G, $\chi^2_\ell(G) \geq \chi^2(G)$. Kostochka and Woodall [21] even conjectured that it is actually an equality, though the conjecture was recently disproved [20].

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [23], and has been actively studied because of the conjecture given below. The *maximum degree* of a graph G is denoted $\Delta(G)$.

Conjecture 1 (*Wegner [23]*). *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ *if* $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ *if* $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ *if* $\Delta(G) \geq 8$.

☆ This work was partially supported by the ANR grant EGOS 12 JS02 002 01.

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http://dx.doi.org/10.1016/j.disc.2013.10.022

This conjecture remains open. However, Havet et al. [18] proved that it holds asymptotically even in the case of list 2-distance coloring, i.e. $\chi_\ell^2(G) \le \frac{3\Delta(G)}{2}(1 + o(1))$.

Note that any graph G satisfies $\chi^2(G) \ge \Delta(G)+1$. It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [22], what conditions on the sparseness of the graph can be sufficient to ensure that the equality holds.

A first measure of the sparseness of a planar graph is its girth. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle. Wang and Lih [22] conjectured that for any integer $k \ge 5$, there exists an integer $D(k)$ such that for every planar graph G verifying $g(G) \ge k$ and $\Delta(G) \ge D(k)$, $\chi^2(G) = \Delta(G)+1$. This was proved by Borodin, Ivanova and Neustroeva [11,12] to be true for $k \ge 7$, even in the case of list-coloring, and false for $k \in \{5, 6\}$. So far, in the case of list coloring, it is known [3,19] that we can choose $D(7) = 16$, $D(8) = 10$, $D(9) = 8$, $D(10) = 6$, and $D(12) = 5$. Borodin, Ivanova and Neustroeva [13] proved that the case $k = 6$ is true on a restricted class of graphs, i.e. for a planar graph G with girth 6 where every edge is incident to a vertex of degree at most two and $\Delta(G) \ge 179$, we have $\chi^2(G) \le \Delta(G) + 1$. Dvořák et al. [15] proved that the case $k = 6$ is true by allowing one more color, i.e. for a planar graph G with girth 6 and $\Delta(G) \ge 8821$, we have $\chi^2(G) \le \Delta(G)+2$. They also conjectured that the same holds for a planar graph G with girth 5 and sufficiently large $\Delta(G)$, but this remains open. Borodin and Ivanova improved [5] Dvořák et al.'s result and extended it to list-coloring [6,7] as follows.

Theorem 1 (*Borodin and Ivanova [5]*). *Every planar graph G with $\Delta(G) \ge 18$ and $g(G) \ge 6$ admits a 2-distance $(\Delta(G) + 2)$-coloring.*

Theorem 2 (*Borodin and Ivanova [7]*). *Every planar graph G with $\Delta(G) \ge 24$ and $g(G) \ge 6$ admits a list 2-distance $(\Delta(G)+2)$-coloring.*

Theorems 1 and 2 are optimal with regard to the number of colors, as shown by the family of graphs presented by Borodin et al. [4], which are of increasing maximum degree, of girth 6 and are not 2-distance $(\Delta + 1)$-colorable. We improve Theorems 1 and 2 as follows.

Theorem 3. *Every planar graph G with $\Delta(G) \ge 17$ and $g(G) \ge 6$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

Another way to measure the sparseness of a graph is through its maximum average degree. The *average degree* of a graph G, denoted $\mathrm{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over all subgraphs H of G. Intuitively, this measures the sparseness of a graph because it states how great the concentration of edges in a same area can be. For example, stating that $\mathrm{mad}(G)$ has to be smaller than 2 means that G is a forest. Using this measure, we prove a more general theorem than Theorem 3.

Theorem 4. *Every graph G with $\Delta(G) \ge 17$ and $\mathrm{mad}(G) < 3$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

Euler's formula links girth and maximum average degree in the case of planar graphs.

Lemma 1 (*Folklore*). *For every planar graph G, $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$.*

By Lemma 1, Theorem 4 implies Theorem 3.

An *injective k-coloring* [17] of G is a (not necessarily proper) coloring of the vertices of G with k colors such that two vertices that have a common neighbor receive distinct colors. We define $\chi_i(G)$ as the smallest k such that G admits an injective k-coloring. A 2-distance k-coloring is an injective k-coloring, but the converse is not true. For example, the cycle of length 5 can be injective colored with 3 colors. The list version of this coloring is a *list injective k-coloring* of G, and $\chi_{i,\ell}(G)$ is the smallest k such that G admits a list injective k-coloring.

Some results on 2-distance coloring have their counterpart on injective coloring with one less color. This is the case of Theorems 1 and 2 [8,9]. The proof of Theorem 4 also works with close to no alteration for list injective coloring, thus yielding a proof that every graph G with $\Delta(G) \ge 17$ and $\mathrm{mad}(G) < 3$ admits a list injective $(\Delta(G) + 1)$-coloring.

In Sections 2 and 3, we introduce the method and terminology. In Sections 4 and 6, we prove Theorem 4 and its counterpart on injective coloring by a discharging method.

2. Method

The discharging method was introduced in the beginning of the 20th century. It has been used to prove the celebrated Four Color Theorem in [1,2]. A discharging method is said to be *local* when the weight cannot travel arbitrarily far. Borodin, Ivanova and Kostochka introduced in [10] the notion of *global* discharging method, where the weight can travel arbitrarily far along the graph.

We prove for induction purposes a slightly stronger version of Theorem 4 by relaxing the constraint on the maximum degree. Namely, we relax it to "For any $k \ge 17$, every graph G with $\Delta(G) \le k$ and $\mathrm{mad}(G) < 3$ verifies $\chi_\ell^2(G) \le k + 2$" so that the property is closed under vertex- or edge-deletion. A graph is *minimal* for a property if it satisfies this property but none of its subgraphs does.

The first step is to consider a minimal counter-example G, and prove that it cannot contain some configurations. To do so, we assume by contradiction that G contains one of the configurations. We consider a particular subgraph H of G, and

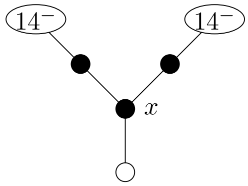

Fig. 1. A weak vertex *x*.

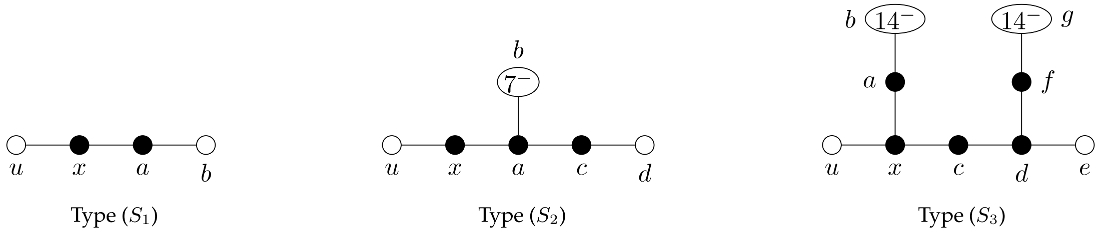

Fig. 2. Support vertices *x*.

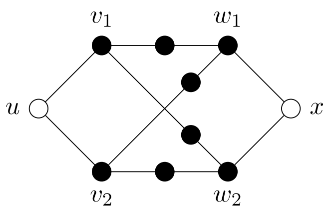

Fig. 3. A locked vertex *u*.

color it by minimality (the maximum average degree of any subgraph of *G* is bounded by the maximum average degree of *G*). We show how to extend the coloring of *H* to *G*, a contradiction.

The second step is to prove that a graph that does not contain any of these configurations has a maximum average degree of at least 3. To that purpose, we assign to each vertex its degree as a weight. We apply discharging rules to redistribute weights along the graph with conservation of the total weight. As some configurations are forbidden, we can then prove that after application of the discharging rules, every vertex has a final weight of at least 3. This implies that the average degree of the graph is at least 3, and hence the maximum average degree is at least 3. So a minimal counter-example cannot exist.

We finally explain how the same proof holds also for list injective $(\Delta + 1)$-coloring.

3. Terminology

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. White vertices may coincide with other vertices of the figure. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label '*i*' means "exactly *i* neighbors", the label 'i^+' (resp. 'i^-') means that it has at least (resp. at most) *i* neighbors.

Let *u* be a vertex. The *neighborhood* $N(u)$ of *u* is the set of vertices that are adjacent to *u*. Let $d(u) = |N(u)|$ be the *degree* of *u*. A *p-link* $x - a_1 - \cdots - a_p - y$, $p \geq 0$, between *x* and *y* is a path between *x* and *y* such that $d(a_1) = \cdots = d(a_p) = 2$. When a *p*-link exists between two vertices *x* and *y*, we say they are *p-linked*. If there is a *p*-link $x - a_1 - \cdots - a_p - y$ between *x* and *y*, we say *x* is *p-linked through* a_1 to *y*. A *partial* 2-*distance list coloring* of *G* is a 2-distance list-coloring of a subgraph *H* of *G*.

A vertex *x* is *weak* when it is of degree 3 and is 1-linked to two vertices of degree at most 14, or twice 1-linked to a vertex of degree at most 14 (see Fig. 1). A weak vertex is represented with a w label inside ($\overline{w}$ if it is not weak).

A vertex *x* is *support* when it is either (see Fig. 2):

Type (S_1): a vertex of degree 2 adjacent to another vertex of degree 2;
Type (S_2): a vertex of degree 2 that is adjacent to a vertex of degree 3 which is adjacent to another vertex of degree 2 and to a vertex of degree at most 7;
Type (S_3): a weak vertex 1-linked to another weak vertex.

A vertex is *positive* when it is of degree at least 4 and is adjacent to a support vertex. A vertex *u* is *locked* if it has two neighbors v_1 and v_2, where v_1 and v_2 are both 1-linked to the same two vertices w_1 and w_2 that have a common neighbor, and $d(v_1) = d(v_2) = d(w_1) = d(w_2) = 3$ (see Fig. 3). This configuration is called a *lock*.

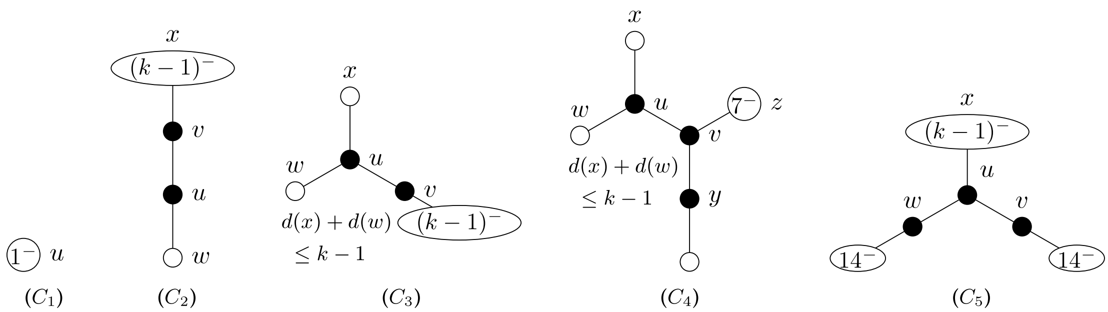

Fig. 4. Forbidden configurations (C_1)–(C_5).

4. Forbidden configurations

In all of the paper, k is a constant integer at least 17 and G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+2)$-list-coloring.

We define configurations (C_1)–(C_{11}) (see Figs. 4–6). Note that configurations similar to Configurations (C_1), (C_2) and (C_4) already existed in the literature, for example in [15].

- (C_1) is a vertex u with $d(u) \leq 1$.
- (C_2) is a vertex u with $d(u) = 2$ that has two neighbors v, w and u is 1-linked through v to a vertex of degree at most $k-1$.
- (C_3) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(w) + d(x) \leq k-1$, and u is 1-linked through v to a vertex of degree at most $k-1$.
- (C_4) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(w) + d(x) \leq k-1$, and v has exactly three neighbors u, y, z with $d(z) \leq 7$ and $d(y) = 2$.
- (C_5) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(x) \leq k-1$ and u is 1-linked through v (resp. through w) to a vertex of degree at most 14. (Note that u is a weak vertex.)
- (C_6) is a vertex u with $d(u) = 4$ that has four neighbors v, w, x, y with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, and u is 1-linked through v to a vertex of degree at most 14.
- (C_7) is a vertex u with $d(u) = 4$ that has four neighbors v, w, x, y with $d(x) + d(y) \leq k-1$ and u is 1-linked through v (resp. through w) to a vertex of degree at most 14.
- (C_8) is a vertex u with $d(u) = 5$ that has five neighbors v, w, x, y, z with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, $d(z) = 2$, and u is 1-linked through v to a vertex of degree at most 7.
- (C_9) is a vertex u with $d(u) = 6$ that has six neighbors v, w, x, y, z, t with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, $d(z) = 2$, $d(t) = 2$, and u is 1-linked through v to a vertex of degree at most 7.
- (C_{10}) is a vertex u with $d(u) = 7$ that has seven neighbors $v, w_1, \ldots, w_6$ with $d(v) \leq 7$ and u is 1-linked through w_i, $1 \leq i \leq 6$, to a vertex of degree at most 3.
- (C_{11}) is a vertex u with $d(u) = k$ that has three neighbors v, w, x with x is a support vertex, v, w are both 1-linked to a same vertex y of degree 3, and v (resp. w) is 1-linked to a vertex of degree at most 14 distinct from y. (Note that v, w are weak vertices.)

Lemma 2. *G does not contain Configurations (C_1)–(C_{11}).*

Proof. Given a partial 2-distance list-coloring of G, a *constraint* of a vertex u is any color appearing on a vertex at distance at most 2 from u in G.

Notation refers to Figs. 4–6.

Claim 1. *G does not contain (C_1).*

Proof. Suppose by contradiction that G contains (C_1). Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints (one for its neighbor and at most $k-1$ for the vertices at distance 2 from u). There are $k+2$ colors available in the list of u, so the coloring of $G \setminus \{u\}$ can be extended to G, a contradiction. □

Claim 2. *G does not contain (C_2).*

Proof. Suppose by contradiction that G contains (C_2). Using the minimality of G, we color $G \setminus \{u, v\}$. Vertex u has at most $k+1$ constraints. Hence we can color u. Then v has at most $k-1+2 = k+1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 3. *G does not contain (C_3).*

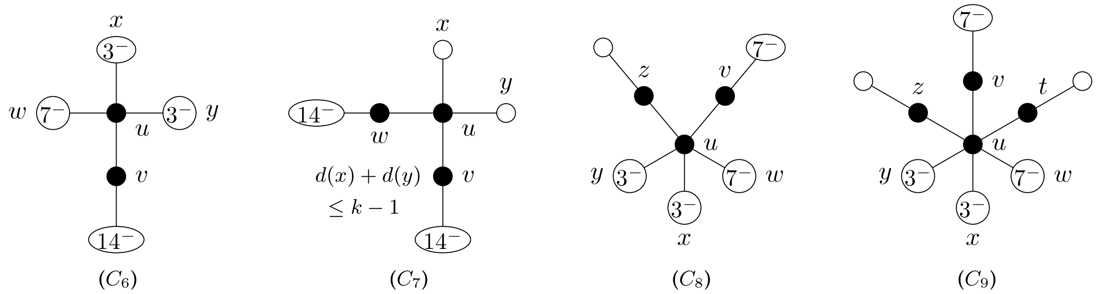

Fig. 5. Forbidden configurations (C_6)–(C_9).

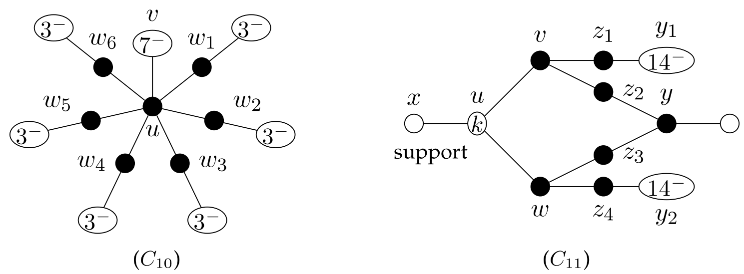

Fig. 6. Forbidden configurations (C_{10}) and (C_{11}).

Proof. Suppose by contradiction that G contains (C_3). Using the minimality of G, we color $G \setminus \{v\}$. Because of u, vertices w and x have different colors. We discolor u. Vertex v has at most $k - 1 + 2 = k + 1$ constraints. Hence we can color v. Vertex u has at most $d(w) + d(x) + 2 \leq k + 1$ constraints. Hence we can color u. So we can extend the coloring to G, a contradiction. □

Claim 4. *G does not contain* (C_4).

Proof. Suppose by contradiction that G contains (C_4). Let e be the edge uv. Using the minimality of G, we color $G \setminus \{e\}$. We discolor u and v. Vertex u has at most $d(w) + d(x) + 2 \leq k + 1$ constraints. Hence we can color u. Vertex v has at most $7 + 3 + 2 \leq k + 1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 5. *G does not contain* (C_5).

Proof. Suppose by contradiction that G contains (C_5). Using the minimality of G, we color $G \setminus \{u, v, w\}$. Vertex u has at most $k - 1 + 2 = k + 1$ constraints. Hence we can color u. Vertices v and w have at most $14 + 3 \leq k + 1$ constraints. Hence we can color v and w. So we can extend the coloring to G, a contradiction. □

Claim 6. *G does not contain* (C_6).

Proof. Suppose by contradiction that G contains (C_6). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex v has at most $14 + 3 \leq k + 1$ constraints. Hence we can color v. Vertex u has at most $2 + 3 + 3 + 7 \leq k + 1$ constraints. Hence we can color u. So we can extend the coloring to G, a contradiction. □

Claim 7. *G does not contain* (C_7).

Proof. Suppose by contradiction that G contains (C_7). Using the minimality of G, we color $G \setminus \{v, w\}$. We discolor u. Vertex u has at most $d(x) + d(y) + 2 \leq k + 1$ constraints. Hence we can color u. Vertices v and w have at most $14 + 4 \leq k + 1$ constraints. Hence we can color v and w. So we can extend the coloring to G, a contradiction. □

Claim 8. *G does not contain* (C_8).

Proof. Suppose by contradiction that G contains (C_8). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex u has at most $7 + 3 + 3 + 2 + 1 \leq k + 1$ constraints. Hence we can color u. Vertex v has at most $7 + 5 \leq k + 1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 9. *G does not contain* (C_9).

Proof. Suppose by contradiction that G contains (C_9). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex u has at most $7 + 3 + 3 + 2 + 2 + 1 \leq k + 1$ constraints. Hence we can color u. Vertex v has at most $7 + 6 \leq k + 1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 10. *G does not contain (C_{10}).*

Proof. Suppose by contradiction that G contains (C_{10}). Using the minimality of G, we color $G \setminus \{u, w_1, \dots, w_6\}$. Vertex u has at most $7 + 6 \leq k + 1$ constraints. Hence we can color v. Each vertex w_i has at most $3 + 7 \leq k + 1$ constraints. Hence we can color $w_1, \dots, w_6$. So we can extend the coloring to G, a contradiction. □

Claim 11. *G does not contain (C_{11}).*

Proof. Suppose by contradiction that G contains (C_{11}). Since x is a support vertex, and u is of degree k, it is of Type (S_1), (S_2) or (S_3) of support vertices with the notation of Fig. 2. Note that some vertices may coincide between Figs. 2 and 6.

We define a set of vertices A as follows:

$$A = \begin{cases} \{a\} & \text{if } x \text{ is of Type } (S_1) \\ \{a, c\} & \text{if } x \text{ is of Type } (S_2) \\ \{a, c\} & \text{if } x \text{ is of Type } (S_3). \end{cases}$$

Using the minimality of G, we color $G \setminus (\{v, w, x, y, z_1, \dots, z_4\} \cup A)$. If x is of Type (S_1) (resp. (S_2)), a (resp. c) has at most $k + 1$ constraints. Hence we can color a (resp. c). For the three types (S_i), x has at most $k - 3 + 1 + 2 = k$ constraints, thus it has at least 2 available colors. Vertex y has at most k constraints, thus it has at least 2 available colors. Both v and w have at most $k - 3 + 1 + 1 \leq k - 1$ constraints, so they have at least 3 available colors in their list.

We now explain how to color v, w, x, y (other uncolored vertices will be colored later). Suppose x and y can be assigned the same color, then both v and w have at least 2 available colors and thus can be colored.

Suppose the lists of available colors of x and y are disjoint. We color v with a color not appearing in the list of x. Then we color y that has $k + 1$ constraints. (Vertex x has still at least 2 available colors.) Then we color w that has $k + 1$ constraints and finally x.

Now we assume that we cannot assign the same color to x and y and that their lists of available colors are not disjoint. This means that x and y are either adjacent or have a common neighbor. So some vertices coincide between Figs. 2 and 6. The different cases where x and y are either adjacent or have a common neighbor are the following:

(S_1) – $b = y$.
(S_2) – $b = y$
– $a = y$ and w.l.o.g $b = z_2$, $c = z_3$ and $d = w$.
(S_3) – $b = y$
– $d = y$, and w.l.o.g. $f = z_2$, $g = v$ and $e = z_3$.

In all these cases, y has at most 1 constraint. So we can color x, v, w, y, in this order as they all have at most $k + 1$ constraints when they are colored.

If x is of Type (S_2) (resp. (S_3)), vertex a (resp. vertices a, c) has at most 11 constraints (resp. 17, 6), so we can color them. The vertices z_i have at most $17 \leq k+1$, so we can color them. Thus the coloring has been extended to G, a contradiction. □

□

5. Structure of support vertices

Let $H(G)$ be the subgraph of G induced by the edges incident to at least a support vertex. We prove several properties of support vertices and of the graph $H(G)$.

Lemma 3. *Each positive vertex is of degree k and each support vertex is adjacent to exactly one positive vertex.*

Proof. By Lemma 2, G does not contain Configurations (C_2), (C_3) and (C_5). So a support vertex is adjacent to a vertex of degree k (Configurations (C_2), (C_3) and (C_5) correspond respectively to support vertices of Types (S_1), (S_2) and (S_3)). By definition, a support vertex has at most one neighbor of degree at least 4, thus it is adjacent to exactly one vertex of degree at least 4 and this vertex has in fact degree k. So all the positive vertices are of degree k and a support vertex is adjacent to exactly one positive vertex. □

Lemma 4. *Each cycle of $H(G)$ with an odd number of support vertices contains a subpath $s_1v_1s_2v_2s_3$ where s_1, s_2, s_3 are support vertices of type (S_3) and v_1, v_2 are vertices of degree 2.*

Proof. Let C be a cycle of $H(G)$ with an odd number of support vertices. Cycle C does not contain just one support vertex, as all its edges have to be adjacent to a support vertex (there is no loop or multiple edge in $H(G)$). So C contains at least three support vertices.

Suppose that C contains no positive vertices. Then it contains no support vertices of type (S_1) or (S_2) as such vertices are of degree 2, so all their neighbors would be on C, and they are adjacent to a positive vertex by Lemma 3. So C contains only support vertices of type (S_3). Let s_1, s_2, s_3 be three support vertices of C appearing consecutively along C. A support vertex of Type (S_3) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of s_i on C are vertices of degree 2 that are not support vertices. As $H(G)$ contains only edges incident to support vertices, there exist v_1, v_2 of degree 2 such that $s_1v_1s_2v_2s_3$ is a subpath of C.

Suppose now that C contains some positive vertices. Let $p_1, \dots, p_\ell$ be the set of positive vertices of C appearing in this order along C while walking in a chosen direction (subscripts are understood modulo ℓ). Let Q_i, $1 \le i \le \ell$, be the subpath of C between p_i and p_{i+1} (in the same chosen direction along C). (Note that if $\ell = 1$, then $Q_1 = C$ is not really a subpath.) As C contains an odd number of support vertices, there exists i such that Q_i contains an odd number of support vertices. If Q_i contains just one support vertex v, then Q_i has length 2, since $H(G)$ contains only edges incident to support vertices. So v is adjacent to two different positive vertices (or has a multiple edge if $\ell = 1$), a contradiction to Lemma 3. So Q_i contains at least 3 support vertices. Let s_1, s_2, s_3 be three support vertices of Q_i appearing consecutively along Q_i.

If one of the s_i is of Type (S_1), let x be such a vertex. With the notation of Fig. 2, vertex x is of degree 2, so its two neighbors u, a are on C, with u a positive vertex and a a support vertex of Type (S_1). Then vertex a is of degree 2 so its neighbor b distinct from x is also on C. Vertex b is positive so Q_i is the path u, x, a, b and contains just two support vertices, a contradiction.

If one of the s_i is of Type (S_2), let x be such a vertex. With the notation of Fig. 2, vertex x is of degree 2, so its two neighbors u, a are on C, with u a positive vertex and a a vertex of degree 3. Vertex a is not adjacent to vertices of degree k so by Lemma 3, it is not a support vertex. Let c' be the neighbor of a on C that is distinct from x. As all the edges of $H(G)$ are incident to support vertices, c' is a support vertex. Since c' is adjacent to a vertex of degree 3 it is a support vertex of Type (S_2) and can play the role of c of Fig. 2. Then c is of degree 2 and its neighbor on C distinct from a is a positive vertex d. So Q_i is the path u, x, a, c, d and contains just two support vertices, a contradiction.

So s_1, s_2, s_3 are all of Type (S_3). A support vertex of Type (S_3) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of s_2 on C are vertices v_1, v_2 of degree 2 that are not support vertices. As $H(G)$ contains only edges incident to support vertices, we can assume w.l.o.g. that $s_1v_1s_2v_2s_3$ is a subpath of C. □

Lemma 5. *$H(G)$ does not contain a 2-connected subgraph of size at least three with exactly two support vertices.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that has exactly two support vertices $S = \{s_1, s_2\}$. We color by minimality $G \setminus (S \cup \{v \in N_G(S) | d_G(v) \le 3\})$. (Note that by Lemma 3, the set $\{v \in N_G(S) | d_G(v) \le 3\}$ corresponds to vertex a of Fig. 2 if the support vertex is of Type (S_1) or (S_2) and to vertices a, c if the support vertex is of Type (S_3).)

We first show how to color S. For that purpose we consider three cases corresponding to the type of s_1.

- *s_1 is of Type (S_1).* Then s_1 is of degree 2, has a positive neighbor u and a support neighbor a of Type (S_1). As s_1 is of degree 2, both its neighbors are in C. So a is a support vertex of C, thus $a = s_2$. Let v be the neighbor of s_2 of degree k. Since C contains no other support vertex and is 2-connected, we must have $u = v$. Then u has two neighbors s_1, s_2 that are not colored, so s_1 and s_2 have at most k constraints, and we can color them.
- *s_1 is of Type (S_2).* Then s_1 is of degree 2, has a positive neighbor u and another neighbor a of degree 3. Vertex a is not a support vertex by Lemma 3 since it has no neighbor of degree k. As s_1 is of degree 2, all its neighbors are in C. Vertices u and a are in C that is 2-connected so they have at least two neighbors in C. Since they are not support vertices, all their neighbors in C are support vertices. So both u and a are adjacent to s_2. Vertex s_2 is support, it is adjacent to a that is of degree 3, so s_2 is of Type (S_2). Then u is of degree k, has two neighbors s_1, s_2 that are not colored, so s_1 and s_2 have at most k constraints, and we can color them.
- *s_1 is of Type (S_3).* Then s_1 is of degree 3, has a positive neighbor u and two other neighbors w, w' of degree 2. Vertices w, w' are not support vertices by Lemma 3 since they have no neighbor of degree k. As s_1 is of degree 3, two of u, w, w' are in C. Let Y be the neighbors of s_1 in C. We can assume by symmetry that either $\{v, w\} \subseteq Y$ or $\{w, w'\} \subseteq Y$. Vertices of Y are in C that is 2-connected so they have at least two neighbors in C. Since they are not support vertices, all their neighbors in C are support vertices. So all the vertices of Y are adjacent to s_2. Vertex s_2 is a support vertex, it is adjacent to w that is non support and of degree 2, so s_2 is of Type (S_3). In both cases ($\{v, w\} \subseteq Y$ or $\{w, w'\} \subseteq Y$), vertices s_1 and s_2 have at most k constraints, and we can color them.

Every vertex of $\{v \in N_G(S) | d_G(v) \le 3\}$ has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction. □

Lemma 6. *Every 2-connected subgraph of $H(G)$ that contains exactly three support vertices is a cycle.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that has exactly three support vertices $S = \{s_1, s_2, s_3\}$ and that is not a cycle.

Suppose by contradiction that C contains no cycle C' with $S \subseteq C' \subseteq C$. As C is 2-connected, by Menger's Theorem there exist two internally vertex-disjoint paths Q, Q' between s_1, s_2. Let C'' be the cycle $Q \cup Q'$. By assumption C'' does not contain s_3. So it contains just two support vertices, a contradiction to Lemma 5. So C contains a cycle C' with $S \subseteq C' \subseteq C$.

By Lemma 4, cycle C' contains a subpath $x_1v_1x_2v_2x_3$ where x_1, x_2, x_3 are support vertices of Type (S_3) and v_1, v_2 are vertices of degree 2. As C contains just three support vertices, we have $S = \{x_1, x_2, x_3\}$. Vertices x_1, x_3 are support vertices of Type (S_3), they are of degree 3 and only adjacent to positive vertices and to vertices of degree 2 so they are not adjacent. The graph $H(G)$ contains only edges incident to support vertices, so there exists a vertex y of C' adjacent to x_1, x_3, and $x_1v_1x_2v_2x_3y$ is the cycle C'. If C' has some chords in $H(G)$, then $H(G)$ contains a cycle with two support vertices only, a contradiction to Lemma 5. So C' is an induced cycle of $H(G)$ and so C' has strictly less vertices than C. Let y' be a vertex of C distinct from $x_1, v_1, x_2, v_2, x_3, y$. Vertex y' is not a support vertex, C is 2-connected and $H(G)$ contains only edges incident to support vertices, so y' is adjacent to at least two vertices in S. Then $H(G)$ contains a cycle with two support vertices only, a contradiction to Lemma 5. □

We need the following lemma from Erdős et al. [16]:

Lemma 7 ([16]). *If G is a 2-connected graph that is neither a clique nor an odd cycle, and L is a list assignment on the vertices of G such that $\forall u \in V(G)$, $|L(u)| \geq d(u)$, then G is L-colorable.*

Lemma 8. *Every 2-connected subgraph of $H(G)$ of size at least three is either a cycle with an odd number of support vertices or a subgraph of a lock of $H(G)$.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that is not a cycle with an odd number of support vertices nor a subgraph of a lock of $H(G)$. Let $S = \{s_1, \dots, s_p\}$ be the support vertices of C. By Lemma 5, $p \geq 3$. Let $\mathcal{S}$ be the graph with $V(\mathcal{S}) = S$ where there is an edge between s_i and s_j if and only if they are adjacent or have a common neighbor in G.

Claim 12. *$\mathcal{S}$ is not a clique of size at least four.*

Proof. Suppose, by contradiction that $\mathcal{S}$ is a clique with $p \geq 4$.

Given a support vertex x, we say that a support vertex x', distinct from x, satisfies the property P_x if it is either adjacent to x in G or has a non-positive common neighbor with x in G. At most two vertices can satisfy P_x (vertex a of Fig. 2 if x is of Type (S_1), vertices b, c if x is of Type (S_2), vertices b, d if x is of Type (S_3)). Note that if x' satisfies P_x, then x satisfies $P_{x'}$.

We claim that there exist two support vertices in S that do not have a positive common neighbor in G. Suppose by contradiction, that every pair of vertices of S has a positive common neighbor. By Lemma 3, every support vertex has at most one positive neighbor, so all the vertices of S are adjacent to the same positive vertex v. As C is 2-connected, there is a path Q in $C \setminus \{v\}$ between s_1, s_2. Let s_i be the first support vertex, distinct from s_1, appearing along Q while starting from s_1 (maybe $i = 2$ if there is no support vertex in the interior of Q). Let Q' be the subpath of Q between s_1 and s_i (maybe $Q = Q'$). Then $Q' \cup \{v\}$ forms a 2-connected subgraph of size ≥ 3 with exactly two support vertices, a contradiction to Lemma 5. So there exist two support vertices x, x' in S that do not have a positive common neighbor in G. Since $\mathcal{S}$ is a clique, vertices x, x' are adjacent or have a common non-positive neighbor, so x satisfies $P_{x'}$ (and x' satisfies P_x).

Suppose there exists a support vertex $y \in S$ that does not satisfy P_x or $P_{x'}$. Since $\mathcal{S}$ is a clique, vertex y has a common positive neighbor z with x and z' with x'. Since x and x' have no positive common neighbor, z and z' are distinct. Thus y has two positive neighbors, a contradiction. So every vertex of $S \setminus \{x, x'\}$ satisfies either P_x or $P_{x'}$. If two vertices y, y' of $S \setminus \{x, x'\}$ satisfy P_x, then at least three vertices, x', y, y' verify P_x, a contradiction. So there is at most one vertex of $S \setminus \{x, x'\}$ satisfying P_x and similarly at most one satisfying $P_{x'}$. So $p \leq 4$ and we can assume, w.l.o.g., that $S = \{x, x', y, y'\}$, where vertex y satisfies P_x and not $P_{x'}$ and vertex y' satisfies $P_{x'}$ and not P_x. Thus x has a common positive neighbor z with y' and x' has a common positive neighbor z' with y. Since x, x' do not have a common positive neighbor, z and z' are distinct. Vertices y, y' have at most one positive neighbor, thus, they do not have a common positive neighbor. Since $\mathcal{S}$ is a clique, y satisfies $P_{y'}$. Let $(y_1, y_2, y_3, y_4) = (x, x', y', y)$ (subscripts are understood modulo 4).

Suppose there exists $i \in \{1, 2, 3, 4\}$ such that y_i, y_{i+1} are adjacent in G. Two support vertices can be adjacent only if they are of Type (S_1). So y_i, y_{i+1} are of Type (S_1) and of degree two. Then y_i is only adjacent to y_{i+1} and to a positive vertex in $\{z, z'\}$. If y_i is adjacent to y_{i-1}, then $y_{i-1} = y_{i+1}$, a contradiction. If y_i is not adjacent to y_{i-1}, then y_{i+1} is a common neighbor of y_i and y_{i-1}. Since y_{i+1} is of degree two and has a positive neighbor, $y_i = y_{i-1}$, a contradiction. So y_i, y_{i+1} are not adjacent in G for any $1 \leq i \leq 4$. Let w_i be a non-positive common neighbor of y_i, y_{i+1}.

Suppose there exists $i \in \{1, 2, 3, 4\}$ such that $d(y_i) = 2$. Then $w_i = w_{i-1}$. So $\{y_{i-1}, y_i, y_{i+1}\} \subseteq N(w_i)$, and w_i is not positive, so $d(w_i) = 3$. Two support vertices can have a common neighbor of degree 3 only if they are both of degree two (Type (S_2)). So $d(y_{i-1}) = d(y_i) = d(y_{i+1}) = 2$. Since y_{i+1} is of degree two and has a positive neighbor, $w_i = w_{i+1}$, so $y_{i+2} \in N(w_i)$, a contradiction. So $d(y_i) \geq 3$ for any $1 \leq i \leq 4$.

Then all the y_i are of Type (S_3), they are of degree three and their non-positive neighbors are of degree two. Thus $d(w_i) = 2$ for any $1 \leq i \leq 4$. So $y_1, \dots, y_4, w_1, \dots, w_4, z, z'$ induce a lock. So all the edges incident to $S = \{y_1, \dots, y_4\} = \{s_1, \dots, s_4\}$ belong to a lock, contradicting the definition of C. □

By Lemma 5, the graph $\mathcal{S}$ is not an edge. If $\mathcal{S}$ is a triangle, then C contains exactly three support vertices and, by Lemma 6, it is a cycle with an odd number of support vertices, a contradiction. So $\mathcal{S}$ is not a triangle. By Claim 12, $\mathcal{S}$ is not a clique of size at least 4. So finally, $\mathcal{S}$ is not a clique.

Suppose, by contradiction, that $\mathcal{S}$ is an odd cycle with ≥ 5 vertices. Then C is a 2-connected graph that is not a cycle, so it contains a vertex v with at least 3 neighbors in C. If v is not a support vertex, then it has at least 3 support neighbors in C that form a triangle in $\mathcal{S}$, a contradiction. So v is a support vertex. Then either v has three neighbors in $\mathcal{S}$, a contradiction to $\mathcal{S}$ being a cycle, or C contains a cycle with two support vertices, a contradiction to Lemma 5. So $\mathcal{S}$ is not an odd cycle.

Suppose, by contradiction, that $\mathcal{S}$ is not 2-connected. Then there exist three support vertices s, s', s'' of S such that s', s'' appear in two different connected components of $\mathcal{S} \setminus \{s\}$. As C is 2-connected, there exists a path Q between s', s'' in $C \setminus \{s\}$. This path Q is composed only of edges incident to support vertices so in $\mathcal{S} \setminus \{s\}$ it corresponds to a path between s', s'', a contradiction. So $\mathcal{S}$ is 2-connected.

We now consider the graph G, we color by minimality $G \setminus (S \cup \{v \in N_G(S) | d_G(v) \leq 3\})$. We show how to color S. In the three Types (S_j), the number of constraints on a support vertex s_i of Type (S_j) is at most $k + 2$ minus the number of its

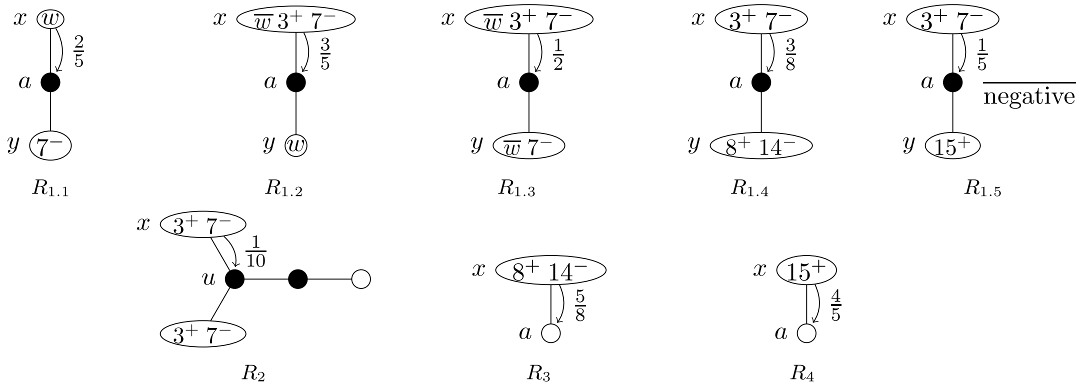

Fig. 7. Discharging rules $R_{1.i}$, R_2, R_3, and R_4.

neighbors in $\mathcal{S}$. So the number of available colors of a support vertex is at least its degree in $\mathcal{S}$. Now Lemma 7 can be applied to $\mathcal{S}$, which is not a clique, not an odd cycle and 2-connected. So we can color S. Every vertex of $\{v \in N_G(S) | d_G(v) \leq 3\}$ has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction. □

A *cactus* is a connected graph in which any two cycles have at most one vertex in common.

Lemma 9. *Every connected component of $H(G)$ is either a cactus where each cycle has an odd number of support vertices or a lock.*

Proof. All the edges of a lock are incident to support vertices of type (S_3) so all the edges of a lock of G appear in $H(G)$. The only vertices of a lock that can have neighbors outside a lock are locked vertices (vertices u and x in Fig. 3). By Lemma 2, graph G does not contain Configuration (C_{11}), so a locked vertex is incident to only two support vertices, the two support vertices of a lock. A lock is a connected component of $H(G)$.

Let C be a connected component of $H(G)$ that is not a lock. By Lemma 8, each 2-connected subgraph of C is a cycle with an odd number of support vertices. So C is a cactus where each cycle of C has an odd number of support vertices. □

6. Discharging rules

A *negative* vertex is a support vertex of type (S_1) or (S_2) or a vertex of degree 2 adjacent to two support vertices of type (S_3). In this case we say that the negative vertex is of type (N_1), (N_2) or (N_3) respectively.

Each vertex has an initial weight (later defined). The discharging rules $R_{1.1}, R_{1.2}, R_{1.3}, R_{1.4}, R_{1.5}, R_2, R_3, R_4$ and R_g (see Fig. 7) defined below explain how vertices will receive and/or give weight. We also use a so-called *common pot* which is empty at the beginning, receives weight from some vertices and gives weight to some others. For any vertex x of degree at least 3,

- Rule R_1 is when $3 \leq d(x) \leq 7$, and x is 1-linked (with a path $x - a - y$) to a vertex y.
  - Rule $R_{1.1}$ is when x is weak with $d(y) \leq 7$. Then x gives $\frac{2}{5}$ to a.
  - Rule $R_{1.2}$ is when x is not weak and y is weak. Then x gives $\frac{3}{5}$ to a.
  - Rule $R_{1.3}$ is when x and y are not weak, with $d(y) \leq 7$. Then x gives $\frac{1}{2}$ to a.
  - Rule $R_{1.4}$ is when $8 \leq d(y) \leq 14$. Then x gives $\frac{3}{8}$ to a.
  - Rule $R_{1.5}$ is when $15 \leq d(y)$ and a is not negative. Then x gives $\frac{1}{5}$ to a.
- Rule R_2 is when $3 \leq d(x) \leq 7$ and x is adjacent to a vertex u of degree 3 that is adjacent to a vertex of degree 2 and a vertex of degree at most 7. Then x gives $\frac{1}{10}$ to u.
- Rule R_3 is when $8 \leq d(x) \leq 14$. Then x gives $\frac{5}{8}$ to each of its neighbors.
- Rule R_4 is when $15 \leq d(x)$. Then x gives $\frac{4}{5}$ to each of its neighbors.
- Rule R_g states that each positive vertex gives $\frac{2}{5}$ to a common pot, and that each negative vertex receives $\frac{1}{5}$ from the common pot.

Lemma 10. *The common pot has a non-negative value after applying R_g.*

Proof. Given a set of vertices X, let $n(X)$ be its number of negative vertices and $p(X)$ its number of positive vertices. To prove that the common pot has a positive value after applying R_g, we show that each connected component C of $H(G)$ satisfies $p(C) \geq \left\lceil \frac{n(C)}{2} \right\rceil$.

Let C be a connected component of $H(G)$. By Lemma 9, C is either a cactus where each cycle has an odd number of support vertices or a lock. If C is a lock, then $n(C) = 4$ and $p(C) = 2$, so we are done. So we can assume that C is a cactus where each cycle has an odd number of support vertices.

Claim 13. *Every connected subgraph C' of C, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in C' and whose negative vertices of Type (N_3) are adjacent to their two neighbors in C', satisfies* $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

Proof. Suppose by contradiction that this is false. Let C' be a connected subgraph of C of minimum number of vertices, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in C', and such that $p(C') < \left\lceil \frac{n(C')}{2} \right\rceil$. The graph C' is a connected subgraph of a cactus so it is also a cactus.

Suppose first that C' contains a pendant vertex u. Let x be the neighbor of the positive vertex u in C'. As $H(G)$ contains only edges incident to support vertices, x is a support vertex. So it is not positive and thus is not a pendant vertex of C'. So x has at least two neighbors in C'. We consider different cases according to the Type of x and its number of neighbors in C'.

- *x is of Type (S_1)*. Then let a be the neighbor of x distinct from u. We have $a \in C'$ and a is a support vertex of Type (S_1). The positive neighbor b of a is in C' by assumption. Let C'' be the graph $C' \setminus \{u, x, a\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C since u, x, a is a subpath of C' where u is pendant and x, a are of degree 2. All the pendant vertices of C'' are positive since the only new possible pendant vertex is b. All the support vertices of C'' are adjacent to their positive neighbor in C'' since the only positive vertex that has been removed is u and its support neighbor x has also been removed. All the negative vertices of Type (N_3) are adjacent to their two neighbors in C' as no support vertex of Type (S_3) has been removed. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') = p(C'') + 1 \geq \left\lceil \frac{n(C'')+2}{2} \right\rceil = \left\lceil \frac{n(C')}{2} \right\rceil$.
- *x is of Type (S_2)*. Then let a be the neighbor of x distinct from u. We have $a \in C'$ and a is of degree 3. Let b, c be the neighbors of a distinct from x. Since a is not positive, it is not a pendant vertex of C', so at least one of b, c is in C'. We assume w.l.o.g. that c is in C'. As $H(G)$ contains only edges incident to support vertices, vertex c is a support vertex of Type (S_2). We consider two cases depending on whether a has its three neighbors in C' or not.

  If $b \in C'$, then let C'' be the graph $C' \setminus \{u, x\}$. We have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

  If $b \notin C'$, then let C'' be the graph $C' \setminus \{u, x, a, c\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.
- *x is of Type (S_3) and has two neighbors in C'*. Then let c be the neighbor of x distinct from u that is in C'. Vertex c is of degree 2, it is not positive, so its neighbor d, distinct from x, is in C'. As $H(G)$ contains only edges incident to support vertices and c is not a support vertex, vertex d is a support vertex and so of Type (S_3). Let e, f be the neighbors of d distinct from c where e is a positive vertex and f is a vertex of degree 2. Vertex e is the positive neighbor of d so it is in C' by assumption. We consider two cases corresponding to whether d has its three neighbors in C' or not. If $f \in C'$, then let C'' be the graph $C' \setminus \{u, x, c\}$. If $f \notin C'$, then let C'' be the graph $C' \setminus \{u, x, c, d\}$. In both cases, we have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.
- *x is of Type (S_3) and has three neighbors in C'*. Then let a, c be the neighbors of x distinct from u. We have a, c in C'. Vertex a (resp. c) is of degree 2, it is not positive, so its neighbor b (resp. d) is in C'. As $H(G)$ contains only edges incident to support vertices and a and c are not support vertices, vertices b and d are support vertices and thus of Type (S_3). The positive neighbor h of b (resp. e of d) is in C', by assumption. We consider several cases corresponding to whether b and d have their three neighbors in C' or not. If b and d both have their three neighbors in C', then let C'' be the graph $C' \setminus \{u, x, c, a\}$. If b has its three neighbors in C' but not d, then let C'' be the graph $C' \setminus \{u, x, c, a, d\}$. If d has its three neighbors in C' but not b, then let C'' be the graph $C' \setminus \{u, x, c, a, b\}$. If none of b and d has its three neighbors in C', then let C'' be the graph $C' \setminus \{u, x, c, a, b, d\}$. In the four cases we have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is not necessarily connected but it is composed of one or two connected subgraphs of C whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality (on each component of C''), we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

Now we can assume that C' contains no pendant vertex. Suppose that C' is a single vertex v. Then v is not support as all support vertices have their positive neighbor in C' and v is not negative of Type (N_3) as negative vertices of Type (N_3) have their two neighbors in C'. So v is not negative and $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil = 0$. Now we can assume that C' is not a single vertex. The graph C' is a cactus, not a single vertex, contains no pendant vertex, so it contains a cycle C'', of size ≥ 3, such that

$C''' = C' \setminus C''$ is connected (note that we may have $C' = C''$ and C''' empty). Cycle C'' is a cycle of C so it has an odd number of support vertices by Lemma 9. Let S be the set of support vertices of C'', with $s = |S|$. By Lemma 4, cycle C'' contains a subpath $s_1v_1s_2v_2s_3$ where s_1, s_2, s_3 are support vertices of Type (S_3) and v_1, v_2 are vertices of degree 2. By assumption, the positive vertex z that is adjacent to s_2 is in C'. It is not in C'' as there is no chord in C''. So the only vertex of C'' that has some neighbors in $C' \setminus C''$ is s_2. So all the positive vertices that are adjacent to $S \setminus \{s_2\}$ are vertices of C' and thus of C''. A positive vertex of C'' has at most two support neighbors in C'' so $p(C'') \geq \lceil \frac{s-1}{2} \rceil$. A support vertex of Type (S_1) or (S_2) is a negative vertex of Type (N_1) or (N_2). A negative vertex of Type (N_3) of C'' is of degree 2 and so has its two neighbors on C'' and these two neighbors are support vertices of Type (S_3). So the number of negative vertices of C'' is at most the number of support vertices of C'' and strictly less if C'' contains a vertex of Type (N_3). Vertex v_1 is of Type (N_3), so $s > n(C'')$ and so $p(C'') \geq \lceil \frac{s-1}{2} \rceil \geq \lceil \frac{n(C'')}{2} \rceil$. The graph C''' is a connected subgraph of C whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in C''' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality we have $p(C''') \geq \lceil \frac{n(C''')}{2} \rceil$. So finally, $p(C') = p(C'') + p(C''') \geq \lceil \frac{n(C''')}{2} \rceil + \lceil \frac{n(C'')}{2} \rceil \geq \lceil \frac{n(C'')+n(C''')}{2} \rceil = \lceil \frac{n(C')}{2} \rceil$. □

Let C' be the graph obtained from C by removing all pendant vertices that are not positive vertices. We claim that C' is a connected subgraph of C, whose pendant vertices are positive vertices, whose support vertices have their positive neighbor in C', whose negative vertices of Type (N_3) are adjacent to their two neighbors in C' and such that $n(C') = n(C)$. As C is connected and only pendant vertices have been removed from C, the graph C' is also connected. All support and negative vertices are of degree 2 or 3 and have all their incident edges in $H(G)$ and thus in C, so there is no pendant vertex of C that is a support or a negative vertex. So no support or negative vertex has been removed from C and $n(C') = n(C)$. A pendant vertex of C that has been removed is not positive, not support, not negative but incident to a support, so it is necessarily a degree 2 vertex a incident to a support vertex x of Type (S_3) (with notations of Fig. 2). When a is removed from C, this does not create any new pendant vertex as x has degree 2 after the removal. All pendant vertices that are not positive are removed from C, no new pendant vertices are created, thus in C all pendant vertices are positive. No positive vertex has been removed and each support vertex is adjacent to its positive neighbor in $H(G)$, so support vertices of C' are adjacent to their positive neighbor in C'. No support vertex has been removed and each negative vertex of Type (N_3) is adjacent to its support neighbors of Type (S_3) in $H(G)$, so negative vertices of Type (N_3) of C' are adjacent to their two neighbors in C'. By Claim 13 applied to C', we have $p(C') \geq \lceil \frac{n(C')}{2} \rceil$. So $p(C) = p(C') \geq \lceil \frac{n(C')}{2} \rceil = \lceil \frac{n(C)}{2} \rceil$ and we are done. □

We now use the discharging rules to prove the following:

Lemma 11. $\text{mad}(G) \geq 3$.

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3, R_4 and R_g. The common pot is empty at the beginning and, by Lemma 10, it has a non-negative value after applying R_g. We show that all the vertices have a weight of at least 3 at the end.

Let u be a vertex of G. By Lemma 2, graph G does not contain Configurations (C_1)–(C_{11}). According to Configuration (C_1), we have $d(u) \geq 2$. We now consider different cases corresponding to the value of $d(u)$.

1. $d(u) = 2$.
   So u has an initial weight of 2 and gives nothing. We show that it receives at least 1, so it has a final weight of at least 3.
   (a) *Assume u is adjacent to a vertex u_2 of degree* 2.
   Then u is a negative vertex of Type (N_1) and receives $\frac{1}{5}$ from the common pot by R_g. According to Configuration (C_2), vertex u is adjacent to a vertex v with $d(v) = k$. Since $k \geq 17$, according to R_4, vertex v gives $\frac{4}{5}$ to u.
   (b) *Assume both neighbors v_1 and v_2 of u are of degree at least* 3.
   Vertex u is not a negative vertex of Type (N_1) since it has no neighbor of degree 2.
      i. *u has two weak neighbors.*
      Then u is a negative vertex of Type (N_3). It receives $\frac{1}{5}$ from the common pot by R_g and $\frac{2}{5}$ from each of its two neighbors by $R_{1.1}$.
      ii *u has one weak neighbor w and one non-weak neighbor v.*
         A. $3 \leq d(v) \leq 7$.
         Vertex u receives $\frac{3}{5}$ from v by $R_{1.2}$ and $\frac{2}{5}$ from w by $R_{1.1}$.
         B. $8 \leq d(v) \leq 14$.
         Vertex u receives $\frac{5}{8}$ from v by R_3 and $\frac{3}{8}$ from w by $R_{1.4}$.
         C. $15 \leq d(v)$.
         Vertex w is weak and v has degree at least 15, so one can check that u is not negative of Type (N_1) or (N_3). According to Configuration (C_3), it is not negative of Type (N_2). So u is not negative and it receives $\frac{1}{5}$ from w by $R_{1.5}$ and $\frac{4}{5}$ from v by R_4.
      iii. *u has two non-weak neighbors v, v'.*
         A. $3 \leq d(v) \leq 7$ *and* $3 \leq d(v') \leq 7$. Vertex u receives $\frac{1}{2}$ from each neighbor by $R_{1.3}$.
         B. $3 \leq d(v) \leq 7$ *and* $8 \leq d(v') \leq 14$.
         Vertex u receives $\frac{5}{8}$ from v' by R_3 and $\frac{3}{8}$ from v by $R_{1.4}$.

C. $3 \leq d(v) \leq 7$ *and* $15 \leq d(v')$.
If u is negative, it receives $\frac{1}{5}$ from the common pot by R_g. If u is non-negative, it receives $\frac{1}{5}$ from v by $R_{1.5}$. In both cases, it receives $\frac{4}{5}$ from v' by R_4.
D. $8 \leq d(v)$ *and* $8 \leq d(v')$.
Vertex u receives at least $\frac{5}{8}$ from each neighbor by R_3 or R_4.

2. $d(u) = 3$.
So u has an initial weight of 3. We show that it has a final weight of at least 3.
(a) *Assume u has three neighbors y_1, y_2 and y_3 of degree* 2.
Let z_i, $1 \leq i \leq 3$, be the neighbors of y_i distinct from u. According to Configuration (C_3), $d(z_1) = d(z_2) = d(z_3) = k$. So y_1, y_2 and y_3 are negative vertices of Type (N_2). So no rule applies to u.
(b) *Assume u has exactly two neighbors y_1 and y_2 of degree* 2.
Let z_i, $1 \leq i \leq 2$, be the neighbors of y_i distinct from u. Let x be the third neighbor of u, $d(x) \geq 3$. According to Configuration (C_3), we are in one of the two following cases:
i. $d(x) \geq k - 2$.
Vertex x gives $\frac{4}{5}$ to u by R_4 and u gives nothing to x.
A. *Assume vertex u is weak.*
Since u is weak, $d(y_i) \leq 14$, so vertex u gives at most $\frac{2}{5}$ to each of y_1, y_2 by $R_{1.1}$ or $R_{1.4}$.
B. *Assume vertex u is not weak.*
Then, w.l.o.g., $d(z_1) \geq 15$. So vertex u gives at most $\frac{1}{5}$ to y_1 by $R_{1.5}$. Vertex u gives at most $\frac{3}{5}$ to y_2 by $R_{1.2}$, $R_{1.3}$, $R_{1.4}$ or $R_{1.5}$.
ii. $d(z_1) = d(z_2) = k$.
A. $d(x) \leq 7$.
According to Configuration (C_4), vertex u gives nothing to x by R_2. Vertices y_1 and y_2 are negative (of Type (N_2)) and u gives nothing to y_1, y_2.
B. $d(x) \geq 8$.
Vertex u gives $\frac{1}{5}$ to y_1 and y_2 by $R_{1.5}$. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.
(c) *Assume u has exactly one neighbor y of degree* 2.
Let z be the neighbor of y distinct from u. Let w and x be the other neighbors of u, where $d(w) \geq d(x) \geq 3$. We consider three cases according to the value of $d(w)$.
i. $15 \leq d(w)$.
Then, vertex u gives at most $\frac{3}{5}$ to y by $R_{1.i}$, $1 \leq i \leq 5$. Vertex u gives at most $\frac{1}{10}$ to x by R_2. Vertex w gives $\frac{4}{5}$ to u by R_4.
ii. $8 \leq d(w) \leq 14$.
According to Configuration (C_4), vertex u gives nothing to x by R_2. Vertex u gives at most $\frac{3}{5}$ to y by $R_{1.i}$, $1 \leq i \leq 5$. Vertex w gives $\frac{5}{8}$ to u by R_3.
iii. $d(w) \leq 7$.
According to Configuration (C_4), vertex u gives nothing to x and w by R_2. According to Configuration (C_3), we have $d(z) = k$. Vertex u gives $\frac{1}{5}$ to y by $R_{1.5}$. Both w and x give $\frac{1}{10}$ to u by R_2.
(d) *Assume all the neighbors of u have degree at least* 3 *and at most* 7.
According to Configuration (C_4), vertex u gives nothing to its neighbors by R_2.
(e) *Assume u has no neighbor of degree* 2 *and at least a neighbor v of degree at least* 8.
Vertex v gives at least $\frac{5}{8}$ to u by R_3 or R_4. Vertex u gives at most $\frac{1}{10}$ to each of its other neighbors by R_2.

3. $d(u) = 4$.
So u has an initial weight of 4. We show that it has a final weight of at least 3.
(a) *Assume u has at least three neighbors y_1, y_2 and y_3 of degree* 2.
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2) \geq d(z_3)$. Let x be the neighbor of u distinct from y_1, y_2 and y_3. We consider three cases depending on $d(z_2)$ and $d(z_3)$.
i. $d(z_2) \leq 14$.
According to Configuration (C_7), we have $d(x) \geq k - 2$. Vertex u gives at most $3 \times \frac{3}{5}$ by $R_{1.i}$, $1 \leq i \leq 5$. Vertex x gives $\frac{4}{5}$ to u by R_4.
ii. $d(z_2) \geq 15$ and $d(z_3) \leq 14$.
According to Configuration (C_6), we have $d(x) \geq 8$. Vertex u gives at most $\frac{1}{5}$ to each of y_1, y_2 by $R_{1.5}$. Vertex u gives at most $\frac{3}{5}$ to y_3 by $R_{1.i}$.
iii. $d(z_3) \geq 15$.
Vertex u gives at most $\frac{1}{5}$ to each of its neighbors by $R_{1.5}$.
(b) *Assume u has exactly two neighbors y_1 and y_2 of degree* 2.
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2)$. Let w and x be the neighbors of u distinct from y_1, y_2. We assume that $d(w) \geq d(x) \geq 3$. We consider two cases depending on $d(z_1)$.

i. $d(z_1) \leq 14$.
According to Configuration (C_7), we have $d(w) \geq 9$. Vertex u gives at most $\frac{3}{5}$ to each of y_1, y_2 by $R_{1.i}$, and at most $\frac{1}{10}$ to x by R_2. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.
ii. $d(z_1) \geq 15$.
Vertex u gives at most $\frac{1}{5}$ to y_1 by $R_{1.6}$, at most $\frac{3}{5}$ to y_2 by $R_{1.i}$, and at most $\frac{1}{10}$ to each of w, x by R_2.

(c) *Assume u has at most one neighbor of degree* 2.
Vertex u gives at most $3 \times \frac{1}{10}$ by R_2, and at most $\frac{3}{5}$ by $R_{1.i}$.

4. $d(u) = 5$.
So u has an initial weight of 5. We show that it has a final weight of at least 3.
(a) *Assume u has at least four neighbors y_1, y_2, y_3 and y_4 of degree* 2.
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2) \geq d(z_3) \geq d(z_4)$. Let x be the neighbor of u distinct from the y_i's. We consider two cases depending on $d(z_4)$.
i. $d(z_4) \leq 7$.
According to Configuration (C_8), we have $d(x) \geq 8$. Vertex u gives at most $\frac{3}{5}$ to each of y_i by $R_{1.i}$. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.
ii. $d(z_4) \geq 8$.
Vertex u gives at most $5 \times \frac{3}{8}$ on total to its neighbors y_i's and x by $R_{1.4}$ or $R_{1.5}$.
(b) *Assume u has at most three neighbors of degree* 2.
Vertex u gives at most $3 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{10}$ by R_2.

5. $d(u) = 6$.
So u has an initial weight of 6. We show that it has a final weight of at least 3.
(a) *Assume u has at least five neighbors $y_1, \dots, y_5$, of degree* 2.
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq \cdots \geq d(z_5)$. Let x be the neighbors of u distinct from y_i's. According to Configuration (C_9), we are in one of the following two cases.
i. $d(z_5) \geq 8$.
Vertex u gives at most $6 \times \frac{3}{8}$ on total to its neighbors by $R_{1.4}$ or $R_{1.5}$.
ii. $d(x) \geq 8$.
Vertex u gives at most $5 \times \frac{3}{5}$ on total to the y_i's.
(b) *Assume u has at most four neighbors of degree* 2.
Vertex u gives at most $4 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{10}$ by R_2.

6. $d(u) = 7$.
So u has an initial weight of 7. We show that it has a final weight of at least 3.
(a) *Assume u has at least six neighbors of degree* 2 *adjacent to vertices of degree at most* 3.
According to Configuration (C_{10}), vertex u has a neighbor v of degree at least 8. Vertex u gives at most $6 \times \frac{3}{5}$ by $R_{1.i}$.
(b) *Assume u has at most five neighbors of degree* 2 *adjacent to vertices of degree at most* 3.
Vertex u gives at most $5 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{2}$.

7. $8 \leq d(u) \leq 14$.
Then Rule R_3 applies to every neighbor of u, and $d(u) - (d(u) \times \frac{5}{8}) \geq 3$.
8. $15 \leq d(u) < k$.
Then Rule R_4 applies to every neighbor of u, and $d(u) - (d(u) \times \frac{4}{5}) \geq 3$.
9. $d(u) = k$.
Then Rule R_4 applies to every neighbor of u and R_g applies to u. We have $k \geq 17$ so $k - (k \times \frac{4}{5} + \frac{2}{5}) \geq 3$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least 3, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G} 3 = 3|V|$. Therefore, $\text{mad}(G) \geq 3$. □

Finally, k is a constant integer greater than 17 and G is a minimal graph such that $\Delta(G) \leq k$, and G admits no 2-distance $(k+2)$-list-coloring. By Lemma 11, we have $\text{mad}(G) \geq 3$. So Theorem 4 is true.

7. Conclusion

We proved that graphs with $\Delta(G) \geq 17$ and maximum average degree less than 3 are list 2-distance $(\Delta(G)+2)$-colorable. The key idea in the proof is to use Brooks' lemma (Lemma 7) instead of the usual special case of an even cycle being 2-choosable. Thus we can prove stronger structural properties, which results in a global arborescent structure that is a cactus. As far as we know, Brooks' lemma has not been used in a global discharging proof before, and it might be useful for other problems. One remaining question would be to determine the maximum $\Delta(G)$ of a graph G with $\text{mad}(G) < 3$ that is not 2-distance $(\Delta(G)+2)$-colorable. By Theorem 4, it cannot be more than 16.

Note that these proofs can be effortlessly transposed to list injective $(\Delta(G)+1)$-coloring. Indeed, every vertex we color has a neighbor that is already colored. This means that in the case of list injective coloring, every vertex we color has at least one constraint less than in the case of list 2-distance coloring. Consequently, $\Delta(G)+1$ colors are enough in the case of list injective coloring, as mentioned in the introduction.

In contrast to Theorem 4, other results have been obtained on the 2-distance coloring of planar graphs of girth at least 6 when more colors are allowed. For example, Bu and Zhu [14] proved that every planar graph G of girth at least 6 was 2-distance $(\Delta(G)+5)$-colorable.

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