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List coloring the square of sparse graphs with large degree[☆]

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ABSTRACT

We consider the problem of coloring the squares of graphs of bounded maximum average degree, that is, the problem of coloring the vertices while ensuring that two vertices that are adjacent or have a common neighbor receive different colors.

Borodin et al. proved in 2004 and 2008 that the squares of planar graphs of girth at least seven and sufficiently large maximum degree Δ are list $(\Delta + 1)$-colorable, while the squares of some planar graphs of girth six and arbitrarily large maximum degree are not. By Euler's Formula, planar graphs of girth at least 6 are of maximum average degree less than 3, and planar graphs of girth at least 7 are of maximum average degree less than $\frac{14}{5} < 3$.

We strengthen their result and prove that there exists a function f such that the square of any graph with maximum average degree $m < 3$ and maximum degree $\Delta \geq f(m)$ is list $(\Delta + 1)$-colorable. Note the planarity assumption is dropped. This bound of 3 is optimal in the sense that the above-mentioned planar graphs with girth 6 have maximum average degree less than 3 and arbitrarily large maximum degree, while their square cannot be $(\Delta + 1)$-colored. The same holds for list injective Δ-coloring.

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1. Introduction

The square of a graph G is defined as a graph with the same set of vertices as G, where two vertices are adjacent if and only if they are adjacent or have a common neighbor in G. A *k-coloring* of the square of a graph G (also known as 2-distance k-coloring of G) is therefore a coloring of the vertices of G with k colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that the square of G admits a k-coloring. For example, the square of a cycle of length 5 cannot be colored with less than 5 colors as any two vertices are either adjacent or have a common neighbor: its square is the clique of size 5.

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [14], and has been active ever since. The *maximum degree* of a graph G is denoted $\Delta(G)$. Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence at least $\Delta(G) + 1$ colors are needed to color the square of G. It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [13], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. The sparseness of a planar graph can for example be measured by its girth. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle.

Conjecture 1 (*Wang and Lih [13]*)**.** *For any integer $k \geq 5$, there exists an integer $M(k)$ such that for every planar graph G satisfying $g(G) \geq k$ and $\Delta(G) \geq M(k)$, $\chi^2(G) = \Delta(G) + 1$.*

Conjecture 1 was proved in [5,7,12] to be true for $k \geq 7$ and false for $k = 6$. An extension of the k-coloring of the square is the *list k-coloring of the square*, where instead of having the same set of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. Given a graph G, we call $\chi^2_\ell(G)$ the minimal integer k such that the square of G admits a list k-coloring for any list assignment. Obviously, coloring is a subcase of list coloring (where the same color list is assigned to every vertex), so for any graph G, we have $\chi^2_\ell(G) \geq \chi^2(G)$. Thus, in the case of list-coloring, Conjecture 1 is also false for $k = 6$, and Borodin, Ivanova and Neustroeva [8] proved it to be true for $k \geq 7$.

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The average degree of a graph G, denoted $\mathrm{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of $\mathrm{ad}(H)$ over every subgraph H of G. Euler's formula links girth and maximum average degree in the case of planar graphs.

Lemma 1 (*Folklore*)**.** *For every planar graph G,* $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$.

The question raised by Conjecture 1 and now resolved could be reworded as follows: what is the minimum k such that any graph G with $g(G) \geq k$ and large enough $\Delta(G)$ (depending only on $g(G)$) satisfies $\chi^2_\ell = \Delta(G) + 1$? A consequence of Lemma 1 is that we can transpose any theorem holding for an upper bound on $\mathrm{mad}(G)$ into a theorem holding for planar graphs with lower-bounded girth. It is then natural to transpose the question to the maximum average degree, as it is a more refined measure of sparseness. More precisely, what is the supremum M such that any graph G with $\mathrm{mad}(G) < M$ and large enough $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi^2_\ell = \Delta(G) + 1$?

The authors [3] proved that $\frac{14}{5} \leq M$, which was recently also proved by Cranston and Škrekovski [11]. We know that $M \leq 3$ due to the family of graphs that appears in [5] (see Fig. 1), which are of maximum average degree < 3, of increasing maximum degree, and whose squares are not $(\Delta + 1)$-colorable. We prove here that $3 \leq M$, thus obtaining the exact value of M, which is 3.

Theorem 1. *There exists a function f such that for any $\epsilon > 0$, every graph G with* $\mathrm{mad}(G) < 3 - \epsilon$ *and* $\Delta(G) \geq f(\epsilon)$ *satisfies* $\chi^2_\ell(G) = \Delta(G) + 1$.

This answers the transposition of Conjecture 1 to graphs with an upper-bounded maximum average degree. As the maximum average degree is not discrete, we obtain a sharper value than for planar graphs of bounded girth.

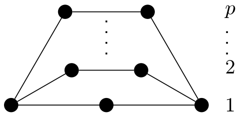

Fig. 1. A graph G_p with $\Delta(G_p) = p$, $\mathrm{mad}(G_p) = 3 - \frac{5}{2p+1}$ and $\chi^2(G_p) = \Delta(G_p) + 2$.

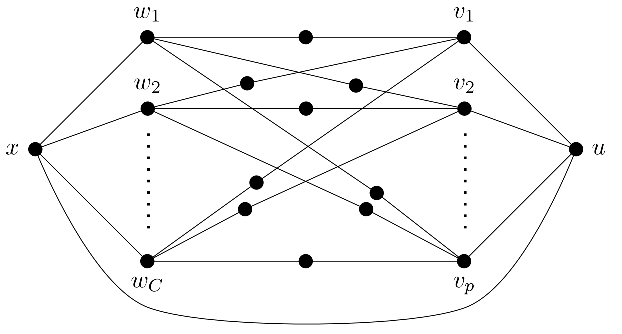

Fig. 2. For $p \geq C$, a graph $G_{p,C}$ with $\Delta(G_{p,C}) = p$, $\mathrm{mad}(G_{p,C}) = \frac{(2C+1)(2p+1)+1}{(C+1)(p+1)+1}$ and $\chi^2(G_{p,C}) = p + C + 1$.

More generally, is it possible to get similar results when allowing an additional constant number of colors, as was done by Wang and Lih in [13] for planar graphs? More precisely, what is, for any $C \geq 2$, the supremum $M(C)$ such that any graph G with $\mathrm{mad}(G) < M(C)$ and sufficiently large $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi_\ell^2(G) \leq \Delta(G) + C$?

The authors proved in [3] that $\lim_{C\to\infty} M(C) = 4$. Interestingly, while graphs with $\mathrm{mad}(G) < 4-\epsilon$ satisfy $\chi_\ell^2(G) \leq \Delta(G) + \mathcal{O}(\frac{1}{\epsilon})$, some graphs with $\mathrm{mad}(G) < 4$ and arbitrarily large maximum degree have $\chi^2(G) \geq \frac{3\Delta(G)}{2}$. This is true even with a restriction to planar graphs with girth at least 4.

Charpentier [10] generalized the family of graphs presented in Fig. 1 to obtain for each C a family of graphs which are of maximum average degree less than $\frac{4C+2}{C+1}$, of increasing maximum degree, and whose square requires $\Delta + C + 1$ colors to be colored (see Fig. 2). Consequently, for every C, we have $M(C) \leq \frac{4C+2}{C+1}$.

This result, and the fact that $\frac{4C+2}{C+1}$ equals $M(C)$ when $C = 1$ and when C tends to infinity, raise the following question.

Question 1. *Is it true that $M(C) = \frac{4C+2}{C+1}$ for any $C \geq 1$?*

Theorem 1 is proved using a discharging method. The discharging method was introduced in the beginning of the 20th century. It is notably known for being used to prove the Four Color Theorem [1,2]. When the discharging rules are local (i.e., the weight cannot travel arbitrarily far), as is most commonly used, we say the discharging method is *local*. Borodin, Ivanova and Kostochka introduced in [6] the notion of *global* discharging, which is when there is no bound on the size of the discharging rules (i.e., the weight can travel arbitrarily far along the graph). When it is mixed, i.e., the discharging rules are of bounded size but take into account structures of unbounded size in the graph, we say the discharging method is *semi-global* (see [4] for a first occurrence of such a proof). Our proof of Theorem 1 is presented in Section 2 as global for simplicity, but could actually be made semi-global by more careful discharging. The global discharging argument is of the same vein as a nice proof of Borodin, Kostochka and Woodall [9] later simplified by Woodall [15]. We explain in Section 3 how this proof can be transposed to injective colorings.

2. Proof of Theorem 1

We prove that there exists a function f such that for any $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 3-\epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi_\ell^2(G) = \Delta(G) + 1$. In the following, we try to simplify the proof rather than improve the function f.

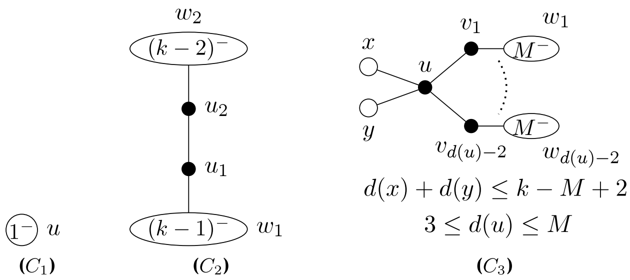

Fig. 3. Forbidden configurations for Theorem 1.

For technical reasons, we will have to consider $\epsilon \leq \frac{1}{20}$. For $\epsilon > \frac{1}{20}$, it suffices to set $f(\epsilon) = f(\frac{1}{20})$. Indeed, if $\epsilon > \frac{1}{20}$, then for every graph with $\text{mad}(G) < 3 - \epsilon$ and $\Delta(G) \geq f(\epsilon)$, we have in particular $\text{mad}(G) < 3 - \frac{1}{20}$ and $\Delta(G) \geq f(\frac{1}{20})$, thus the conclusion holds. From now on, we consider $\epsilon \leq \frac{1}{20}$.

Let $f : \epsilon \mapsto \frac{3}{\epsilon^2}$. Assume by contradiction that there exists a constant $\frac{1}{20} \geq \epsilon > 0$ and a graph Γ with $\text{mad}(\Gamma) < 3-\epsilon$ and $\Delta(\Gamma) \geq f(\epsilon)$ that satisfies $\chi^2_\ell(\Gamma) > \Delta(\Gamma)+1$. There is a *minimal* subgraph G of Γ such that $\chi^2_\ell(G) > \Delta(\Gamma) + 1$, in the sense that the square of every proper subgraph of G is list $(\Delta(\Gamma) + 1)$-colorable. For $k = \Delta(\Gamma)$, the graph G satisfies $\Delta(G) \leq k$ and $\chi^2_\ell(G) > k + 1$, while the square of all its proper subgraphs are list $(k + 1)$-colorable. We aim at proving that $\text{mad}(G) \geq 3 - \epsilon$, a contradiction to the fact that G is a subgraph of Γ with $\text{mad}(\Gamma) < 3 - \epsilon$.

Let $M = \frac{6}{\epsilon}$. Note that since $\epsilon \leq \frac{1}{20}$, we have $k = \Delta(\Gamma) \geq f(\epsilon) = \frac{3}{\epsilon^2} \geq \frac{18}{\epsilon} = 3 \times M$.

In Section 2.1, we introduce the terminology and notation. In Section 2.4, we use the structural observations from Sections 2.2 and 2.3 to derive with a discharging argument that such a graph has maximum average degree at least $3 - \epsilon$, which concludes the proof.

2.1. Terminology and notations

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label 'i' means "exactly i neighbors", the label 'i^+' (resp. 'i^-') means that it has at least (resp. at most) i neighbors. Note that the white vertices may coincide with other vertices.

A *constraint* of a vertex u is an already colored vertex that is adjacent to or has a common neighbor with u. Two constraints with the same color count as one.

Given a vertex u, the *neighborhood* $N(u)$ is the set of vertices that are adjacent to u. For $p \geq 1$, a *p-link* $x - a_1 - \cdots - a_p - y$ between x and y is a path such that $d(a_1) = \cdots = d(a_p) = 2$. When a p-link exists between two vertices x and y, we say they are *p-linked*. Given a subset X and a vertex u, we denote by $d_X(u)$ the number of neighbors of u in X, regardless of whether u belongs to X. Given a subset X, we informally refer to the set of vertices adjacent to at least one vertex in X by "the neighbors of X".

2.2. Forbidden configurations

We define configurations (C_1)–(C_3) (see Fig. 3).

- (C_1) is a vertex u of degree 0 or 1.
- (C_2) is a vertex w_1 of degree at most $k - 1$ that is 2-linked (through w_1-u_1-u_2-w_2) to a vertex w_2 of degree at most $k - 2$.
- (C_3) is a vertex u with $3 \leq d(u) \leq M$ that is 1-linked (through u-v_i-w_i) to $(d(u) - 2)$ vertices $(w_i)_{1 \leq i \leq d(u)-2}$ of degree at most M, and such that the sum of the degrees of its two other neighbors x and y is at most $k - M + 2$.

Lemma 2. *Graph G cannot contain any of Configurations (C_1)–(C_3).*

Proof. We assume G contains a configuration, apply the minimality to color a subgraph of G, and prove this coloring can be extended to the whole graph, a contradiction.

Claim 1. *G cannot contain (C_1).*

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints. There are $k+1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. □

Claim 2. *G cannot contain (C_2).*

Proof. Using the minimality of G, we color $G \setminus \{u_1, u_2\}$. The vertex u_1 has at most $|\{w_2\}| + d(w_1) \leq 1 + (k-1) \leq k$ constraints. Hence we can color u_1. Then u_2 has at most $|\{w_1, u_1\}| + d(w_2) \leq 2 + (k-2) \leq k$ constraints, so we can extend the coloring of $G \setminus \{u_1, u_2\}$ to G. □

Claim 3. *G cannot contain (C_3).*

Proof. Using the minimality of G, we color $G \setminus \{v_1, \ldots, v_{d(u)-2}\}$. We did not delete u in order to obtain a coloring where x and y receive different colors, but u might have the same color as some w_i, so it needs to be recolored. The vertex u has at most $M - 2 + d(x) + d(y) \leq k$ constraints, hence we can recolor u. Then every v_i has at most $M + M \leq k$ constraints, so we can extend the coloring of $G \setminus \{v_1, \ldots, v_{d(u)-2}\}$ to G. □

This concludes the proof of Lemma 2. □

2.3. Global structure

We define three sets V_1, V_2 and T that will outline some global structure on G. We build step-by-step the set V_1 as follows.

Any vertex u of degree at most $M - 1$ belongs to V_1 if it has $d(u) - 1$ neighbors $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are of degree at most $M - 1$, and at most one of $\{w_1, \ldots, w_{d(u)-1}\}$ does not belong to V_1.

Thus, at first, the only vertices in V_1 are those of degree 2 which are adjacent to a vertex of degree 2 whose other neighbor is of degree at most $M - 1$. Note that the set is well-defined as a vertex that satisfies at some point the requirements to be in V_1 will always satisfy them, and the order in which vertices are declared to be in V_1 has absolutely no influence on the set V_1 as it is when no more vertex can be added (equivalently, when all the vertices satisfying the requirements are already in V_1).

As for V_2, any vertex u of degree at most $M - 1$ belongs to V_2 if it has $d(u) - 1$ neighbors $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are of degree at most $M - 1$, and all of $\{w_1, \ldots, w_{d(u)-1}\}$ belong to V_1. Note that V_2 is a subset of V_1.

We define T as the set of vertices of degree 2 whose both neighbors are in V_1. See Fig. 4 for examples of vertices in V_1, V_2 or T. In the figures, we denote by a label V_1 (resp. V_2, T) the fact that a vertex belongs to V_1 (resp. V_2, T). Similarly, we denote by a label $\neg V_1$ a vertex that does not belong to V_1. Since $V_2 \subset V_1$, we omit the label V_1 on vertices labeled V_2.

Lemma 3. *The vertices of V_1 satisfy the following:*

- *Every vertex of V_1 has exactly one neighbor of degree at least $k - M$.*
- *The set V_1 is a stable set.*
- *The sets V_1 and T are disjoint.*

Proof. Assume by contradiction that a vertex u of V_1 has no neighbor of degree at least $k - M$. Then u is adjacent to $d(u) - 1$ vertices $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors are of degree at most $M - 1$, and to another vertex w of degree at most $k - M - 1$. We consider two cases depending on whether $d(u) = 2$.

- If $d(u) = 2$, then the other neighbor of v_1 is a vertex of degree at most $M - 1 \leq k - 1$ that is 2-linked to w, which is a vertex of degree at most $k - M - 1 \leq k - 2$. By Claim 2 in Lemma 2, Configuration (C_2) is not contained in G, a contradiction.

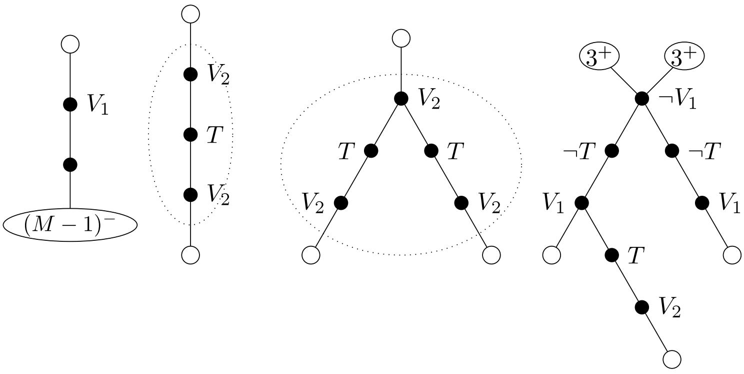

Fig. 4. Examples of vertices in V_1, V_2 or T.

- If $d(u) \geq 3$, then u is a vertex with $3 \leq d(u) \leq M-1 \leq M$ that is 1-linked (through v_i, for $1 \leq i \leq d(u)-2$) to $d(u)-2$ vertices of degree at most $M-1 \leq M$, and such that the sum of the degrees of its two other neighbors w and $v_{d(u)-1}$ is at most $k-M-1+2 \leq k-M+2$. By Claim 3 in Lemma 2, Configuration (C_3) is not contained in G, a contradiction.

Therefore every vertex u of V_1 has a neighbor of degree at least $k-M$. By the definition of V_1, all the other neighbors of u are of degree 2. Thus u has a unique neighbor of degree at most $k-M$.

Since $k \geq 2M$ then $k-M > M-1$ and vertex u has no neighbor v of degree $3 \leq d(v) \leq M-1$. Consequently, two vertices u, v of V_1 that are adjacent must both be of degree 2. By definition of V_1, the other neighbors of u and v must be of degree at most $M-1$, a contradiction. It follows that V_1 is a stable set in G and thus $T \cap V_1 = \emptyset$. □

Any connected component C of $G[V_1 \cup T]$ is a *weak component of* G if every vertex belongs to V_2 or T (in other words, if no vertex of C belongs to V_1 and not to V_2). The only apparent weak components on Fig. 4 are encircled. The *strength* of a component of $G[V_1 \cup T]$ is the number of vertices of V_1 it contains. Let C_w be the set of weak components of G of strength less than $\frac{1}{\epsilon}$. Let S_w be the set of vertices of V_2 that belong to an element of C_w. Let U be the set of vertices of degree at least $k-M$ with a neighbor in S_w.

We first need the following two results.

Theorem 2 (*[9], Theorem 3*)**.** *For any bipartite multigraph G, if L is a color assignment such that $\forall (u,v) \in E$, $|L(u,v)| \geq \max(d(u), d(v))$, then G is L-edge-choosable.*

Lemma 4. *Let H be a bipartite multigraph with vertex set $V(H) = A \cup B$, and $A \neq \emptyset$, $B \neq \emptyset$. For $\alpha > 0$, if for every subset $B' \subseteq B$ and $A' = N(B') \subseteq A$, there exists a vertex $u \in A'$ with $d_{B'}(u) < \alpha$, then $\alpha|A| > |B|$.*

Proof. By induction on $|B|$. If $|B| < \alpha$, since $|A| \geq 1$, the conclusion holds. If $|B| \geq \alpha$, there exists $u \in A$ with $d(u) < \alpha$. We apply the induction hypothesis to the graph $H \setminus (\{u\} \cup N(u))$. It follows that $\alpha(|A|-1) > |B| - \alpha$, hence the result. □

Lemma 5. *The graph G satisfies $|C_w| \leq \frac{1}{\epsilon} \times |\{v \in V \mid d_G(v) \geq k-M\}|$.*

Proof. Assume by contradiction that $|C_w| > \frac{1}{\epsilon} \times |\{v \in V \mid d(v) \geq k-M\}|$.

Recall that by Lemma 3, every vertex of $S_w \subseteq V_1$ has a unique neighbor in U. Let D be the bipartite multigraph whose vertex set is $V(D) = U \cup C_w$, and whose edge set is in bijection with S_w: for every element $v \in S_w$, we add an edge (u, w), where u is the element of U adjacent to v and w is the element of C_w to which v belongs.

For $A = \{v \in V \mid d(v) \geq k-M\}$, $B = C_w$ and $\alpha = \frac{1}{\epsilon}$, we have $|B| > \alpha|A|$. So by Lemma 4, there is a subset C'_w of C_w such that, for U' the neighbors of C'_w in U, the subgraph D' induced in D by $C'_w \cup U'$ satisfies $\forall u \in U'$, $d_{D'}(u) \geq \frac{1}{\epsilon}$.

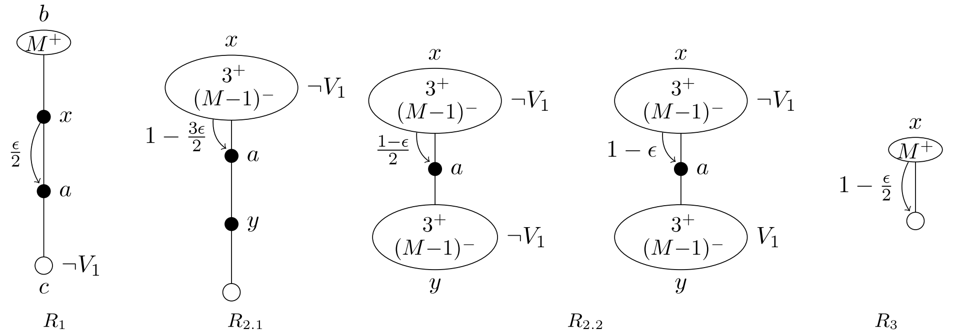

Fig. 5. Discharging rules R_1, R_2, and R_3 for Theorem 1.

Let S'_w (resp. T') be the set of vertices of S_w (resp. T) that belong to an element of C'_w.

We color by minimality $G \setminus (S'_w \cup T')$. Note that every vertex v of S'_w, belonging to V_2, is adjacent to exactly one vertex u of degree at least $k-M$, and that all its other neighbors $v_1, \ldots, v_{d(u)-1}$ are vertices of T whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are in V_1. Since the element C of $C'_w \subseteq C_w$ to which v belongs is a connected component of $G[V_1 \cup T]$, all the v_i's and w_i's belong to $C \in C'_w$. Consequently, for every i, we have $v_i \in T'$ and $w_i \in S'_w$. Thus v has at most $k + 1 - d_{D'}(u)$ constraints, hence v has at least $d_{D'}(u)$ colors available. To color the vertices of S'_w, it is sufficient to list-color the edges of D', where every edge is assigned the same list of colors as the vertex of S'_w it is in bijection with.

By definition of C_w and since $C'_w \subseteq C_w$, every element of C'_w contains at most $\frac{1}{\epsilon}$ vertices of V_2, so it has degree at most $\frac{1}{\epsilon}$ in D thus in D'. Moreover, every vertex of U' has degree at least $\frac{1}{\epsilon}$ in D'. Thus for every edge (u, v) of D', with $u \in U$ and $v \in C'_w$, we have $\max(d_{D'}(u), d_{D'}(v)) = d_{D'}(u)$. So D' is a bipartite multigraph whose every edge has a list assignment of size at least $\max(d_{D'}(u), d_{D'}(v))$. We apply Theorem 2 to color the vertices of S'_w.

It then remains to color the vertices of T'. These are vertices of degree 2 whose both neighbors are in S'_w. But all the vertices of S'_w are of degree at most M. So the vertices of T' have at most $2 \times M \leq k$ constraints, and we can color the vertices of T', a contradiction. □

2.4. Discharging rules

We introduce four discharging rules R_1, R_2, R_3 and R_g ('g' stands for 'global'), as follows (see Fig. 5). We will use them in the case where the initial weight of a vertex v is $d(v)-3+\epsilon$. The weight of a subset of vertices is the sum of the weights of the vertices it contains. During the discharging process, a subset of vertices (here, a weak component) may receive some charge: the question of which vertices in that subset actually receive this charge is of no importance. Indeed, we later consider only the weight of the component, and do not care for the details inside.

Here each connected component of $G[T \cup V_1]$ (and in particular each weak component of G) behaves as a single entity. For any vertex x,

- Rule R_1 is when $d(x) = 2$ and its two neighbors a and b are such that $d(a) = 2$ and $d(b) \geq M$, and the other neighbor c of a is not in V_1. Then x gives $\frac{\epsilon}{2}$ to a.
- Rule R_2 is when $3 \leq d(x) \leq M - 1$ and $x \notin V_1$. If x has a neighbor a of degree 2 whose other neighbor is y,
  - Rule $R_{2.1}$ is when $d(y) = 2$. Then x gives $1 - \frac{3\epsilon}{2}$ to a.
  - Rule $R_{2.2}$ is when $3 \leq d(y) < M$. If $y \notin V_1$, then x gives $\frac{1-\epsilon}{2}$ to a. If $y \in V_1$, then x gives $1 - \epsilon$ to a.
- Rule R_3 is when $M \leq d(x)$. Then x gives $1 - \frac{\epsilon}{2}$ to each of its neighbors.
- Rule R_g states that every vertex of degree at least $k - M$ gives an additional $\frac{1}{\epsilon}$ to an initially empty common pot, and every weak component of G of strength less than $\frac{1}{\epsilon}$ receives 1 from this pot.

We use these discharging rules to prove the following lemma:

Lemma 6. *Graph G satisfies* $\mathrm{mad}(G) \geq 3 - \epsilon$.

Proof. We attribute to each vertex v a weight equal to $d(v) - 3 + \epsilon$, and apply discharging rules R_1, R_2, R_3 and R_g. We show that all the vertices of $G \setminus (T \cup V_1)$ have a non-negative weight in the end, and that each connected component of $G[T \cup V_1]$ has a non-negative total weight.

By Lemma 5, the common pot has a non-negative value, and Rule R_g is valid.

Let x be a vertex of $G \setminus (T \cup V_1)$. By Configuration (C_1), we have $d(x) \geq 2$.

1. $d(x) = 2$.
   The vertex x has an initial weight of $-1 + \epsilon$. We prove that it receives at least $1 - \epsilon$. Let u_1 and u_2 be its two neighbors. We consider two cases depending on whether one of them is of degree at least M.
   (a) $d(u_1) \geq M$ or $d(u_2) \geq M$.
   Consider w.l.o.g. that $d(u_1) \geq M$. By R_3, vertex u_1 gives $1 - \frac{\epsilon}{2}$ to x. The vertex x gives at most $\frac{\epsilon}{2}$ to u_2 by R_1. So x receives at least $1 - \epsilon$.
   (b) $d(u_1) < M$ and $d(u_2) < M$.
   Assume that u_1 or u_2 is of degree 2. Consider w.l.o.g. that $d(u_1) = 2$. Then u_1 belongs to V_1 by definition, and the other neighbor of u_1 is of degree at least M. Since $u_1 \in V_1$ and $x \notin T$, then $u_2 \notin V_1$ and we have $M \geq d(u_2) \geq 3$. By R_1 and $R_{2.1}$, vertex u_1 gives $\frac{\epsilon}{2}$ to x, and u_2 gives $1 - \frac{3\epsilon}{2}$. So x receives $1 - \epsilon$ and gives no weight away. If both u_1 and u_2 have degree at least three, then since $x \notin T$, at most one of u_1 and u_2 is in V_1 and $R_{2.2}$ applies. So vertices u_1 and u_2 give a total of $1 - \epsilon$ to x, and x gives no weight away.
2. $3 \leq d(x) \leq M - 1$.
   The vertex x has an initial weight of $d(x) - 3 + \epsilon \geq \epsilon$. Let $u_1, \ldots, u_q$ denote its neighbors of degree 2 whose other neighbor is of degree at most $M - 1$, where $u_1, \ldots, u_p$ denote its neighbors of degree 2 whose other neighbor belongs to V_1 (note that p may be equal to 0 when x has no such neighbor, and that q may be equal to p). We consider two cases depending on q.
   (a) $q \leq d(x) - 3$.
   Then x gives at most $(d(x) - 3) \times (1 - \epsilon) \leq d(x) - 3 + \epsilon$ by R_2.
   (b) $q \geq d(x) - 2$.
   Then, by Configuration (C_3), vertex x has a neighbor v with $d(v) \geq \frac{k-M+2}{2} \geq M$ (recall that $k \geq 3 \times M$). By Rule R_3, vertex x receives $1 - \frac{\epsilon}{2}$ from v. We consider two cases depending on p.
      i. $p \leq d(x) - 3$. By Rule R_2, x gives at most $(d(x)-3) \times (1-\epsilon) + 2 \times \frac{1-\epsilon}{2} \leq d(x) - 3 + \epsilon + (1 - \frac{\epsilon}{2})$.
      ii. $p \geq d(x) - 2$. Since $x \notin V_1$, we have $p = q = d(x) - 2$. By Rule R_2, x gives at most $(d(x) - 2) \times (1 - \epsilon) \leq d(x) - 3 + \epsilon + (1 - \frac{\epsilon}{2})$.
3. $M \leq d(x) \leq k - M - 1$.
   By Rule R_3, vertex x gives at most $d(x) \times (1 - \frac{\epsilon}{2})$. Since $M = \frac{6}{\epsilon}$, we have $d(x) \times \frac{\epsilon}{2} \geq 3 \geq 3 - \epsilon$, so x has a non-negative final weight.
4. $k - M \leq d(x)$.
   By Rules R_3 and R_g, vertex x gives at most $\frac{1}{\epsilon} + d(x) \times (1 - \frac{\epsilon}{2})$. Since $k \geq \frac{3}{\epsilon^2}$, $M = \frac{6}{\epsilon}$ and $\epsilon \leq \frac{1}{20}$, we have $d(x) \times \frac{\epsilon}{2} \geq (\frac{3}{\epsilon^2} - \frac{6}{\epsilon}) \times \frac{\epsilon}{2} = \frac{3}{2\epsilon} - 3 \geq \frac{1}{\epsilon} + 10 - 3 \geq \frac{1}{\epsilon} + 3 - \epsilon$, so x has a non-negative final weight.

Therefore, every vertex of $G \setminus (T \cup V_1)$ has a non-negative final weight. It remains to consider vertices of $G[T \cup V_1]$. Let C be a connected component of $G[T \cup V_1]$. Let s be the strength of C. Note that $s \geq 1$.

If $s = 1$, then C consists of a single vertex u of degree 2 in G and that is adjacent to a vertex v of degree at least $k - M$ and 1-linked to a vertex w of degree less than M. Thus, by R_1 and R_3, vertex u has an initial weight of $-1 + \epsilon$, receives $1 - \frac{\epsilon}{2}$ from v, and gives $\frac{\epsilon}{2}$ to its neighbor of degree 2: its final weight is 0. We assume from now on that $s \geq 2$.

No vertex of C gives weight. Indeed, a vertex of C can only send charge according to R_1 since all the other rules are for vertices of degree at least M or vertices of degree at least 3 but not in V_1. If a vertex x of C sends some charge according to R_1, then $d(x) = 2$ and its two neighbors a and b are such that $d(b) \geq M$ and $d(a) = 2$, where the other neighbor c of a is not in V_1. Since $d(b) \geq M$, we have $b \notin V_1$ and $x \notin T$, so $x \in V_1$. Then, since V_1 is a stable set by Lemma 3, we have $a \notin V_1$, and $s = 1$, a contradiction with our assumption.

We denote by $N(C)$ the set of vertices that do not belong to C but are adjacent to a vertex in C. Since every vertex in $C \cap V_1$ has a neighbor of degree at least $k - M$ (thus not in C), and the vertices in $C \cap V_1$ that are not in V_2 have a neighbor of degree $2 < k - M$ that is not in C, we have $\sum_{v \in V_1 \cap C} d_{N(C)}(v) \geq s + |C \cap (V_1 \setminus V_2)|$. Also, every vertex u in $C \cap V_1$ receives $1 - \frac{\epsilon}{2}$ from its neighbor of degree at least $k - M$. Thus the weight W of C (without taking R_g into account) is as follows.

$$\begin{aligned}
W &\geq \sum_{v \in C}(d(v) - 3 + \epsilon) + s \times \left(1 - \frac{\epsilon}{2}\right) \\
&\geq \sum_{v \in T \cap C}(d(v) - 3 + \epsilon) + \sum_{v \in V_1 \cap C}(d(v) - 3 + \epsilon) + s \times \left(1 - \frac{\epsilon}{2}\right) \\
&\geq \sum_{v \in T \cap C}(-1 + \epsilon) + \sum_{v \in V_1 \cap C} d_C(v) + \sum_{v \in V_1 \cap C} d_{N(C)}(v) + \sum_{v \in V_1 \cap C}(-3 + \epsilon) + s \times \left(1 - \frac{\epsilon}{2}\right) \\
&\geq \sum_{v \in T \cap C}(-1 + \epsilon) + \sum_{v \in V_1 \cap C} d_C(v) + (s + |C \cap (V_1 \setminus V_2)|) + s \times (-3 + \epsilon) + s \times \left(1 - \frac{\epsilon}{2}\right).
\end{aligned}$$

Remember that the vertex set of C is the union of $V_1 \cap C$ and $T \cap C$, which are stable sets. Also, the two neighbors of a vertex in T belong to C, so $\sum_{v \in V_1 \cap C} d_C(v) = \sum_{v \in T \cap C} d_C(v) = 2|T \cap C|$. Since C is a connected component, we have $|T \cap C| \geq |V \cap C| - 1 = s - 1$. Then,

$$\begin{aligned}
W &\geq |T \cap C| \times (-1 + \epsilon) + 2|T \cap C| + |N(C) \setminus U| + s \times \left(-1 + \frac{\epsilon}{2}\right) \\
&\geq \left(-1 - \epsilon + \frac{3\epsilon s}{2}\right) + |N(C) \setminus U|.
\end{aligned}$$

We consider three cases depending on whether C is weak and $s < \frac{1}{\epsilon}$.

1. *C is a weak component of G and $s < \frac{1}{\epsilon}$.*
   By R_g, component C receives an extra weight of 1. Thus, its final weight is $1 + W \geq 1 + (-1 - \epsilon + \frac{3\epsilon s}{2}) = -\epsilon + \frac{3\epsilon s}{2} > 0$.
2. *C is a weak component of G and $s \geq \frac{1}{\epsilon}$.*
   Then the final weight of C is $W \geq -1 - \epsilon + \frac{3\epsilon s}{2} \geq -1 - \epsilon + \frac{3\epsilon}{2 \times \epsilon} \geq 0$.
3. *C is not a weak component of G.*
   There is at least a vertex v in $(V_1 \cap C) \setminus V_2$. Then the final weight of C is $W \geq (-1 - \epsilon + \frac{3\epsilon s}{2}) + 1 \geq 0$.

Consequently, after application of the discharging rules, every vertex v of $G \setminus \{V_1 \cup T\}$ has a non-negative final weight, and every connected component C of $G[V_1 \cup T]$ has a non-negative final total weight, meaning that $\sum_{v \in G}(d(v) - 3 + \epsilon) \geq 0$. Therefore, $\mathrm{mad}(G) \geq 3 - \epsilon$. This completes the proof of Lemma 6, and thus of Theorem 1. □

3. List injective coloring

A *list injective k-coloring* of a graph is a (not necessarily proper) list k-coloring of its vertices such that two vertices with a common neighbor are of different color, or, in other words, such that no vertex has two neighbors with the same color. We denote by $\chi_\ell^i(G)$ the minimum k such that a graph G admits a list injective k-coloring. Note that the proof for Theorem 1 also work, with close to no alteration, for list injective coloring with one color less. Indeed, the discharging part does not depend on the problem considered, and the configuration part can easily be checked to work also for this as, though one less color is available, every critical vertex has at least one less constraint since already colored neighbors do not count anymore. There is no reason to think that this would be the case for any discharging proof about list coloring of the square, but it happens to be the case most often.

We thus obtain the following theorem.

Theorem 3. *There exists a function f such that for any $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 3 - \epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi_\ell^i(G) = \Delta(G)$.*

Theorem 3 is optimal in the same sense as Theorem 1 by the graph family described in Fig. 1.

4. Conclusion

For any $C \geq 1$, we asked for the supremum $M(C)$ such that any graph G with $\mathrm{mad}(G) < M(C)$ and sufficiently large $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi_\ell^2(G) \leq \Delta(G) + C$. It was already known [3] that $\lim_{C\to\infty} M(C) = 4$. We proved here that $M(1) = 3$, and conjectured that $M(C) = \frac{4C+2}{C+1}$.

It might be a good approach to try, for every fixed C, to adapt the same proof outline. However, rather than prove incremental results, it would be more interesting to look for a general proof that would work for every C, or maybe only for every large enough C.

The proof that $\lim_{C\to\infty} M(C) = 4$ [3] is quite short and simple. However, finding the exact value of $M(C)$ might still be difficult for large C. Indeed, the proof does not even involve lower bounds on $\Delta(G)$.

This leads to a similar question (with no constraint on $\Delta(G)$).

Question 2. *What is, for any $C \geq 1$, the maximum $m(C)$ such that any graph G with $\mathrm{mad}(G) < m(C)$ satisfies $\chi_\ell^2(G) \leq \Delta(G) + C$?*

Obviously, we have $m(C) \leq M(C) < 4$ for every C. Our proof that $\lim_{C\to\infty} M(C) = 4$ was actually a proof that $\lim_{C\to\infty} m(C) = 4$. Note that $m(1) = m(2) = 2$, as a cycle C_5 of length five has $\mathrm{mad}(C_5) = 2$, $\Delta(C_5) = 2$ and $\chi^2(C_5) = 5$, while any graph G with $\mathrm{mad}(G) < 2$ is a forest and thus satisfies $\chi_\ell^2(G) = \Delta(G) + 1$. So there is a significant gap for $C = 1, 2$. What about general C? Does there exist some C for which $m(C) = M(C)$?

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