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Partitioning a triangle-free planar graph into a forest and a forest of bounded degree

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Abstract

We prove that every triangle-free planar graph can have its set of vertices partitioned into two sets, one inducing a forest and the other a forest with maximum degree at most 5. We also show that if for some d, there is a triangle-free planar graph that cannot be partitioned into two sets, one inducing a forest and the other a forest with maximum degree at most d, then it is an NP-complete problem to decide if a triangle-free planar graph admits such a partition.

Keywords: triangle-free planar graph, vertex partition, forest, bounded degree.

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1 Introduction

We only consider finite simple graphs, without loops nor multi-edges. Our planar graphs are supposed to be embedded in the plane. We say that a graph G can be partitioned into k subgraphs H_1, H_2, ..., H_k if its set of vertices can be partitioned into k subsets V_1, ..., V_k such that V_i induces H_i in G for all i. A *j-degenerate* graph is a graph G such that all subgraph H of G has a vertex of degree at most j. A *forest* is an acyclic graph, that is a 1-degenerate graph. Empty graphs are exactly 0-degenerate graphs.

The Four Color Theorem [1,2] states that every planar graph admits a proper 4-coloring, i.e. every planar graph can be partitioned into four empty graphs. Borodin [3] proved that every planar graph admits an acyclic coloring with at most five colors. This implies that every planar graph can be partitioned into an empty graph and two forests. Poh [7] proved that every planar graph can be partitioned into three forests with maximum degree at most 2. Thomassen proved that a planar graph can be partitioned into a forest and a 2-degenerate graph [9], and into an empty graph and a 3-degenerate graph [10]. However, there are planar graphs that cannot be partitioned into two forests [6]. Borodin and Glebov [4] proved that planar graphs of girth at least 5 (that is planar graphs with no triangles nor cycles of length 4) can be partitioned into an empty graph and a forest.

Let us consider the problem of partitioning a planar graph into two sets such that each set avoids a given graph H as a subgraph. Broersma et al. [5] proved that for any given graph H, the problem is NP-complete if H is a tree with at least two edges, and polynomial time solvable otherwise.

We focus on triangle-free planar graphs. Raspaud and Wang [8] proved that every planar graph with no triangles at distance at most 2 (and thus in particular every triangle-free planar graph) can be partitioned into two forests. However, it is not known whether every triangle-free planar graph can be partitioned into an empty graph and a forest. We consider the following questions:

Question 1.1 *Can every triangle-free planar graph be partitioned into an empty graph and a forest?*

Question 1.2 *What is the lowest d such that every triangle-free planar graph can be partitioned into a forest and a forest with maximum degree at most d?*

Proving $d = 0$ in Question 1.2 would prove Question 1.1. Let $\mathcal{F}$ be the set of the forests, and $\mathcal{F}_d$ be the set of the forests with maximum degree at most d. A *$(\mathcal{F}, \mathcal{F}_d)$-partition* of G is a vertex-patition (F, D) such that $G[F] \in \mathcal{F}$

and $G[D] \in \mathcal{F}_d$.

Our main result is:

Theorem 1.3 *For any triangle-free planar graph G, the vertex set of G can be partitioned in two sets F and D such that F induces a forest and D induces a forest with maximum degree at most 5.*

In other words, every triangle-free planar graph admits a $(\mathcal{F}, \mathcal{F}_5)$-partition, so $d \leq 5$ in Question 1.2. Our proof uses the discharging method. It is constructive and immediately yields an algorithm for finding a $(\mathcal{F}, \mathcal{F}_5)$-partition of a triangle-free planar graph in quadratic time. Since not every triangle-free planar graph can be partitioned into two graphs of bounded degree, our result is tight in some sense.

We also showed that if for some d, there exists a triangle-free planar graph that does not admit a $(\mathcal{F}, \mathcal{F}_d)$-partition, then deciding if a triangle-free planar graph admits such a partition is NP-complete. That is, if the answer to Question 1.2 is some $k \geq 0$, then for all $d < k$, deciding if a triangle-free planar graph admits a $(\mathcal{F}, \mathcal{F}_d)$-partition is NP-complete. We proved this by reduction to Planar 3-Sat.

2 Sketch of the proof

Let G be a planar graph.

We call a vertex of degree k, at least k, and at most k, a *k-vertex*, a *k^+-vertex*, and a *k^--vertex* respectively, and by extension, for any fixed vertex v, we call a neighbor of v of degree k, at least k, and at most k, a *k-neighbor*, a *k^+-neighbor*, and a *k^--neighbor* of v respectively. We call a face of length ℓ, at least ℓ, and at most ℓ a *ℓ-face*, a *ℓ^+-face*, and a *ℓ^--face* respectively. We say that a vertex of G is *big* if it is a 8^+-vertex and *small* otherwise. By extension, a big neighbor of a vertex v is a 8^+-neighbor of v and a small neighbor of v is a 7^--neighbor of v. Two neighbors u and w of a vertex v are *consecutive* if uvw forms a path on the boundary of a face.

We prove Theorem 1.3 by contradiction. Let $G = (V, E)$ be a counter-example to Theorem 1.3 of minimum order.

Graph G is connected, otherwise at least one of its connected components would be a counter-example to Theorem 1.3, contradicting the minimality of G.

Let us state a series of lemmas on the structure of G, that correspond to forbidden configurations in G.

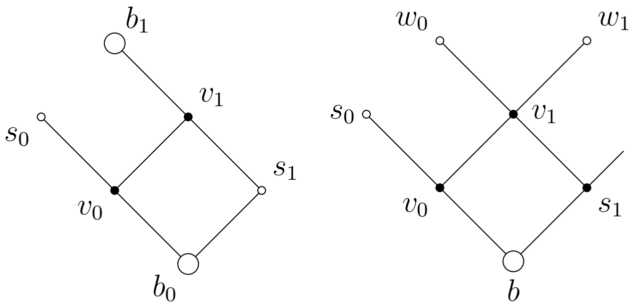

Fig. 1. The forbidden configurations of Lemma 2.4 and Lemma 2.5. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented.

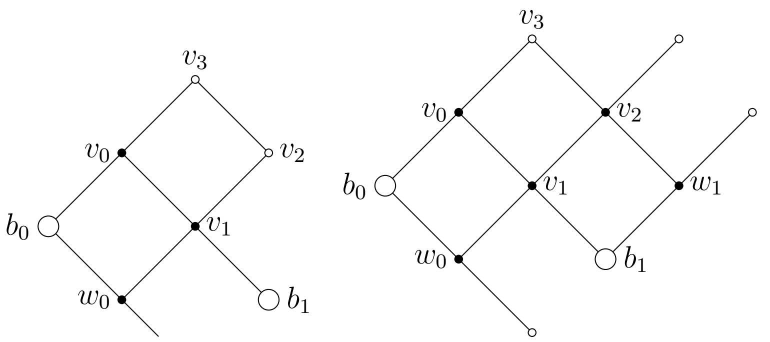

Fig. 2. Configuration 2.6 and the forbidden configuration of Lemma 2.7.

Lemma 2.1 *There are no* 2^-*-vertices in* G*.*

Lemma 2.2 *Every* 3*-vertex in* G *has at least one big neighbor.*

Lemma 2.3 *Every* 4*-vertex or* 5*-vertex in* G *has at least one* 4^+*-neighbor.*

Lemma 2.4 *The following configuration does not occur in* G*: two adjacent* 3*-vertices* v_0 *and* v_1 *such that for* $i \in \{0, 1\}$*,* v_i *has a big neighbor* b_i *and a small neighbor* s_i*, and such that* $v_0v_1s_1b_0$ *bounds a* 4*-face of* G*.(See Figure 1, left.)*

Lemma 2.5 *The following configuration does not occur in* G*: a* 3*-vertex* v_0 *adjacent to a* 4*-vertex* v_1 *such that* v_0 *has a big neighbor* b *and a small neighbor* s_0*, and* v_1 *has three other small neighbors* s_1*,* w_0*, and* w_1 *such that* $v_0v_1s_1b$ *bounds a* 4*-face of* G *and* s_1 *has degree* 3*. (See Figure 1, right.)*

We define a specific configuration:

Configuration 2.6 *Two 4-faces $b_0v_0v_1w_0$ and $v_0v_1v_2v_3$, such that b_0 is a big vertex, v_0 and w_0 are 3-vertices, v_1 is a 4-vertex, v_2 and v_3 are small vertices, and the fourth neighbor of v_1, say b_1, is a big vertex. (See Figure 2, left.)*

Lemma 2.7 *The following configuration is forbidden: Configuration 2.6 with the added condition that there is a 4-face $b_1v_1v_2w_1$ with w_1 a 3-vertex, v_2 a 4-vertex, and the fourth neighbor of v_2, the third neighbor of w_1, and the third neighbor of w_0 are small vertices. (See Figure 2, right.)*

We now apply a discharging procedure: first, for all $j \geq 0$, every j-vertex v has a charge equal to $c(v) = j - 4$, and every j-face f has a charge equal to $c(f) = j - 4$. Observe that, since G is triangle-free, every face has a non-negative initial charge, and by Lemma 2.1, the vertices that have negative initial charges are exactly the 3-vertices of G, and they have an initial charge of -1. By Euler's formula, we have $\sum_{x \in V(G) \cup F(G)} c(x) = -8$. Here is our discharging procedure:

Discharging procedure:

- *Step 1*: Every big vertex gives $\frac{1}{2}$ to each of its small neighbors. Furthermore, for every 4-face $uvwx$ where u and v are big, and w and x are small, v gives $\frac{1}{4}$ to x.
- *Step 2*: Every 4-vertex v gives $\frac{1}{4}$ to each of its small neighbors that are consecutive to a big vertex, and $\frac{1}{2}$ to each of its small neighbors that are consecutive to two big vertices, except when v corresponds to v_1 in Configuration 2.6.

  Consider the case where v corresponds to v_1 in Configuration 2.6. We use the notations of Configuration 2.6. If w_0 has two big neighbors, then v_1 gives $\frac{1}{4}$ to v_0 and v_2. Otherwise, it gives $\frac{1}{4}$ to w_0 and v_0.

  Every small 5^+-vertex that has a big neighbor gives $\frac{1}{4}$ to each of its small neighbors, and an additional $\frac{1}{4}$ for each small vertex that is consecutive to at least one big vertex. Every small 5^+-vertex that has no big neighbors gives $\frac{1}{4}$ to each of its 3-neighbors.
- *Step 3*: For every 4-face $uvwx$, with u a big vertex, v a 3-vertex, w a 4-vertex and x a small vertex such that x gave charge to w in Step 2, w gives $\frac{1}{4}$ to v.
- *Step 4*: Every 5^+-face that has a big vertex in its boundary gives $\frac{1}{4}$ to each of the small vertices in its boundary. Every 5^+-face that has no big vertices in its boundary gives $\frac{1}{5}$ to each of the vertices in its boundary.
- *Step 5*: For every 4-face $uvwx$, with u a big vertex, v a 3-vertex, w a 4-

vertex and x a 3-vertex such that the other face that has vw in its boundary is a 5^+-face, w gives $\frac{1}{5}$ to v.

Observe that during the procedure, no charges are created and no charges disappear; hence the total charge is kept fixed. For every vertex or face x, let $c'(x)$ be the charge at the end of the discharging procedure.

The conclusion arises from the fact that every vertex and every face has a non-negative charge at the end of the procedure. That leads to the following contradiction with Euler's formula:

$$0 \leq \sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) = -8$$

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