

Partitioning sparse graphs into an independent set and a forest of bounded degree

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EXTENDED ABSTRACT

Abstract

An $(\mathcal{I}, \mathcal{F}_d)$ -partition of a graph is a partition of the vertices of the graph into two sets I and F , such that I is an independent set and F induces a forest of maximum degree at most d . We showed that, for all $2 \leq M < 3$ and $d \geq \frac{2}{3-M} - 2$, a graph with maximum average degree less than M admits an $(\mathcal{I}, \mathcal{F}_d)$ -partition. Additionally, we proved that, for all $\frac{8}{3} \leq M < 3$ and $d \geq \frac{1}{3-M}$, a graph with maximum average degree less than M admits an $(\mathcal{I}, \mathcal{F}_d)$ -partition.

1 Introduction

In this abstract, all the considered graphs are simple graphs, without loops or multi-edges. For i classes of graphs $\mathcal{G}_1, \dots, \mathcal{G}_i$, a $(\mathcal{G}_1, \dots, \mathcal{G}_i)$ -partition of a graph G is a partition of the vertices of G into i sets V_1, \dots, V_i such that, for all $1 \leq j \leq i$, the graph $G[V_j]$ induced by V_j belongs to \mathcal{G}_j . In the following we will consider the following classes of graphs:

- \mathcal{F} the class of forests,
- \mathcal{F}_d the class of forests with maximum degree at most d ,
- Δ_d the class of graphs with maximum degree at most d ,
- \mathcal{I} the class of empty graphs (i.e. graphs with no edges).

For example, an $(\mathcal{I}, \mathcal{F}, \Delta_2)$ -partition of G is a vertex-partition into three sets V_1, V_2, V_3 such that $G[V_1]$ is an empty graph, $G[V_2]$ is a forest, and $G[V_3]$ is a graph with maximum degree at most 2. Note that $\Delta_0 = \mathcal{F}_0 = \mathcal{I}$ and $\Delta_1 = \mathcal{F}_1$. The *average degree* of a graph G with n vertices and m edges, denoted by $\text{ad}(G)$, is equal to $\frac{2m}{n}$. The *maximum average degree* of a graph G , denoted by $\text{mad}(G)$, is the maximum of $\text{ad}(H)$ over all subgraphs H of G . The *girth* of a graph G is the length of a smallest cycle in G , and infinity if G has no cycle. Many results on partitions of sparse graphs appear in the literature, where a graph is said to be sparse if it has a low maximum average degree, or if it is planar and has a large girth. The study of partitions of sparse graphs started with the Four Colour Theorem [1, 2], which states that every planar graph admits an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$ -partition. Borodin [3] proved that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$ -partition, and Borodin and Glebov [4] proved that every planar graph with girth at least 5 admits an $(\mathcal{I}, \mathcal{F})$ -partition. Poh [9] proved that every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$ -partition. More recently, Borodin and Kostochka [7] showed that for all $j \geq 0$ and $k \geq 2j + 2$, every graph G with $\text{mad}(G) < 2 \left(2 - \frac{k+2}{(j+2)(k+1)} \right)$ admits a (Δ_j, Δ_k) -partition. In particular, every graph G with $\text{mad}(G) < \frac{8}{3}$ admits an (\mathcal{I}, Δ_2) -partition, and every graph G with $\text{mad}(G) < \frac{14}{5}$ admits an (\mathcal{I}, Δ_4) -partition. With Euler's formula, this yields that planar graphs with girth at least 7 admit (\mathcal{I}, Δ_4) -partitions, and that planar graphs with girth at least 8 admit (\mathcal{I}, Δ_2) -partitions. Borodin and Kostochka [6] proved that every graph G with $\text{mad}(G) < \frac{12}{5}$ admits an (\mathcal{I}, Δ_1) -partition, which implies that every planar graph with girth at least 12 admits an (\mathcal{I}, Δ_1) -partition. This last result was improved by Kim, Kostochka and Zhu [8], who proved that every triangle-free graph with maximum

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$ $(\mathcal{I}, \mathcal{F}, \mathcal{F})$ $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$	The Four Color Theorem [1, 2] Borodin [3] Poh [9]
Planar graphs with girth 5	$(\mathcal{I}, \mathcal{F})$	Borodin and Glebov [4]
Planar graphs with girth 6	no (\mathcal{I}, Δ_d)	Borodin et al. [5]
Planar graphs with girth 7	(\mathcal{I}, Δ_4) $(\mathcal{I}, \mathcal{F}_5)$	Borodin and Kostochka [7] Present paper
Planar graphs with girth 8	(\mathcal{I}, Δ_2) $(\mathcal{I}, \mathcal{F}_3)$	Borodin and Kostochka [7] Present paper
Planar graphs with girth 10	$(\mathcal{I}, \mathcal{F}_2)$	Present paper
Planar graphs with girth 11	(\mathcal{I}, Δ_1)	Kim, Kostochka and Zhu [8]

Table 1: Known results on planar graphs.

average degree at most $\frac{11}{9}$ admits an (\mathcal{I}, Δ_1) -partition, and thus that every planar graph with girth at least 11 admits an (\mathcal{I}, Δ_1) -partition. In contrast with these results, Borodin, Ivanova, Montassier, Ochem and Raspaud [5] proved that for every d , there exists a planar graph of girth at least 6 that admits no (\mathcal{I}, Δ_d) -partition.

In this paper, we focus on $(\mathcal{I}, \mathcal{F}_d)$ -partitions of sparse graphs ; this is a follow-up of the previous studies. Note that, if a graph admits an $(\mathcal{I}, \mathcal{F}_d)$ -partition, then it admits an (\mathcal{I}, Δ_d) -partition, and that an $(\mathcal{I}, \mathcal{F}_1)$ -partition is the same as an (\mathcal{I}, Δ_1) -partition. Therefore the previous results imply the following for $(\mathcal{I}, \mathcal{F}_d)$ -partitions:

- for every d , there exists a planar graph of girth at least 6 that admits no $(\mathcal{I}, \mathcal{F}_d)$ -partition;
- every planar graph with girth at least 11 admits an $(\mathcal{I}, \mathcal{F}_1)$ -partition.

Here are the main results of our paper:

Theorem 1 *Let $2 \leq M < 3$ be a real number and $d \geq \frac{2}{3-M} - 2$ be an integer. Every graph G with $\text{mad}(G) < M$ admits an $(\mathcal{I}, \mathcal{F}_d)$ -partition.*

Theorem 2 *Let $\frac{8}{3} \leq M < 3$ be a real number and $d \geq \frac{1}{3-M}$ be an integer. Every graph G with $\text{mad}(G) < M$ admits an $(\mathcal{I}, \mathcal{F}_d)$ -partition.*

By a direct application of Euler's formula, every planar graph with girth at least g has maximum average degree less than $\frac{2g}{g-2}$. That yields the following corollary:

Corollary 3 *Let G be a planar graph with girth at least g .*

1. *If $g \geq 7$, then G admits an $(\mathcal{I}, \mathcal{F}_5)$ -partition.*
2. *If $g \geq 8$, then G admits an $(\mathcal{I}, \mathcal{F}_3)$ -partition.*
3. *If $g \geq 10$, then G admits an $(\mathcal{I}, \mathcal{F}_2)$ -partition.*

Corollaries 3.2 and 3.3 are obtained from Theorem 2, whereas Corollary 3.1 is obtained from Theorem 1. See Table 1 for an overview of the results on vertex partitions of planar graphs presented above.

2 Sketch of the proofs

The proofs of Theorems 1 and 2 work on the same model, though the proof of Theorem 2 has some additional arguments. We will focus here on the proof of Theorem 1. The proof uses the discharging method. Let G be a counter-example to Theorem 1 with the minimum number of vertices. For all k , a vertex of degree k , at least k , or at most k in G is a k -vertex, a k^+ -vertex, or a k^- -vertex respectively. A $(d+1)^-$ -vertex is a *small* vertex, and a $(d+2)^+$ -vertex is a *big* vertex. We prove some properties on the structure of G .

Lemma 4 *There are no 1^- -vertices in G .*

Lemma 5 *There are no 2-vertices adjacent to two small vertices in G .*

Lemma 6 *There are no sets B of small 3^+ -vertices in G inducing a tree such that every vertex that is not in B and has a neighbour in B is of degree 2.*

Note that Lemma 6 allows us to forbid configurations of unbounded size. Let us define the notion of *light forest* in the graph G that is useful both in defining some forbidden configurations and in the discharging procedure. Let B be a (maximal) set of small 3^+ -vertices such that $G[B]$ is a tree with only one edge that links a vertex of B to a 3^+ -vertex $u \in V(G) \setminus B$, and u is a big vertex. Such a set B is called a *bud* with father u . A 2-vertex adjacent to a small vertex is called a *leaf*. Let us build the *light forest* L , by the following three steps:

1. While there are leaves that are not in L , do the following. Pick a leaf v , and let u be the big vertex adjacent to v (that exists by Lemma 5). Add to L the vertex v , the edge uv , and the vertex u (if it is not already in L). Also set that u is the *father* of v (and v is a *son* of u). See Figure 1, left.
2. While there are buds that are not in L , do the following. Pick a bud B . Let u be the father of B , and let v be the vertex of B adjacent to u . Add $G[B]$ to L , as well as the edge uv , and the vertex u (if it is not already in L). The vertex u is the father of v , and the father/son relationship in B is that of the tree $G[B]$ rooted at v . See Figure 1, middle.
3. While, for some k , there exists a big k -vertex $w \in L$ that has $k-1$ sons in L and whose last neighbour is a 2-vertex that is not in L , do the following. Let v be the 2-vertex adjacent to w that is not in L , and let u be the neighbour of v distinct from w . Note that v is a non-leaf 2-vertex (since it was not added to L in Step 1), therefore u is a big vertex. Add to L the vertex v , the edges uv and vw , and the vertex u (if it is not already in L). We set that v is the father of w , and that u is the father of v . See Figure 1, right.

The following lemma uses the notion of light forest to define forbidden configurations of unbounded size.

Lemma 7 *For all k , the light forest L of G does not contain a big k -vertex with k sons.*

Discharging procedure:

Let $\epsilon = 3 - M$. Recall that $d \geq \frac{2}{3-M} - 2 = \frac{2}{\epsilon} - 2$, therefore $\epsilon \geq \frac{2}{d+2} > 0$. We start by assigning to each k -vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G , the sum of the charges of the vertices is negative. The initial charge of each 3^+ -vertex is at least ϵ , and thus is positive.

Every big vertex gives charge $1 - \epsilon$ to each of its neighbours that are its sons in L , does not give anything to its father in L (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other neighbours with degree 2.

One can prove that, at the end of the procedure, every vertex has a non-negative charge, that is a contradiction.

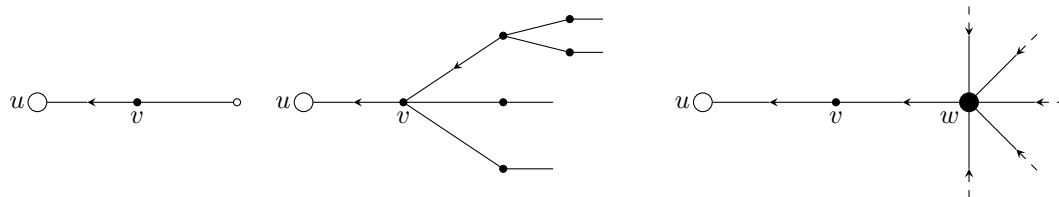


Figure 1: The construction of the light forest L . The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented. The dashed lines are the continuation of the light forest. The arrows point from son to father in L .

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