



# A lower bound on the order of the largest induced forest in planar graphs with high girth<sup>☆</sup>



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## ARTICLE INFO

### Article history:

Received 8 April 2015

Received in revised form 14 June 2016

Accepted 17 June 2016

Available online 27 July 2016

### Keywords:

Induced forest

Feedback vertex set

Large girth

Planar graph

## ABSTRACT

We give here new lower bounds on the size of a largest induced forest in planar graphs with high girth. This is equivalent to upper bounds on the size of a smallest feedback vertex set. In particular, we prove that a planar graph with girth  $g$  and size  $m$  has a feedback vertex set of size at most  $\frac{4m}{3g}$ , improving the trivial bound of  $\frac{2m}{g}$ . We also prove that every 2-connected graph with maximum degree 3 and order  $n$  has a feedback vertex set of size at most  $\frac{n+2}{3}$ .

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## 1. Introduction

In this article we only consider finite simple graphs.

Let  $G$  be a graph. A *feedback vertex set* or *decycling set*  $S$  of  $G$  is a subset of the vertices of  $G$  such that removing the vertices of  $S$  from  $G$  yields an acyclic graph. Thus  $S$  is a feedback vertex set of  $G$  if and only if the graph induced by  $V(G) \setminus S$  in  $G$  is an induced forest of  $G$ . The FEEDBACK VERTEX SET decision problem (given a graph  $G$  and an integer  $k$ , decide whether there is a decycling set of  $G$  of size  $k$ ) is known to be NP-complete, even restricted to the case of planar graphs, bipartite graphs or perfect graphs [10]. It is thus legitimate to seek bounds for the size of a decycling set or an induced forest. The smallest size of a decycling set of  $G$  is called the *decycling number* of  $G$ , and the highest order of an induced forest of  $G$  is called the *forest number* of  $G$ , denoted respectively by  $\phi(G)$  and  $a(G)$ . Note that the sum of the decycling number and the forest number of  $G$  is equal to the order of  $G$  (i.e.  $|V(G)| = a(G) + \phi(G)$ ).

Mainly, the community focuses on the following challenging conjecture due to Albertson and Berman [2]:

**Conjecture 1** (Albertson and Berman [2]). *Every planar graph  $G$  of order  $n$  admits an induced forest of order at least  $\frac{n}{2}$ , that is  $a(G) \geq \frac{n}{2}$ .*

Conjecture 1, if true, would be tight (for  $n \geq 3$  multiple of 4) because of the disjoint union of complete graphs on four vertices (Akiyama and Watanabe [1] gave examples showing that the conjecture differs from the optimal by at most one half for all  $n$ ), and would imply that every planar graph has an independent set on at least a quarter of its vertices, the only

<sup>☆</sup> This research was partially supported by ANR EGOS project, under contract ANR-12-JS02-002-01.

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Fig. 1. An outerplanar graph  $G$  with  $a(G) = \frac{2|V(G)|}{3}$ .

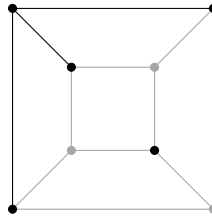


Fig. 2. The cube  $Q$  admits an induced forest on five of its vertices, but no induced forest on six or more of its vertices, i.e.  $a(Q) = 5$ .

known proof of which relies on the Four-Color Theorem. The best known lower bound to date for the forest number of a planar graph is due to Borodin and is a consequence of the acyclic 5-colorability of planar graphs [6]. We recall that an acyclic  $k$ -coloring is a proper vertex coloring using  $k$  colors such that the graph induced by the vertices of any two color classes is a forest. From Borodin's result one can obtain the following theorem:

**Theorem 2** (Borodin [6]). *Every planar graph of order  $n$  admits an induced forest of order at least  $\frac{2n}{5}$ .*

Hosono [9] showed the following theorem and showed that the bound is tight.

**Theorem 3** (Hosono [9]). *Every outerplanar graph of order  $n$  admits an induced forest of order at least  $\frac{2n}{3}$ .*

The tightness of the bound is shown by the example in Fig. 1.

Akiyama and Watanabe [1], and Albertson and Haas [3] independently raised the following conjecture:

**Conjecture 4** (Akiyama and Watanabe [1], and Albertson and Haas [3]). *Every bipartite planar graph of order  $n$  admits an induced forest of order at least  $\frac{5n}{8}$ .*

This conjecture, if true, would be tight for  $n$  multiple of 8: for example, if  $G$  is the disjoint union of  $k$  cubes, then we have  $a(G) = 5k$  and  $G$  has order  $8k$  (see Fig. 2). Motivated by Conjecture 4, Alon [4] proved the following theorem using probabilistic methods:

**Theorem 5** (Alon [4]). *There exist some absolute constants  $b > 0$  and  $b' > 0$  such that:*

- For every bipartite graph  $G$  with  $n$  vertices and average degree at most  $d (\geq 1)$ ,  $a(G) \geq (\frac{1}{2} + e^{-bd^2})n$ .
- For every  $d \geq 1$  and all sufficiently large  $n$ , there exists a bipartite graph with  $n$  vertices and average degree at most  $d$  such that  $a(G) \leq (\frac{1}{2} + e^{-b'\sqrt{d}})n$ .

The lower bound was later improved by Conlon et al. [7] to  $a(G) \geq (1/2 + e^{-b''d})n$  for a constant  $b''$ .

Conjecture 4 also led to research on lower bounds of the forest number of triangle-free planar graphs (as a superclass of bipartite planar graphs). Alon et al. [5] proved the following theorem and a corollary:

**Theorem 6** (Alon et al. [5]). *Every triangle-free graph of order  $n$  and size  $m$  admits an induced forest of order at least  $n - \frac{m}{4}$ .*

**Corollary 7** (Alon et al. [5]). *Every triangle-free cubic graph of order  $n$  admits an induced forest of order at least  $\frac{5n}{8}$ .*

Theorem 6 is tight because of the union of cycles of length 4.

The *girth* of a graph is the length of a shortest cycle. A forest has infinite girth. In a planar graph with girth at least  $g$ , order  $n$ , and size  $m$  with at least one cycle, the number of faces is at most  $2m/g$  (since all the faces' boundaries have length at least  $g$ ). Then, by Euler's formula,  $2m/g \geq m - n + 2$ , and thus  $m \leq (g/(g-2))(n-2)$ . In particular, triangle-free planar graphs of order  $n \geq 3$  have size at most  $2n-4$ . As a consequence of Theorem 6, for a triangle-free planar graph  $G$  of order  $n$ ,  $a(G) \geq n/2$ . Salavatipour proved a better lower bound [13]:  $a(G) \geq \frac{17n+24}{32}$ . In a companion paper, the authors strengthen this bound as follows:

**Theorem 8** ([8]). *Every triangle-free planar graph of order  $n \geq 1$  admits an induced forest of order at least  $\frac{6n+7}{11}$ .*

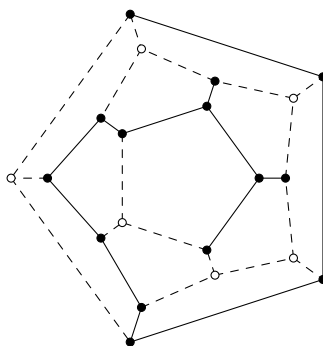


Fig. 3. The dodecahedron  $D$  admits an induced forest on fourteen of its vertices, but no induced forest on fifteen or more of its vertices, i.e.  $a(D) = 14$ .

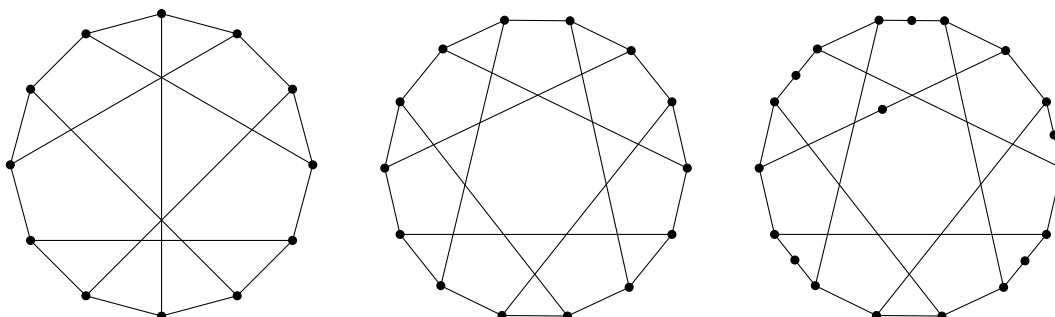


Fig. 4. Examples showing that Conjecture 12 does not extend to non-planar graphs for  $g = 5, 6, 7$ .

Kowalik et al. [11] made the following conjecture on planar graphs of girth at least 5:

**Conjecture 9** (Kowalik et al. [11]). *Every planar graph with girth at least 5 and order  $n$  admits an induced forest of order at least  $7n/10$ .*

This conjecture, if true, would be tight for  $n$  multiple of 20, as shown by the example of the union of dodecahedrons, given by Kowalik et al. [11] (see Fig. 3).

A first step toward Conjecture 9 was done in a companion paper [8]; moreover a generalization for higher girth was given:

**Theorem 10** ([8]). *Every planar graph with girth at least 5 and order  $n \geq 1$  admits an induced forest of order at least  $\frac{44n+50}{69}$ .*

**Theorem 11** ([8]). *Every planar graph with girth at least  $g \geq 5$  and order  $n \geq 1$  admits an induced forest of order at least  $n - \frac{(5n-10)g}{23(g-2)}$ .*

For planar graphs with given girth, we conjecture the following:

**Conjecture 12.** *Let  $G$  be a planar graph of size  $m$  and girth  $g$ . There exists a feedback vertex set  $S$  of  $G$  of size at most  $\frac{m}{g}$ .*

Note that Conjecture 12 is true for  $g = 4$  even for non-planar graphs by Theorem 6. However, it is false for non-planar graphs for  $g = 5, 6, 7$ , as shown in Fig. 4. If Conjecture 12 is true, then it is tight for  $m$  multiple of  $g$  due to the union of disjoint cycles of length  $g$ . It is easy to prove that  $G$  admits a feedback vertex set of size at most  $\frac{2m}{g}$  (removing a vertex that is in the boundary of at least two faces decreases the number of faces by one, and this can be applied recursively).

The main result of this paper is a first non-trivial step toward Conjecture 12:

**Theorem 13.** *Let  $G$  be a planar graph of size  $m$  and girth  $g$ . There exists a feedback vertex set  $S$  of  $G$  of size at most  $\frac{4m}{3g}$ .*

Theorem 13 is the best result so far for  $g \geq 7$ , and gives  $a(G) \geq \frac{(3g-10)n+8}{3(g-2)}$  using  $m \leq (n-2)\frac{g}{g-2}$  (Theorem 11 is better for  $g = 6$ ). We summarize the previous results in Table 1.

For comparison, Speckenmeyer [14] proved that every subcubic connected (but not necessarily planar) graph of size  $m$  and girth  $g$  has a feedback vertex set of size at most  $\frac{g+1}{6g-3}m + \frac{g-1}{2g-1}$ . Liu and Zhao [12] improved this bound to  $\frac{g}{6(g-1)}m + \frac{g-3}{2g-2}$  for most graphs of this class.

**Table 1**  
Lower bounds on the forest numbers for planar graphs with given girth.

Planar graph with girth $g$	Forest number
4	$\frac{6n+7}{11}$
5	$\frac{44n+50}{69}$
6	$\frac{31n+30}{46}$
$g \geq 7$	$\frac{(3g-10)n+8}{3(g-2)}$

**Theorem 13** will be proven in Section 3. For this, we will use **Theorem 14** (proven in Section 2) that is of independent interest. Let  $\mathcal{C}_{2,3-}$  be the family of 2-connected graphs of maximum degree at most 3. Note that graphs in  $\mathcal{C}_{2,3-}$  need not be planar.

**Theorem 14.** Every graph in  $\mathcal{C}_{2,3-}$  of order  $n$  has a feedback vertex set of size at most  $\frac{n+2}{3}$ .

**Theorem 14** is tight for the complete graph on 4 vertices. Moreover, consider any 3-regular graph  $G$ , and consider the graph  $H$  obtained from  $G$  by replacing each vertex by a triangle (as the cube connected cycles obtained from the hypercube). Graph  $H$  has  $3|V(G)|$  vertices, and cannot have a feedback vertex set of less than  $|V(G)|$  vertices, since a feedback vertex set of  $H$  contains at least one vertex of each added triangle. Hence there is a graph of order  $n$  without a feedback vertex set of size less than  $\frac{n}{3}$  for an arbitrary large  $n$ .

Finally, if we replace the 2-connectedness condition by simply connected, then  $\frac{3n}{8} + \frac{1}{4}$  becomes a tight bound [5]. One can observe that without connectedness condition, the disjoint union of complete graphs on four vertices has a smallest feedback vertex set of size  $\frac{n}{2}$ .

**Notation.** Consider a graph  $G = (V, E)$ . For a set  $S \subseteq V$ , let  $G - S$  be the graph obtained from  $G$  by removing the vertices of  $S$  and all the edges that are incident to a vertex of  $S$ . If  $x \in V$ , then we denote  $G - \{x\}$  by  $G - x$ . For a set  $S$  of vertices such that  $S \cap V = \emptyset$ , let  $G + S$  be the graph constructed from  $G$  by adding the vertices of  $S$ . If  $x \notin V$ , then we denote  $G + \{x\}$  by  $G + x$ . For a set  $F$  of pairs of vertices of  $G$  such that  $F \cap E = \emptyset$ , let  $G + F$  be the graph constructed from  $G$  by adding the edges of  $F$ . If  $e$  is a pair of vertices of  $G$  and  $e \notin E$ , then we denote  $G + \{e\}$  by  $G + e$ . For a set  $W \subseteq V$ , we denote by  $G[W]$  the subgraph of  $G$  induced by  $W$ . We call a vertex of degree  $d$ , at least  $d$ , and at most  $d$ , a  $d$ -vertex, a  $d^+$ -vertex, and a  $d^-$ -vertex respectively. Similarly, we call a cycle of length  $\ell$ , at least  $\ell$ , and at most  $\ell$  a  $\ell$ -cycle, a  $\ell^+$ -cycle, and a  $\ell^-$ -cycle respectively, and by extension a face of length  $\ell$ , at least  $\ell$ , and at most  $\ell$  a  $\ell$ -face, a  $\ell^+$ -face, and a  $\ell^-$ -face respectively. For a face  $f$  of a plane graph  $G$ , we denote the boundary of  $f$  by  $G[f]$ . We say that two faces are adjacent if their boundaries share (at least) an edge. We say that two cycles are adjacent if they share at least an edge. An edge cut-set of a graph  $G$  is a minimal set of edges  $F$  such that  $G \setminus F$  is disconnected. If an edge cut-set is a singleton, then its element is a cut edge. A vertex cut-set of a graph  $G$  is a set  $X$  of vertices of  $G$  such that  $G \setminus X$  is disconnected. If a vertex cut-set is a singleton, then its element is a cut vertex.

**2. Proof of Theorem 14**

We recall that  $G = (V, E)$  is called  $k$ -connected if  $|V| > k$  and  $G - X$  is connected for every set  $X \subseteq V$  with  $|X| < k$ . Also  $G = (V, E)$  is called  $k$ -edge connected if  $|V| > 1$  and the deletion of any set of at most  $(k - 1)$  edges leads to a connected graph.

Let us consider  $H = (V, E)$  a counter-example to **Theorem 14** of minimum order, and let  $n = |V| \geq 3$  be the order of  $H$ . Let us prove some lemmas on the structure of  $H$ .

**Lemma 15.** Graph  $H$  is cubic.

**Proof.** Suppose there is a vertex  $v$  of degree at most 2 in  $H$ . As  $H$  is 2-connected,  $v$  has degree 2. Let  $u$  and  $w$  be the two neighbors of  $v$  in  $H$ . Suppose  $uw \notin E$ . Let  $H' = H - v + uw$ . Since  $u$  and  $w$  have degree at least 2 ( $H$  is 2-connected),  $|V(H')| \geq 3$ . Then graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  has a feedback vertex set  $S$  of size  $|S| \leq \frac{n-1+2}{3} \leq \frac{n+2}{3}$ , and  $S$  is also a feedback vertex set of  $H$ , a contradiction. Therefore  $uw \in E$ . If both  $u$  and  $w$  have degree 2, then  $H = C_3$  and  $H$  admits a feedback vertex set of size  $1 \leq \frac{n+2}{3} = \frac{5}{3}$ , a contradiction. If one of  $u$  and  $w$  has degree 2 and the other one has degree 3, then  $H$  is not 2-connected, a contradiction. Therefore both  $u$  and  $w$  have degree 3. Note that more generally, we proved that there are no two adjacent vertices of degree 2 in  $H$ . Let  $u'$  and  $w'$  be the third neighbors of  $u$  and  $w$  respectively. If  $u' = w'$ , then  $V = \{u, v, w, u'\}$  ( $H$  is 2-connected), and  $H$  admits a feedback vertex set of size  $1 \leq \frac{n+2}{3} = 2$  ( $\{u\}$  for example), a contradiction. Thus  $u'$  and  $w'$  are distinct. Suppose  $u'w' \in E$ . Let  $H' = H - \{u, v, w\}$ . If  $|V(H')| < 3$ , then  $u'$  and  $w'$  are adjacent vertices of degree 2 in  $H$  and we fall into a previous case. Therefore  $|V(H')| \geq 3$ . Then graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  has a feedback vertex set  $S'$  of size  $|S'| \leq \frac{n-3+2}{3}$ . The set  $S = S' \cup \{u\}$  is a feedback vertex set of  $H$  of size  $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$ , a contradiction. Therefore  $u'w' \notin E$ . Let  $H' = H - \{u, v, w\} + u'w'$ . Graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  has a feedback vertex set  $S'$  of size  $|S'| \leq \frac{n-3+2}{3}$ . The set  $S = S' \cup \{u\}$  is a feedback vertex set of  $H$  of size  $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$ , a contradiction.  $\square$

In the following, we will use the fact that  $H$  is cubic without referring to Lemma 15.

**Lemma 16.** *There are no adjacent triangles in  $H$ .*

**Proof.** Assume that there are two triangles  $xyz$  and  $xyz'$  sharing an edge  $xy$  in  $H$ . If  $zz' \in E$ , then  $H = K_4$  ( $H$  is connected), which contradicts the fact that  $H$  is a counter-example to Theorem 14. Therefore  $zz' \notin E$ . Let  $v$  be the neighbor of  $z$  distinct from  $x$  and  $y$ . Observe that  $vz' \notin E$ , since  $H$  is cubic and 2-connected. Let  $H' = H - \{x, y, z\} + vz'$ . Graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  has a feedback vertex set  $S'$  of size  $|S'| \leq \frac{n-3+2}{3}$ . The set  $S = S' \cup \{x\}$  is a feedback vertex set of  $H$  of size  $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$ , a contradiction.  $\square$

**Lemma 17.** *There is no triangle that shares an edge with a 4-cycle in  $H$ .*

**Proof.** By Lemma 16, there is no triangle that shares two edges with a 4-cycle in  $H$ . Assume that there are a triangle  $xyw$  and a 4-cycle  $vzxy$  that share the edge  $xy$ .

Suppose first that there is a vertex  $z'$  adjacent to  $v$  and  $w$ . If  $zz' \in E$ , then  $V = \{v, w, x, y, z, z'\}$  ( $H$  is connected), i.e.  $H$  is the prism, and  $\{y, z\}$  is a feedback vertex set of  $H$ , thus  $H$  is not a counter-example to Theorem 14, a contradiction. Therefore  $zz' \notin E$ . Let  $z''$  be the third neighbor of  $z'$ . Let  $H' = H - \{x, w, y\} + zz''$ . Graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $|S'| \leq \frac{n-3+2}{3}$ . The set  $S = S' \cup \{y\}$  is a feedback vertex set of  $H$  of size  $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$ , a contradiction.

Therefore there is no vertex adjacent to  $v$  and  $w$ . Let  $w'$  be the neighbor of  $w$  distinct from  $x$  and  $y$ . By Lemma 16,  $w' \notin \{v, z\}$ . Let  $H'' = H - \{x, y, w\} + vw'$ . Graph  $H''$  is in  $\mathcal{C}_{2,3-}$ . By minimality of  $H$ ,  $H''$  admits a feedback vertex set  $S''$  of size at most  $|S''| \leq \frac{n-3+2}{3}$ . The set  $S = S'' \cup \{x\}$  is a feedback vertex set of  $H$  of size  $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$ , a contradiction.  $\square$

**Lemma 18.** *There are no two 4-cycles that share two edges in  $H$ .*

**Proof.** Let  $uvwx$  and  $vzxy$  be two 4-cycles of  $H$ . Let  $u', w'$  and  $y'$  be the third neighbors of  $u, w$  and  $y$  respectively. By Lemma 16, they are distinct from the vertices defined previously. If  $u' = w' = y'$ , then  $H = K_{3,3}$  admits a feedback vertex set of size  $2 \leq \frac{6+2}{3} = \frac{n+2}{3}$  (for example  $\{u, y\}$ ), a contradiction.

Suppose  $u' \neq w' \neq y' \neq u'$ . Let  $H' = H - \{u, v, w, y\} + \{u'x, w'x, y'x\}$ . If  $H'$  is not 2-connected, then w.l.o.g.  $x$  separates  $u'$  and  $w'$  in  $H'$ , and thus  $u$  separates  $u'$  and  $w'$  in  $H$ , a contradiction. Therefore  $H'$  is in  $\mathcal{C}_{2,3-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $|S'| \leq \frac{n-4+2}{3}$ . The set  $S = S' \cup \{v\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-4+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction.

Thus w.l.o.g.,  $u' = y' \neq w'$ . Let  $z$  be the neighbor of  $u'$  distinct from  $u$  and  $y$ . Observe that  $z$  is distinct from  $w'$  since  $H$  is cubic and 2-connected. Let  $H' = H - \{u, v, w, x, y, u'\}$  if  $zw' \in E$  and  $H' = H - \{u, v, w, x, y, u'\} + zw'$  otherwise. Graph  $H'$  is in  $\mathcal{C}_{2,3-}$  since  $H$  is. By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $|S'| \leq \frac{n-6+2}{3}$ . The set  $S = S' \cup \{v, x\}$  is a feedback vertex set of  $H$  of size  $|S'| + 2 \leq \frac{n-6+2}{3} + 2 \leq \frac{n+2}{3}$ , a contradiction.  $\square$

The following lemma is folklore, but here is a proof for the sake of completeness.

**Lemma 19.** *For every  $k \in \{1, 2, 3\}$ , a graph with maximum degree at most 3 is  $k$ -connected if and only if it is  $k$ -edge-connected.*

**Proof.** Let  $G$  be a graph with maximum degree at most 3. One can easily check that the result holds for the complete graph on at most four vertices.

Suppose now that  $G$  is not complete. Let  $C_v$  be a vertex cut-set of  $G$  and  $C_e$  be an edge cut-set of  $G$ , both of minimum size. If we show that  $|C_v| = |C_e|$ , then the lemma holds.

Let  $V_1$  and  $V_2$  be the vertex sets of the two connected components of  $G - C_v$ . We have  $V_1 \cup V_2 = V(G)$ . By minimality of  $|C_e|$ , every edge of  $C_e$  has an endvertex in  $V_1$  and the other one in  $V_2$ . Suppose every vertex of  $V_1$  is adjacent to every vertex of  $V_2$  in  $G$ . We have  $|C_e| = |V_1| |V_2| \geq |V_1| + |V_2| - 1 = |V(G)| - 1$ . Moreover, for any vertex in  $G$ , the set of the edges incident to this vertex is an edge cut-set of  $G$ . Therefore, since  $G$  is not complete, by minimality of  $C_e$ ,  $|C_e| \leq |V(G)| - 2$ , a contradiction. Therefore there are two vertices  $v_1 \in V_1$  and  $v_2 \in V_2$  such that  $v_1v_2 \notin E(G)$ . Let  $C'_v = \{x \neq v_1 | \exists y \in V_2, xy \in C_e\} \cup \{y | v_1y \in C_e\}$ . Note that  $|C'_v| = |\{x \neq v_1 | \exists y \in V_2, xy \in C_e\}| + |\{y | v_1y \in C_e\}| \leq |C_e|$ . For each edge in  $C_e$ , one of the endvertices of this edge is in  $C'_v$ . As neither  $v_1$  nor  $v_2$  is in  $C'_v$ ,  $C'_v$  separates  $v_1$  from  $v_2$  in  $G$ . Therefore  $|C_v| \leq |C'_v|$ , and thus  $|C_v| \leq |C_e|$ .

Let  $W_1$  and  $W_2$  be the vertex sets of two connected components of  $G - C_e$ . Let  $x \in C_v$ . Since  $x$  has degree at most 3,  $x$  has at most one neighbor in  $W_1$  or at most one neighbor in  $W_2$ , and it has at least one neighbor in  $W_1$  and one in  $W_2$  by minimality of  $C_v$ . Let  $y_x$  be the neighbor of  $x$  that is in  $W_1$  if there is only one neighbor of  $x$  in  $W_1$ , and the neighbor of  $x$  in  $W_2$  otherwise, and  $e_x = xy_x$ . Observe that this defines a unique edge  $e_x$  for every  $x \in C_v$ . Let  $C'_e = \{e_x | x \in C_v\}$ . Assume  $C'_e$  does not separate  $W_1$  and  $W_2$ . There are  $v_1 \in W_1$  and  $v_2 \in W_2$  such that there is a path  $P$  from  $v_1$  to  $v_2$  in  $H - C'_e$ . Let us consider  $v_1$  and  $v_2$  such that  $P$  has minimal length. Then there are  $w_1$  and  $w_2$  in  $C_v$  such that  $v_1w_1 \in E(P)$  and  $v_2w_2 \in E(P)$ . If  $w_1 = w_2$ , then either  $v_1w_1 \in C'_e$  or  $v_2w_2 \in C'_e$ , a contradiction. If  $w_1 \neq w_2$ , then  $w_1$  has a neighbor in  $V(G) \setminus (W_1 \cup W_2)$ , so it has only one neighbor in  $W_1$ , that is  $v_1$ , so  $v_1w_1 \in C'_e$ , a contradiction. Therefore  $C'_e$  separates  $W_1$  and  $W_2$ . We have  $|C'_e| = |C_v|$ , thus  $|C_e| \leq |C_v|$ . Finally, since  $|C_v| \leq |C_e|$ ,  $|C_v| = |C_e|$ .  $\square$

**Lemma 20.** Graph  $H$  is 3-connected.

**Proof.** Suppose by contradiction that  $H$  is not 3-connected. By Lemma 15,  $|V(H)| \geq 4$ . By hypothesis and Lemma 19,  $H$  is 2-edge-connected but not 3-edge-connected. Let  $\{e, f\}$  be an edge cut-set of  $H$  that induces two connected components  $V_1$  and  $V_2$  such that  $|V_1|$  is minimum.

We will now prove the two following properties:

- $P_e$ : The deletion of any edge in  $H[V_1]$  preserves the 2-edge connectivity of  $H$ .  
By contradiction, suppose there is an edge  $e'$  that has both of its endvertices in  $V_1$  such that  $H - e'$  is not 2-edge-connected (but connected since  $H$  is 2-edge-connected). Let  $f'$  be a cut edge of  $H - e'$ . If  $f'$  has at least one of its endvertices in  $V_1$ , then one of the connected components of  $H - \{e', f'\}$  is strictly included in  $V_1$ , a contradiction with the minimality of  $|V_1|$ . Therefore,  $f'$  has both of its endvertices in  $V_2$ . Neither  $e$  nor  $f$  is a cut edge of  $H - e'$ , otherwise we fall into the previous case. Thus  $e'$  is not a cut edge of  $H[V_1]$ . In particular, there is a path in  $H \setminus \{f', e'\}$  that connects the two endvertices of  $e'$ . However,  $e'$  is a cut edge of  $H \setminus f'$ , a contradiction.

- $P_v$ : For every vertex  $v$  in  $V_1$  that has all of its neighbors in  $V_1$ ,  $H - v$  is 2-edge-connected, and thus 2-connected by Lemma 19.

Suppose there is a vertex  $v \in V_1$  that is not incident to an edge of  $\{e, f\}$  such that  $H - v$  is not 2-edge-connected. Let  $f'$  be a cut edge of  $H - v$ . As vertex  $v$  has degree 3, there is an edge  $e'$  incident to  $v$  such that  $H - \{e', f'\}$  is disconnected. As  $v$  is not incident to an edge of  $\{e, f\}$ ,  $e'$  has both of its endvertices in  $V_1$ , a contradiction with  $P_e$ .

Let  $v \in V_1$  and  $u \in V_2$  be the endpoints of  $e$ . Let  $w$  and  $x$  be the two neighbors of  $v$  distinct from  $u$ . Vertices  $w$  and  $x$  are in  $V_1$ , otherwise w.l.o.g.  $f = vw$ , and  $vx$  is a cut edge of  $H$ , a contradiction.

Let us show that  $wx \notin E$ . By contradiction assume that  $wx \in E$ . Let  $w'$  be the neighbor of  $w$  distinct from  $v$  and  $x$ , and  $x'$  be the neighbor of  $x$  distinct from  $v$  and  $w$ . By Lemmas 16 and 17,  $w'$ ,  $x'$  and  $u$  are distinct and pairwise not adjacent. Moreover, if  $w' \notin V_1$  or  $x' \notin V_1$ , say  $w' \notin V_1$ , then  $f = ww'$ , and thus  $xx'$  is a cut edge of  $H$ , a contradiction. Hence  $v, w, x, w'$  and  $x'$  are all in  $V_1$ , and thus, by  $P_v$ ,  $H - w$  is 2-connected. Let  $H' = H - \{v, w, x\} + ux'$ . Graph  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{w\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction.

Let  $w_0$  and  $w_1$  be the two neighbors of  $w$  distinct from  $v$ . If  $w_0$  or  $w_1$  is in  $V_2$ , say  $w_0 \in V_2$ , then  $ww_0 = f$ , and  $\{vx, ww_1\}$  is an edge cut-set of  $H$ , contradicting the minimality of  $|V_1|$ . Therefore  $w_0$  and  $w_1$  are in  $V_1$ .

Let us show that  $w_0w_1 \notin E$ . By contradiction assume that  $w_0w_1 \in E$ . Let  $w'_0$  be the neighbor of  $w_0$  distinct from  $w$  and  $w_1$ , and  $w'_1$  be the neighbor of  $w_1$  distinct from  $w$  and  $w_0$ . By Lemmas 16 and 17,  $w'_0$  and  $w'_1$  are distinct and not adjacent. Vertices  $v, w, w_0$  and  $w_1$  are all in  $V_1$ , thus, by  $P_v$ ,  $H - w$  is 2-connected. Let  $H' = H - \{w, w_0, w_1\} + w'_0w'_1$ . Graph  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{w\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction.

Let  $w_{00}$  and  $w_{01}$  be the two neighbors of  $w_0$  distinct from  $w$ . Let us show that  $w_{00}w_{01} \notin E$ . By contradiction assume that  $w_{00}w_{01} \in E$ . By Lemma 17,  $w_{00}, w_{01} \neq x$ . Let  $w'_{00}$  be the neighbor of  $w_{00}$  distinct from  $w_0$  and  $w_{01}$ , and  $w'_{01}$  be the neighbor of  $w_{01}$  distinct from  $w_0$  and  $w_{00}$ . By Lemmas 16 and 17,  $w'_{00}$  and  $w'_{01}$  are distinct and not adjacent. Suppose  $w_{00}$  or  $w_{01}$  is in  $V_2$ , say  $w_{00} \in V_2$ . Then  $w_0w_{00} = f$ , and  $e, f$  is not an edge cut-set of  $H$  (since  $w_0w_{00}w_{01}$  is a triangle), a contradiction. Therefore  $w, w_0, w_{00}$  and  $w_{01}$  are in  $V_1$ , and thus, by  $P_v$ ,  $H - w_0$  is 2-connected. Let  $H' = H - \{w_0, w_{00}, w_{01}\} + w'_{00}w'_{01}$ . Graph  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{w_0\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction.

Let  $w_{10}$  and  $w_{11}$  be the two neighbors of  $w_1$  distinct from  $w$ . By symmetry,  $w_{10}w_{11} \notin E$ . Suppose  $\{w_{00}, w_{01}\} = \{w_{10}, w_{11}\}$ ; say  $w_{00} = w_{10}$  and  $w_{01} = w_{11}$ . Lemma 18 leads to a contradiction. Therefore the pairs  $\{w_{00}, w_{01}\}$  and  $\{w_{10}, w_{11}\}$  are not equal. As  $v, w, w_0$  and  $w_1$  are in  $V_1$ , by  $P_v$ ,  $H - w$  is 2-connected. Let  $H' = H - \{w, w_0, w_1\} + \{w_{00}w_{01}, w_{10}w_{11}\}$ . Graph  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{w\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction, which completes the proof.  $\square$

**Lemma 21.** There is no triangle in  $H$ .

**Proof.** Suppose there is a triangle  $uvw$  in  $H$ . Let  $u', v'$  and  $w'$  be the third neighbor of  $u, v$  and  $w$  respectively. By Lemmas 16 and 17,  $u', v'$  and  $w'$  are distinct and non-adjacent. Let  $H' = H - \{u, v, w\} + u'v'$ . Observe that by Lemma 20,  $H - w$  is 2-connected. Therefore  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{w\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction.  $\square$

Let  $v$  be a vertex of  $H$ , and  $x$  and  $y$  be two neighbors of  $v$ . They are not adjacent by Lemma 21. Let  $x_0, x_1, y_0$  and  $y_1$  be the two other neighbors of  $x$  and  $y$  respectively. Vertices  $x_0$  and  $x_1$  are not adjacent by Lemma 21, and similarly  $y_0$  and  $y_1$  are not adjacent. The pairs  $\{x_0, x_1\}$  and  $\{y_0, y_1\}$  are distinct by Lemma 18. Let  $H' = H - \{v, x, y\} + \{x_0x_1, y_0y_1\}$ . By Lemma 20,  $H'$  is in  $\mathcal{C}_{2,3^-}$ . By minimality of  $H$ ,  $H'$  admits a feedback vertex set  $S'$  of size at most  $\frac{n-3+2}{3}$ . The set  $S = S' \cup \{v\}$  is a feedback vertex set of  $H$  of size  $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$ , a contradiction. That completes the proof of Theorem 14.

### 3. Proof of Theorem 13

Let  $g \geq 3$  be a fixed integer. For  $G$  a planar graph,  $\omega : E(G) \rightarrow \mathbb{N}$  a weight function, and  $F \subseteq E(G)$ , we denote  $\sum_{e \in F} \omega(e)$  by  $\omega(F)$ , and  $\sum_{e \in E(G)} \omega(e)$  by  $\omega(G)$ . We will prove the following claim:

**Claim 22.** *Let  $G$  be a planar graph, and  $\omega : E(G) \rightarrow \mathbb{N}$  a weight function such that for each cycle  $C$  of  $G$ ,  $\omega(C) \geq g$ . There exists a feedback vertex set  $S$  of  $G$  of size at most  $\frac{4\omega(G)}{3g}$ .*

Observe that fixing  $\omega$  constant equal to 1 in Claim 22 yields Theorem 13. Let us consider any embedding of the graph  $G$  in the plane.

Let  $G$  be a 2-connected plane graph. Three faces  $f_0, f_1$  and  $f_2$  of  $G$  are said to be *mergeable* if:

1. there exists a vertex  $v$  that is in the boundary of  $f_0, f_1$  and  $f_2$ .
2. w.l.o.g.  $f_0$  and  $f_1$  (resp  $f_1$  and  $f_2$ ) have at least one common edge in their boundary.

Given three mergeable faces  $f_0, f_1$  and  $f_2$ , the *merge* of  $f_0, f_1$  and  $f_2$  consists in removing the edges belonging to the boundary of two faces among  $f_0, f_1$  and  $f_2$  as well as the vertices that end up being isolated. The common vertex  $v$  of  $f_0, f_1$  and  $f_2$  is called the *crucial* vertex of the merge. A merge is *nice* if the sum of the weights of the edges removed is at least  $\frac{3g}{4}$ . Observe that a merge cannot decrease the minimum weight of a cycle in  $G$ , since we only delete vertices and edges. See Fig. 5 for an example of the merge of three faces.

**Lemma 23.** *Let  $G$  be a 2-connected plane graph, and let  $G'$  be a graph obtained from  $G$  by applying a merge with the crucial vertex  $v$ . If  $S'$  is a feedback vertex set of  $G'$ , then  $S' \cup \{v\}$  is a feedback vertex set of  $G$ .*

**Proof.** Let  $C$  be a cycle of  $G$  that contains an edge  $e \in E(G) \setminus E(G')$ . Edge  $e$  is in the boundary of two of the faces that are merged, say  $f_0$  and  $f_1$ . Cycle  $C$  separates  $f_0$  and  $f_1$ . Therefore it contains all the vertices of  $V(G[f_0]) \cap V(G[f_1])$ . In particular, it contains  $v$ .

Therefore each cycle of  $G$  is either entirely in  $G'$ , or it contains  $v$ . Thus as  $V(G') \setminus S'$  induces a forest in  $G'$ ,  $V(G) \setminus (S' \cup \{v\})$  induces a forest in  $G$ .  $\square$

**Lemma 24.** *Let  $G$  be a 2-connected plane graph, and  $G'$  be obtained from  $G$  by applying a nice merge. If graph  $G'$  satisfies Claim 22, then graph  $G$  also satisfies Claim 22.*

**Proof.** Let  $v$  be the crucial vertex of the merge. We have  $\omega(G') \leq \omega(G) - \frac{3g}{4}$ . Since  $G'$  verifies Claim 22, there exists a feedback vertex set  $S'$  of  $G'$  such that  $|S'| \leq \frac{4\omega(G')}{3g} \leq \frac{4\omega(G)}{3g} - 1$ . Then  $S = S' \cup \{v\}$  is a feedback vertex set of  $G$  (by Lemma 23), and  $|S| \leq \frac{4\omega(G)}{3g} - 1 + 1 = \frac{4\omega(G)}{3g}$ , which completes the proof.  $\square$

Let us assume by contradiction that there are couples  $(G, \omega)$  that do not satisfy Claim 22. Among all counterexamples  $(G, \omega)$  to Claim 22 minimizing  $\omega(G)$ , we consider a couple  $(G, \omega)$  minimizing  $\sum_{v \in V(G)} (\max\{0.5, d(v) - 2.5\})$ .

**Lemma 25.** *Graph  $G$  is 2-connected.*

**Proof.** By contradiction, assume  $G$  is not 2-connected. Graph  $G$  has at least 2 vertices, otherwise it would satisfy Claim 22. Let  $S$  be a minimal vertex cut-set of  $G$ . We have  $|S| \leq 1$ . Let  $V_1$  and  $V_2$  be non-empty sets of vertices separated by  $S$ .

Let  $\omega_1 = \omega(G[V_1 \cup S])$  and  $\omega_2 = \omega(G[V_2 \cup S])$ . By minimality of  $(G, \omega)$ , there exist  $S_1 \subseteq V_1 \cup S$  and  $S_2 \subseteq V_2 \cup S$  which are feedback vertex sets of  $G[V_1 \cup S]$  and  $G[V_2 \cup S]$  respectively, such that  $|S_1| \leq \frac{4\omega_1}{3g}$  and  $|S_2| \leq \frac{4\omega_2}{3g}$ . Now  $S_1 \cup S_2$  is a feedback vertex set of  $G$ , and  $|S_1 \cup S_2| \leq \frac{4\omega_1}{3g} + \frac{4\omega_2}{3g} = \frac{4\omega(G)}{3g}$ . Thus  $G$  satisfies Claim 22, a contradiction.  $\square$

**Lemma 26.** *No nice merges can be done in  $G$ .*

**Proof.** It follows from Lemma 24 and the minimality of  $(G, \omega)$ .  $\square$

**Lemma 27.** *Every face in  $G$  has at least three  $3^+$ -vertices in its boundary.*

**Proof.** Let us assume that there is a face  $f$  in  $G$  with at most two  $3^+$ -vertices in its boundary. Face  $f$  is adjacent to at most two other faces in  $G$ . Suppose  $f$  is adjacent to exactly one face, say  $f'$ . As  $G$  is 2-connected by Lemma 25,  $G[f]$  and  $G[f']$  are cycles. As  $f$  is adjacent only to  $f'$ ,  $E(G[f]) \subseteq E(G[f'])$ , and thus  $G[f] = G[f']$ . So two faces of  $G$  have exactly the same boundary, so  $G$  is a cycle, and it satisfies Claim 22, a contradiction.

Thus  $f$  is adjacent to exactly two other faces, say  $f_0$  and  $f_1$ . Then  $E(G[f]) \subseteq E(G[f_0]) \cup E(G[f_1])$ , and  $E(G[f]) \cap E(G[f_0]) \neq \emptyset \neq E(G[f]) \cap E(G[f_1])$ . As  $G[f]$  is a cycle, there is a vertex  $v$  in  $V(G[f])$  incident to an edge in  $E(G[f]) \cap E(G[f_0])$  and to an edge in  $E(G[f]) \cap E(G[f_1])$ . Merging the faces  $f, f_0$  and  $f_1$  with crucial vertex  $v$  is nice, since we remove all the edges of  $G[f]$  and  $\omega(f) \geq g \geq \frac{3g}{4}$ . This leads to a contradiction with Lemma 26.  $\square$

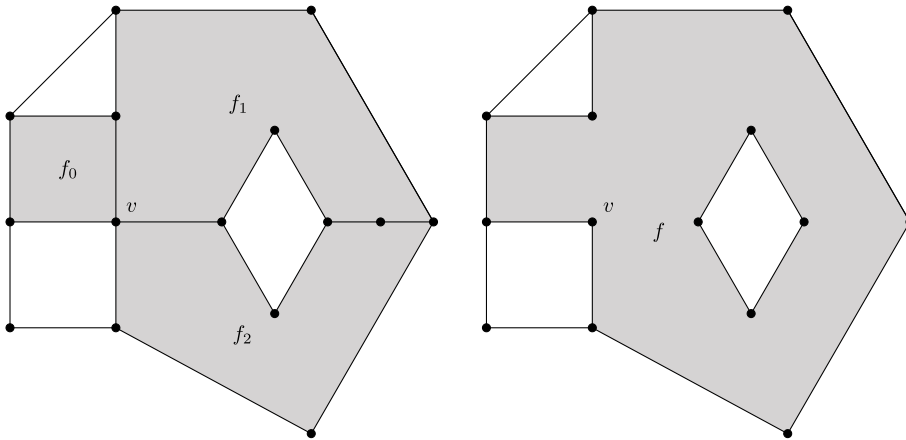


Fig. 5. The merge of faces  $f_0, f_1$  and  $f_2$  into  $f$  with crucial vertex  $v$ .

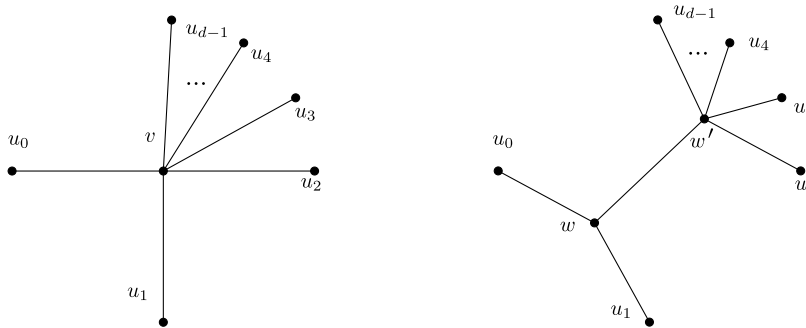


Fig. 6. The construction of Lemma 28.

**Lemma 28.** *There are no  $4^+$ -vertices in  $G$ .*

**Proof.** Suppose  $v$  is a  $d$ -vertex in  $G$  with  $d \geq 4$ . Let  $u_0, \dots, u_{d-1}$  be the neighbors of  $v$ . Let  $G' = G - v + \{w, w'\} + \{wu_0, wu_1, ww', w'u_2, \dots, w'u_{d-1}\}$ ,  $\omega(wu_0) = \omega(vu_0)$ ,  $\omega(wu_1) = \omega(vu_1)$ ,  $\omega(w'u_2) = \omega(vu_2), \dots, \omega(w'u_{d-1}) = \omega(vu_{d-1})$ , and  $\omega(ww') = 0$ . See Fig. 6 for an illustration of this construction. Clearly,  $\omega(G') = \omega(G)$ . As we removed a  $d$ -vertex, added a 3-vertex and a  $(d - 1)$ -vertex, and did not change the degree of the other vertices,  $\sum_{v \in V(G')} (\max\{0.5, d(v) - 2.5\}) = \sum_{v \in V(G)} (\max\{0.5, d(v) - 2.5\}) - 0.5$ .

It is easy to see that for any cycle  $C'$  of  $G'$ , there is a cycle in  $G$  that has the same weight, so  $\omega(C') \geq g$ .

By minimality of  $(G, \omega)$ , let  $S'$  be a feedback vertex set of  $G'$  with  $|S'| \leq \frac{4\omega(G')}{3}$ . For any cycle  $C$  of  $G$  there is a cycle  $C'$  of  $G'$  such that  $C = C'$  or  $V(C) = (V(C') \setminus \{w, w'\}) \cup \{v\}$ . If  $w \in S'$  or  $w' \in S'$ , then let  $S = S' \setminus \{w, w'\} \cup \{v\}$  and otherwise let  $S = S'$ . Then  $|S| \leq |S'| \leq \frac{4\omega(G')}{3} = \frac{4\omega(G)}{3}$ , and  $S$  is a feedback vertex set of  $G$ , a contradiction.  $\square$

**Lemma 29.** *Every cycle has at least three 3-vertices in  $G$ .*

**Proof.** Let  $C$  be a cycle of  $G$ . By Lemma 28, every vertex in  $V(C)$  has degree at most 3. Suppose  $C$  is a separating cycle. By Lemma 25, graph  $G$  is 2-connected, so at least two vertices of  $V(C)$  have a neighbor in the interior of  $C$ , and at least two vertices of  $V(C)$  have a neighbor in the exterior of  $C$ . Therefore  $C$  has at least four 3-vertices. Now if  $C$  bounds a face, then Lemma 27 concludes the proof.  $\square$

**Lemma 30.** *Graph  $G$  is cubic (i.e. 3-regular).*

**Proof.** Suppose  $v$  is a  $2^-$ -vertex in  $G$ . Vertex  $v$  has degree 2 by Lemma 25. Let  $u$  and  $w$  be the two neighbors of  $v$ . By Lemma 29,  $uw \notin E(G)$ .

Let  $G' = G - v + uw$  and  $\omega(uw) = \omega(uv) + \omega(vw)$ . See Fig. 7 for an illustration of this construction. Clearly,  $\omega(G') = \omega(G)$ . As we removed a 2-vertex and did not change the degree of the other vertices,  $\sum_{v \in V(G')} (\max\{0.5, d(v) - 2.5\}) = \sum_{v \in V(G)} (\max\{0.5, d(v) - 2.5\}) - 0.5$ .

Let  $C'$  be any cycle of  $G'$ . If  $uw \notin E(C')$ , then  $C'$  is a cycle of  $G$ , and so  $\omega(C') \geq g$ . Otherwise,  $C = C' - uw + v + \{uv, vw\}$  is a cycle of  $G$ , and  $\omega(C) = \omega(C')$ , so  $\omega(C') \geq g$ .



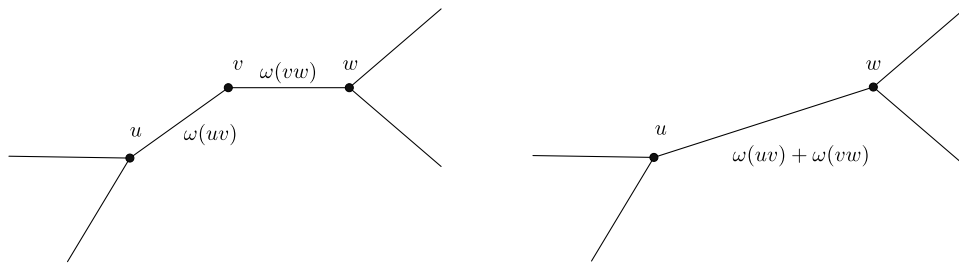


Fig. 7. The construction of Lemma 30.

For any cycle  $C$  of  $G$  there is a cycle  $C'$  of  $G'$  that contains all the vertices of  $V(C) \setminus \{v\}$ . By minimality of  $(G, \omega)$ , let  $S'$  be a feedback vertex set of  $G'$  with  $|S'| \leq \frac{4\omega(G')}{3} = \frac{4\omega(G)}{3}$ . The set  $S'$  is a feedback vertex set of  $G$ , a contradiction.  $\square$

By Lemmas 25 and 30, graph  $G$  is a 2-connected cubic graph. By Theorem 14,  $G$  admits a feedback vertex set of order at most  $\frac{|V(G)|+2}{3}$ . Let us denote by  $n$  the order of  $G$ , by  $m$  the size of  $G$  and by  $f$  the number of faces of  $G$ .

By Euler's formula, we have  $n - m + f = 2$ . We have  $3n = 2m$  as  $G$  is cubic. Therefore,  $f = 2 + m - n = 2 + \frac{n}{2}$ , i.e.  $n = 2(f - 2)$ . Therefore  $G$  has a feedback vertex set  $S$  of size  $|S| \leq \frac{2f-4+2}{3} \leq \frac{2f}{3}$ . As each face has weight at least  $g$ , we have  $gf \leq 2\omega(G)$ , so  $|S| \leq \frac{4\omega(G)}{3g}$ , a contradiction, completing the proof of Theorem 13.

## References

- [1] J. Akiyama, M. Watanabe, Maximum induced forests of planar graphs, *Graphs Combin.* 3 (1987) 201–202.
- [2] M.O. Albertson, D.M. Berman, A conjecture on planar graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), *Graph Theory and Related Topics*, 1979.
- [3] M. Albertson, R. Haas, A Problem Raised at the DIMACS Graph Coloring Week, New Jersey, 1998.
- [4] N. Alon, Problems and results in extremal combinatorics–I, *Discrete Math.* 273 (1) (2003) 31–53.
- [5] N. Alon, D. Mubayi, R. Thomas, Large induced forests in sparse graphs, *J. Graph Theory* 38 (2001) 113–123.
- [6] O.V. Borodin, A proof of Grünbaum's conjecture on the acyclic 5-colorability of planar graphs, *Dokl. Akad. Nauk SSSR* 231 (1) (1976) 18–20 (in Russian).
- [7] D. Conlon, J. Fox, B. Sudakov, Short proofs of some extremal results, *Combin. Probab. Comput.* 23 (2014) 8–28.
- [8] F. Dross, M. Montassier, A. Pinlou, Large induced forests in planar graphs with girth 4 or 5, 2014, arXiv:1409.1348.
- [9] K. Hosono, Induced forests in trees and outerplanar graphs, *Proc. Fac. Sci. Tokai Univ.* 25 (1990) 27–29.
- [10] R.M. Karp, *Reducibility Among Combinatorial Problems*, Springer, 1972.
- [11] L. Kowalik, B. Lužar, R. Škrekovski, An improved bound on the largest induced forests for triangle-free planar graphs, *Discrete Math. Theoret. Comput. Sci.* 12 (1) (2010) 87–100.
- [12] J. Liu, C. Zhao, A new bound on the feedback vertex sets in cubic graphs, *Discrete Math.* 148 (1996) 119–131.
- [13] M.R. Salavatipour, Large induced forests in triangle-free planar graphs, *Graphs Combin.* 22 (2006) 113–126.
- [14] E. Speckenmeyer, On the feedback vertex sets and nonseparating independent sets in cubic graphs, *J. Graph Theory* 12 (1988) 405–412.