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A lower bound on the order of the largest induced linear forest in triangle-free planar graphs

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ABSTRACT

We prove that every triangle-free planar graph of order n and size m has an induced linear forest with at least $\frac{9n-2m}{11}$ vertices, and thus at least $\frac{5n+8}{11}$ vertices. Furthermore, we show that there are triangle-free planar graphs on n vertices whose largest induced linear forest has order $\lceil \frac{n}{2} \rceil + 1$.

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1. Introduction

In this article, we only consider simple finite graphs. All considered planar graphs are supposed to be embedded in the plane.

We look into the problem of finding large induced forests in planar graphs. Albertson and Berman [2] conjectured that every planar graph admits an induced forest on at least half of its vertices. This conjecture, if true, would be tight, as shown by the disjoint union of copies of the complete graph on four vertices. One of the motivations of this conjecture is that it would imply that every planar graph admits an independent set on at least one fourth of its vertices, the only known proof of which relies on the Four Colour Theorem. However, this conjecture appears to be very hard to prove. The best known result for planar graphs is that every planar graph admits an induced forest on at least two fifths of its vertices. This is a consequence of the theorem of 5-acyclic colourability of planar graphs of Borodin [4].

The conjecture of Albertson and Berman has been proved and strengthened for smaller classes of graphs. For example, Hosono [7] showed that every outerplanar graph admits an induced forest on at least two thirds of its vertices, which is tight. Akiyama and Watanabe [1], and Albertson and Haas [3] independently conjectured that every bipartite planar graph admits an induced forest on at least five eighths of its vertices, which is tight. For triangle-free planar graphs (and thus in particular for bipartite planar graphs), the present authors showed that every triangle-free planar graph of order n and size m admits an induced forest of order at least $(38n - 7m)/44$, and thus at least $(6n + 7)/11$ [6].

An interesting variant of this problem is to look for large induced forests with bounded maximum degree. A forest with maximum degree at most 2 is called a *linear forest*.

The problem for linear forests was solved for outerplanar graphs by Pelsmajer [8]: every outerplanar graph admits an induced linear forest on at least four sevenths of its vertices, and this is tight. More generally, the problem for a forest of maximum degree at most d, with $d \geq 2$, was solved for graphs with treewidth at most k for all k by Chappell and Pelsmajer [5].

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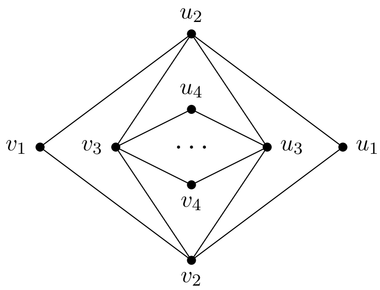

Fig. 1. The graph G_k of Claim 3.

Their result in particular extends the results of Hosono and Pelsmajer on outerplanar graphs to series–parallel graphs, and generalizes it to graphs of bounded treewidth.

In this paper we focus on linear forests. Chappell conjectured that every planar graph admits an induced linear forest on more than four ninths of its vertices. Again, this would be tight if true. Poh [9] proved that every planar graph can have its vertices be partitioned into three sets, each inducing a linear forest, and thus that every planar graph admits an induced linear forest on at least one third of its vertices. In this paper, we prove and strengthen Chappell's conjecture for a smaller class of graphs, the class of triangle-free planar graphs. Observe that planar graphs with arbitrarily large girth can have an arbitrarily large treewidth, so in this setting the best result known to date is that every triangle-free planar graph admits an induced linear forest on at least one third of its vertices.

We prove the following theorem:

Theorem 1. *Every triangle-free planar graph of order n and size m admits an induced linear forest of order at least* $\frac{9n-2m}{11}$.

Thanks to Euler's formula, we can derive the following corollary:

Corollary 2. *Every triangle-free planar graph of order* $n \geq 2$ *admits an induced linear forest of order at least* $\frac{5n+8}{11}$.

For a graph $G = (V, E)$, and $S \subset V$, let $G[S]$ denote the subgraph of G induced by S. Note that we cannot hope to get a better lower bound than $\frac{n}{2} + 1$. Indeed, we prove the following claim:

Claim 3. *For all integer* $n \geq 2$*, there exists a triangle-free planar graph of order n whose largest induced linear forest has order* $\lceil \frac{n}{2} \rceil + 1$.

Proof. Let us build such a graph for $n = 2k$. For odd n, adding an isolated vertex to the graph of order $n - 1$ yields the result.

Let G_k be defined by $G_k = (\bigcup_{1 \leq i \leq k}\{u_i, v_i\}, \bigcup_{1 \leq i \leq k-1}\{u_iu_{i+1}, u_iv_{i+1}, v_iu_{i+1}, v_iv_{i+1}\})$, as represented in Fig. 1. Let us prove by induction on $k \geq 1$ that the largest induced linear forest of G_k has order $k + 1$.

- For $k = 1$, G_1 is the graph with two vertices and no edge, and G_1 is its own largest induced linear forest, with order $2 = k + 1$.
- For $k = 2$, G_2 is a cycle of length 4, any three vertices of G_2 induce a linear forest of order $3 = k + 1$, and G_2 is not a linear forest (thus it has no induced linear forest of order 4).
- For $k = 3$, $\{u_1, v_1, u_3, v_3\}$ induces a linear forest in G_3, and it is easy to check that no five vertices of G_3 induce a linear forest.
- Suppose $k \geq 4$. By induction hypothesis, G_{k-1}, G_{k-2}, and G_{k-3} have a largest induced forest of order k, $k - 1$, and $k - 2$ respectively. Adding u_k and v_k to any induced linear forest of G_{k-2} leads to an induced linear forest of G_k, thus G_k has an induced linear forest of order $k + 1$. All that remains to prove is that G_k has no induced linear forest of order at least $k + 2$.

  Let $F \subset V(G_k)$ be a set inducing a linear forest of G_k. Let us prove that $|F| \leq k+1$. As $F\backslash\{u_k, v_k\}$ induces a linear forest in G_{k-1}, we have $|F\backslash\{u_k, v_k\}| \leq k$. Similarly, $|F\backslash\{u_k, v_k, u_{k-1}, v_{k-1}\}| \leq k-1$ and $|F\backslash\{u_k, v_k, u_{k-1}, v_{k-1}, u_{k-2}, v_{k-2}\}| \leq k-2$. If $|\{u_k, v_k\} \cap F| \leq 1$, then $|F| = |\{u_k, v_k\} \cap F| + |F\backslash\{u_k, v_k\}| \leq k + 1$. Suppose now that $|\{u_k, v_k\} \cap F| > 1$, i.e. $\{u_k, v_k\} \subset F$. At most one of u_{k-1} and v_{k-1} is in F, otherwise $G_k[F]$ has a cycle. If $\{u_{k-1}, v_{k-1}\} \cap F = \emptyset$, then $|F| = |\{u_k, v_k\} \cap F| + |\{u_{k-1}, v_{k-1}\} \cap F| + |F\backslash\{u_k, v_k, u_{k-1}, v_{k-1}\}| \leq 2 + k - 1 = k + 1$. Assume now that $\{u_{k-1}, v_{k-1}\} \cap F \neq \emptyset$, w.l.o.g. $u_{k-1} \in F$. We have $\{u_{k-2}, v_{k-2}\} \cap F = \emptyset$, otherwise u_{k-1} has degree at least 3 in $G_k[F]$. Hence, $|F| = |\{u_k, v_k\} \cap F| + |\{u_{k-1}, v_{k-1}\} \cap F| + |\{u_{k-2}, v_{k-2}\} \cap F| + |F\backslash\{u_k, v_k, u_{k-1}, v_{k-1}, u_{k-2}, v_{k-2}\}| \leq 2+1+k-2 = k + 1$. $\square$

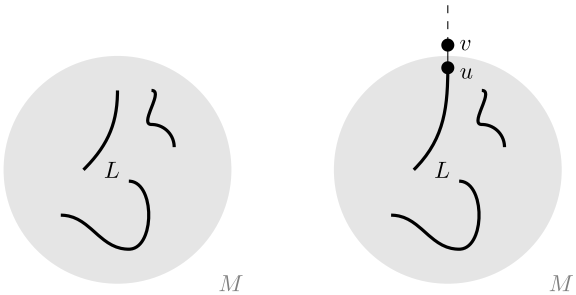

Fig. 2. The situation in Observation 5 (left) and Observation 6 (right). Thick lines are paths, normal lines are edges and dashed lines are edges that may or may not be present.

2. Proof of Theorem 1

Consider a graph $G = (V, E)$. For a set $S \subset V$, let $G - S$ be the graph obtained from G by removing the vertices of S and all the edges incident to a vertex of S. If $x \in V$, then we denote $G - \{x\}$ by $G - x$. For a set S of vertices such that $S \cap V = \emptyset$, let $G + S$ denote the graph obtained from G by adding the vertices of S. If $x \notin V$, then we denote $G + \{x\}$ by $G + x$. For a set F of pairs of vertices of G such that $F \cap E = \emptyset$, let $G + F$ be the graph constructed from G by adding the edges of F. If e is a pair of vertices of G and $e \notin E$, then we denote $G + \{e\}$ by $G + e$. If $x \in V$, then we denote the neighbourhood of x, that is the set of the vertices adjacent to x, by $N(x)$. For a set $S \subset V$, we denote the neighbourhood of S, that is the set of vertices in $V \backslash S$ that are adjacent to at least an element of S, by $N(S)$. We denote $|V|$ by $|G|$ and $|E|$ by $\|G\|$.

We call a vertex of degree d, at least d, and at most d, a *d-vertex*, a *d^+-vertex*, and a *d^--vertex* respectively. We call a cycle of length l an *l-cycle*, and a face with a boundary of length l an *l-face*.

Let $\mathcal{P}_4$ be the class of triangle-free planar graphs. Let $G = (V, E)$ be a counter-example to Theorem 1 with the minimum order. Let $n = |G|$ and $m = \|G\|$. We will use the schemes presented in Observations 4–6 many times throughout this paper.

Observation 4. *Let α, β, γ be integers satisfying $\alpha \geq 1$, $\beta \geq 0$, $\gamma \geq 0$. Let $H^* \in \mathcal{P}_4$ be a graph with $|H^*| = n - \alpha$ and $\|H^*\| \leq m - \beta$. By minimality of G, H^* admits an induced linear forest of order at least $\frac{9}{11}(n-\alpha) - \frac{2}{11}(m-\beta)$. Given an induced linear forest F^* of H^* of order $|F^*| \geq \frac{9}{11}(n-\alpha) - \frac{2}{11}(m-\beta)$, if there is an induced linear forest F of G of order $|F| \geq |F^*| + \gamma$, then as $|F| < \frac{9}{11}n - \frac{2}{11}m$, we have $\gamma < \frac{9}{11}\alpha - \frac{2}{11}\beta$.*

Observation 5. *Suppose $L \subset V$ is a non-empty set of vertices inducing a linear forest in G, and M is a set of vertices such that $M \cap L = \emptyset$ and $M \supset N(L)$. Let $G' = G - M - L$. See Fig. 2 (left) for an illustration. By minimality of G, G' admits a linear forest F' with $|F'| \geq \frac{9}{11}|G'| - \frac{2}{11}\|G'\|$. Observe that $F = G[V(F') \cup L]$ is an induced linear forest of G. As G is a counter-example to Theorem 1, $|F| < \frac{9}{11}|G| - \frac{2}{11}\|G\|$. Therefore $|L| = |F| - |F'| < \frac{9}{11}(|M| + |L|) - \frac{2}{11}(\|G\| - \|G'\|)$.*

Observation 6. *Suppose $L \subset V$ induces a linear forest in G. Suppose there is a set of vertices M and two vertices $u \in L$ and v such that $M \cap L = \emptyset$, $\{v\} = N(L) \setminus M$, and $\{u\} = N(v) \cap L$. Let $G' = G - M - L$. Suppose v is a 1^--vertex in G' and u is a 1^--vertex in $G[L]$. See Fig. 2 (right) for an illustration. By minimality of G, G' admits a linear forest F' with $|F'| \geq \frac{9}{11}|G'| - \frac{2}{11}\|G'\|$. Observe that $F = G[V(F') \cup L]$ is an induced linear forest of G. As G is a counter-example to Theorem 1, $|F| < \frac{9}{11}|G| - \frac{2}{11}\|G\|$. Therefore $|L| = |F| - |F'| < \frac{9}{11}(|M| + |L|) - \frac{2}{11}(\|G\| - \|G'\|)$.*

Now we want to prove some structural properties of G, so that we can later show that the counter-example G does not exist, and thus that Theorem 1 is true. First note that G is connected, otherwise one of its components would be a smaller counter-example to Theorem 1. Then note that every vertex of G has degree at most 4. Otherwise, by considering a vertex of degree at least 5 and by Observation 4 applied to $H^* = G - v$ with $(\alpha, \beta, \gamma) = (1, 5, 0)$ and $F = F^*$, we have $0 < \frac{9}{11} - 5\frac{2}{11}$, a contradiction.

Let us define the notion of a *chain* (or *simple chain*) of G which is a quadruplet $C = (P, N, u, v)$ such that:

- $P \subset V$, $N \subset V \setminus P$, $u \in P$, and $v \in V \backslash (N \cup P)$;
- $G[P]$ is a linear forest;
- vertex u is a 1^--vertex of $G[P]$, and $N(v) \cap P = \{u\}$;
- $N(P) \subset N \cup \{v\}$ in G;
- vertex v is a 2^--vertex in $G - (N \cup P)$.

See Fig. 3 (left) for an illustration. We will use the following notation for a chain $C = (P, N, u, v)$ of G:

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;

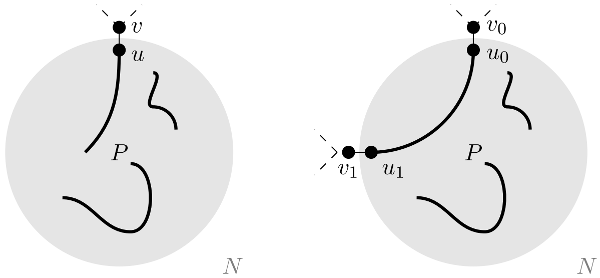

Fig. 3. A simple chain (left) and a double chain (right).

- $d(C)$ is the degree of v in $G - C$ (thus $d(C) \leq 2$);
- $\|C\| = \|G\| - \|G - C\|$.

We will now prove the following lemma:

Lemma 7. *For every chain $C = (P, N, u, v)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$.*

Proof. Let us consider by contradiction a chain $C = (P, N, u, v)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$ maximizing $|C|$.

- Suppose $d(C) = 0$. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of N. By Observation 5 applied to $L = P \cup \{v\}$ and $M = N$, we have $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}\|C\|$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, it follows that $1 < \frac{9}{11} - \frac{2}{11}\frac{1}{2}$, a contradiction.
- Suppose $d(C) = 1$. The set P induces a linear forest, and its neighbourhood is a subset of $N \cup \{v\}$. Furthermore, $N(v) \cap P = \{u\}$, $N(u) \cap (G - C) = \{v\}$, and u and v are 1-vertices in P and $G - C$ respectively. By Observation 6 applied to $L = P$ and $M = N$, we have $|P| < \frac{9}{11}|C| - \frac{2}{11}\|C\|$, thus $|P| < \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, a contradiction.
- Suppose $d(C) = 2$. Let w_0 and w_1 be the neighbours of v in $G - C$.
  - Suppose one of the w_i's, say w_0, has degree 1 in $G - C$. Let $G' = (G - C) - \{v, w_0, w_1\}$. The set $P \cup \{v, w_0\}$ induces a linear forest and its neighbourhood is a subset of $N \cup \{w_1\}$. By Observation 5 applied to $L = P \cup \{v, w_0\}$ and $M = N \cup \{w_1\}$, we have $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + 2)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, it follows that $2 < \frac{9}{11}3 - \frac{2}{11}\frac{5}{2}$, a contradiction.
  - Suppose the w_i's both have degree 2 in $G - C$. Observe that they are not adjacent since G is triangle-free. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{w_0, w_1\}$. Furthermore, $N(w_1) \cap (P \cup \{v\}) = \{v\}$, $N(v) \cap ((G - C) - \{v, w_0\}) = \{w_1\}$, and v and w_1 are 1-vertices in $P \cup \{v\}$ and $G - C - \{v, w_0\}$ respectively. By Observation 6 applied to $L = P \cup \{v\}$ and $M = N \cup \{w_0\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}\frac{7}{2}$, a contradiction.
  - Suppose the w_i's both have degree 4 in $G - C$. Again they are not adjacent since G is triangle-free. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{w_1, w_0\}$. By Observation 5 applied to $L = P \cup \{v\}$ and $M = N \cup \{w_0, w_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + 8)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}3 - \frac{2}{11}\frac{17}{2}$, a contradiction.
  - Suppose one of the w_i's, say w_0, is a 3^--vertex in $G - C$ and the other one is a 3^+-vertex in $G - C$. Let $C' = (P \cup \{v\}, N \cup \{w_1\}, v, w_0)$. Then C' is a chain of G, and by maximality of $|C|$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + \frac{7}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction. □

Let us now define a new notion quite similar to the notion of chain. A *double chain* of G is a sextuplet $C = (P, N, u_0, u_1, v_0, v_1)$, so that:

- $P \subset V$, $N \subset V \setminus P$, $u_0 \in P$, $u_1 \in P$, $v_0 \in V \setminus (N \cup P)$ and $v_1 \in V \setminus (N \cup P)$;
- $v_0 \neq v_1$;
- $G[P]$ is a linear forest;
- u_0 and u_1 are 1^--vertices of $G[P]$ if they are distinct, a 0-vertex of $G[P]$ if they are equal, and for $i \in \{0, 1\}$, $N(v_i) \cap P = \{u_i\}$;
- $N(P) \subset N \cup \{v_0\} \cup \{v_1\}$;
- v_0 and v_1 are 2^--vertices in $G - (N \cup P)$.

See Fig. 3 (right) for an illustration. We will use the following notation for a double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G:

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;

- $d_0(C)$ is the degree of v_0 in $G - C$ (thus $d_0(C) \leq 2$);
- $d_1(C)$ is the degree of v_1 in $G - C$ (thus $d_1(C) \leq 2$);
- $\|C\| = \|G\| - \|G - C\|$.

A double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G such that v_0 and v_1 are on different components of $G - C$ is called a *separating double chain* of G.

We will now prove the following lemmas:

Lemma 8. *For every double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$.*

Proof. Let us consider by contradiction a double chain $C = (P, N, u_0, u_1, v_0, v_1)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$ maximizing $|C|$.

Suppose first that v_0 and v_1 are not adjacent.

- Suppose $d_0(C) = 0$. Then $(P \cup \{v_0\}, N, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}(\|C\| - \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $1 < \frac{9}{11} - \frac{5}{2}\frac{2}{11}$, a contradiction.
- Suppose $d_0(C) = 1$. Let w be the neighbour of v_0 in $G - C$. Then $(P \cup \{v_0\}, N \cup \{w\}, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}\frac{7}{2}$, a contradiction.
- Suppose $d_0(C) = 2$. Let w_0 and w_1 be the neighbours of v_0 in $G - C$.
  - Suppose one of the w_i's, say w_0, has degree 1 in $G - C$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + \frac{3}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}\frac{9}{2}$, a contradiction.
  - Suppose that the w_i's both have degree 2 in $G - C$. Note that they are not adjacent since G is triangle-free. They may, however, be adjacent to v_1.
    Suppose both of the w_i's are adjacent to v_1. The set $P \cup \{v_0, v_1\}$ induces a linear forest in G, and its neighbourhood is a subset of $N \cup \{w_0, w_1\}$. By Observation 5 applied to $L = P \cup \{v_0, v_1\}$ and $M = N \cup \{w_0, w_1\}$, we have $|P| + 2 < \frac{9}{11}(|C| + 4) - \frac{2}{11}(\|C\| + 4)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, it follows that $2 < \frac{9}{11}4 - \frac{2}{11}7$, a contradiction.
    Therefore one of the w_i's, say w_0, is not adjacent to v_1. Let x be the neighbour of w_0 in $G - C$ distinct from v_0. Suppose x has degree 4 in $G - C - \{v_0, w_1\}$. Now $(P \cup \{v_0, w_0\}, N \cup \{w_1, x\}, u_1, v_1)$ is a chain of G. By Lemma 7, $|P| + 2 < \frac{9}{11}(|C| + 4) - \frac{2}{11}(\|C\| + \frac{13}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}\frac{19}{2}$, a contradiction.
    Therefore x is a 3^--vertex in $G - C - \{v_0, w_1\}$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, w_0, u_1, x, v_1)$ is a double chain of G, so by maximality of $|C|$, $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + 4 - 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}4$, a contradiction.
  - Suppose that the w_i's are 4-vertices in $G - C$. Now $(P \cup \{v_0\}, N \cup \{w_0, w_1\}, u_1, v_1)$ is a chain of G. By Lemma 7, $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + \frac{15}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $1 < \frac{9}{11}3 - \frac{2}{11}\frac{21}{2}$, a contradiction.
  - Suppose one of the w_i, say w_0, is a 3^--vertex in $G - C$ and the other one is a 3^+-vertex in $G - C$. Then $(P \cup \{v_0\}, N \cup \{w_1\}, v_0, u_1, w_0, v_1)$ is a double chain of G. By maximality of $|C|$, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + 4 - 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.

Now v_0 and v_1 are adjacent.

- Suppose $d_0(C) = 1$ or $d_1(C) = 1$, say $d_0(C) = 1$. The set $P \cup \{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1\}$. By Observation 5 applied to $L = P \cup \{v_0\}$ and $M = N \cup \{v_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + 1)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.
- Now $d_0(C) = 2$ and $d_1(C) = 2$. Let w be the neighbour of v_0 in $(G - C) - v_1$. Note that w is not adjacent to v_1, otherwise $v_0 v_1 w$ would be a triangle in G.
  Suppose w is a 2^--vertex in $(G - C)$. The set $P \cup \{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1, w\}$. Furthermore, $N(w) \cap (P \cup \{v_0\}) = \{v_0\}$, $N(v_0) \cap V((G - C) - \{v_1\}) = \{w\}$, and v_0 and w are 1^--vertices in $G[P \cup \{v_0\}]$ and $(G - C) - \{v_0\}$ respectively. By Observation 6 applied to $L = P \cup \{v_0\}$ and $M = N \cup \{v_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}6$, a contradiction.
  Now w is a 3^+-vertex in $(G - C)$. The set $P \cup \{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1, w\}$. By Observation 5 applied to $L = P \cup \{v_0\}$ and $M = N \cup \{v_1, w\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + 5)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 3)$, it follows that $1 < \frac{9}{11}3 - \frac{2}{11}8$, a contradiction. $\square$

Lemma 9. *For every separating double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$.*

Proof. Let us consider by contradiction a separating double chain $C = (P, N, u_0, u_1, v_0, v_1)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$ maximizing $|C|$.

- Suppose $d_0(C) = 0$. Then $(P \cup \{v_0\}, N, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}(\|C\| - \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $1 < \frac{9}{11} - \frac{2}{11}\frac{1}{2}$, a contradiction.

- Suppose $d_0(C) = 1$. Let w be the neighbour of v_0 in $G-C$. Suppose w is a 4-vertex in $G-C$. Then $(P\cup\{v_0\}, N\cup\{w\}, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + \frac{7}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}\frac{9}{2}$, a contradiction.
  Now w is a 3^--vertex in $G - C$. Let $C' = (P \cup \{v_0\}, N, v_0, u_1, w, v_1)$. One can see that C' is a separating double chain of G, and by maximality of $|C|$, $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}(\|C\|)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $1 < \frac{9}{11} - \frac{2}{11}$, a contradiction.
- Suppose $d_0(C) = 2$. Let w_0 and w_1 be the neighbours of v_0 in $G - C$.
  - Suppose one of the w_i's, say w_0, has degree 1 in $G - C$. We have a simple chain $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, u_1, v_1)$. By Lemma 7, $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(\|C\| + \frac{3}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}\frac{5}{2}$, a contradiction.
  - Suppose the w_i's both have degree 2 in $G - C$. Note that they are not adjacent since G is triangle-free. Let x be the second neighbour of w_0 in $G - C$. Suppose x is a 4-vertex in $G - C - \{w_1\}$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1, x\}, u_1, v_1)$ is a simple chain of G. By Lemma 7, $|P| + 2 < \frac{9}{11}(|C| + 4) - \frac{2}{11}(\|C\| + \frac{13}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}\frac{15}{2}$, a contradiction. Now x is a 3^--vertex in $G - C - \{w_1\}$, so $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, w_0, u_1, x, v_1)$ is a separating double chain of G. By maximality of $|C|$, $|P|+2 < \frac{9}{11}(|C|+3) - \frac{2}{11}(\|C\|+3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\|-1)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}4$, a contradiction.
  - Suppose the w_i's have degree 4 in $G - C$. Again they are not adjacent since G is triangle-free. We have a simple chain $(P\cup\{v_0\}, N\cup\{w_0, w_1\}, u_1, v_1)$. By Lemma 7, $|P|+1 < \frac{9}{11}(|C|+3) - \frac{2}{11}(\|C\|+\frac{15}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\|-1)$, we have $1 < \frac{9}{11}3 - \frac{2}{11}\frac{17}{2}$, a contradiction.
  - Suppose one of the w_i's, say w_0, is a 3^--vertex in $G-C$ and the other one is a 3^+-vertex in $G-C$. Then $(P\cup\{v_0\}, N\cup \{w_1\}, v_0, u_1, w_0, v_1)$ is a separating double chain. By maximality of $|C|$, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(\|C\| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(\|C\| - 1)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction. □

Let us now prove some lemmas on the structure of G.

Lemma 10. *Graph G has no 2^--vertex.*

Proof. As G is connected, if it has a 0-vertex, then G is the graph with one vertex and it satisfies Theorem 1, a contradiction.

By contradiction, suppose $u \in V$ is a 1-vertex. Let v be the neighbour of u. If v is a 3^--vertex in G, then $(\{u\}, \emptyset, u, v)$ is a chain of G, thus by Lemma 7, $1 < \frac{9}{11} - \frac{2}{11}(1 - \frac{1}{2})$, a contradiction.

Now v is a 4-vertex. Let $H^* = G - \{u, v\}$. Graph H^* has $n - 2$ vertices and $m - 4$ edges. Adding vertex u to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (2, 4, 1)$, $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.

Therefore G has no 1^--vertex. Suppose now that $u \in V$ is a 2-vertex. Let v_0 and v_1 be the two neighbours of u.

Suppose v_0 and v_1 are both 3^--vertices. We have a double chain $(\{u\}, \emptyset, u, u, v_0, v_1)$, and thus by Lemma 8, $1 < \frac{9}{11}1 - \frac{2}{11}(4 - 3)$, a contradiction.

Suppose v_0 or v_1, say v_0, is a 4-vertex, and the other one is a 3^--vertex. We have a simple chain $(\{u\}, \{v_0\}, u, v_1)$. By Lemma 7, $1 < \frac{9}{11}2 - \frac{2}{11}(5 - \frac{1}{2})$, a contradiction.

Now v_0 and v_1 are 4-vertices. Let $H^* = G - \{u, v_0, v_1\}$. Graph H^* has $n - 3$ vertices and $m - 8$ edges. Adding vertex u to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (3, 8, 1)$, $1 < \frac{9}{11}3 - \frac{2}{11}8$, a contradiction. □

Lemma 11. *Graph G has no 3-vertex adjacent to another 3-vertex and two 4-vertices.*

Proof. By contradiction, suppose G has a 3-vertex u, adjacent to a 3-vertex v and two 4-vertices w_0 and w_1. We have a simple chain $(\{u\}, \{w_0, w_1\}, u, v)$. By Lemma 7, $1 < \frac{9}{11}3 - \frac{2}{11}(9 - \frac{1}{2})$, a contradiction. □

Lemma 12. *Graph G has no 3-vertex adjacent to two other 3-vertices and a 4-vertex.*

Proof. Let u be a 3-vertex adjacent to two 3-vertices v_0 and v_1, and to a 4-vertex w. Let x_0 and x_1 be the two neighbours of v_0 distinct from u. Note that x_0 and x_1 may be adjacent to w. Note that x_0 and x_1 are 3^+-vertices in G by Lemma 10, and thus 1^+-vertices in $G' = G - \{u, w, v_0\}$ since they are not adjacent to u.

Suppose that x_0 and x_1 are 2^+-vertices in G'. We have a simple chain $(\{u, v_0\}, \{x_0, x_1, w\}, u, v_1)$ in G. By Lemma 7, $2 < \frac{9}{11}5 - \frac{2}{11}(12 - \frac{1}{2})$, a contradiction.

Suppose one of the x_i's, say x_0, is a 2^+-vertex in G', and the other one is a 1-vertex in G'. We have a double chain $(\{u, v_0\}, \{w, x_0\}, u, v_0, v_1, x_1)$. By Lemma 8, $2 < \frac{9}{11}4 - \frac{2}{11}(10 - 3)$, a contradiction.

Now the x_i's are 1-vertices in G'. By Lemma 10, the x_i's are 3-vertices in G, and thus are both adjacent to w. By planarity of G, one of the x_i's, say x_0, is not adjacent to v_1. Let y be the neighbour of x_0 in G'. By Lemmas 10 and 11, y is a 3-vertex in G. We have a simple chain $(\{u, v_0, x_0\}, \{w, v_1, x_1\}, x_0, y)$. By Lemma 7, $3 < \frac{9}{11}6 - \frac{2}{11}(11 - \frac{1}{2})$, a contradiction. □

Lemma 13. *Graph G has no two adjacent 3-vertices.*

Proof. By Lemma 10, every vertex in G has degree 3 or 4. By Lemmas 11 and 12, there is no 3-vertex adjacent to a 3-vertex and a 4-vertex in G. Suppose by contradiction that there are two adjacent 3-vertices in G. Then as G is connected, G only has 3-vertices.

Suppose there is a 4-cycle $u_0u_1u_2u_3$ in G. For all i, let v_i be the third neighbour of u_i. Since G has no triangle, the only vertices among the u_i's and v_i's that may not be distinct are v_0 and v_2 on the one hand, and v_1 and v_3 on the other hand. Suppose $v_0 = v_2$ and $v_1 = v_3$. Let $H^* = G - \{u_0, u_1, u_2, u_3\}$. Graph H^* has $n-4$ vertices and $m-8$ edges. As v_0 and v_1 are 1-vertices in H^*, that, by planarity, are separated by $u_0u_1u_2u_3$ in G, adding vertices u_0 and u_1 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (4, 8, 2)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}8$, a contradiction. Now w.l.o.g. v_0 and v_2 are distinct. We have a double chain $(\{u_0, u_1, u_2\}, \{u_3, v_1\}, u_0, u_2, v_0, v_2)$. By Lemma 8, $3 < \frac{9}{11}5 - \frac{2}{11}(9-3)$, a contradiction.

Now there is no 4-cycle in G. Suppose there is a 5-cycle $u_0u_1u_2u_3u_4$ in G. For all i, let v_i be the third neighbour of u_i. Now all the v_i's are distinct, otherwise there is a 4-cycle and we fall into the previous case. We have a double chain $(\{u_0, u_1, u_2, u_3\}, \{u_4, v_1, v_2\}, u_0, u_3, v_0, v_3)$. By Lemma 8, $4 < \frac{9}{11}7 - \frac{2}{11}(14-3)$, a contradiction.

Now G is a 3-regular planar graph with girth at least 6, which contradicts Euler's formula. □

Lemma 14. *There is no 4-cycle with at least two 3-vertices in G.*

Proof. By contradiction, suppose there is such a 4-cycle $u_0u_1u_2u_3$. By Lemmas 10 and 13, this cycle has exactly two 3-vertices and two 4-vertices, and the two 3-vertices are not adjacent. W.l.o.g. u_0 and u_2 are 3-vertices, and u_1 and u_3 are 4-vertices. Let v_0 and v_2 be the third neighbours of u_0 and u_2 respectively. By Lemma 13, v_0 and v_2 are 4-vertices.

Suppose that u_0 and u_2 have three neighbours in common, u_1, u_3, and $v = v_0 = v_2$. Let $H^* = G - \{u_0, u_1, u_2, u_3, v\}$. Graph H^* has $n-5$ vertices and $m-12$ edges. Adding vertices u_0 and u_2 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (5, 12, 2)$, $2 < \frac{9}{11}5 - \frac{2}{11}12$, a contradiction.

Now v_0 and v_2 are distinct. Suppose that $v_0v_2 \in E$. We have a chain $(\{u_0, u_2\}, \{u_1, u_3, v_2\}, u_0, v_0)$. By Lemma 7, $2 < \frac{9}{11}5 - \frac{2}{11}(13 - \frac{1}{2})$, a contradiction.

Now $v_0v_2 \notin E$. Let $H^* = G - \{u_0, u_1, u_2, u_3, v_0, v_2\}$. Graph H^* has $n-6$ vertices and $m-16$ edges. Adding vertices u_0 and u_2 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (6, 16, 2)$, $2 < \frac{9}{11}6 - \frac{2}{11}16$, a contradiction. □

Lemma 15. *There is no 4-face with exactly one 3-vertex in G.*

Proof. By contradiction, suppose there is a 4-face $u_0u_1u_2u_3$, such that u_0 is a 3-vertex and the other u_i are 4-vertices. Let v be the third neighbour of u_0. Note that v is a 4-vertex by Lemma 13.

Suppose first that $vu_2 \in E$. By planarity of G, $\{u_0, v, u_2\}$ separates the vertices u_1 and u_3. Therefore $(\{u_0\}, \{v, u_2\}, u_0, u_0, u_1, u_3)$ is a separating double chain of G. By Lemma 9, $1 < \frac{9}{11}3 - \frac{2}{11}(9-1)$, a contradiction.

Now $vu_2 \notin E$. Let w_0 and w_1 be the neighbours of u_1 distinct from u_0 and u_2.

- Suppose w_0 and w_1 are adjacent to u_3. Let $H^* = G - \{u_0, u_1, u_2, u_3, v, w_0, w_1\}$. By Lemma 10, the w_i's are 3^+-vertices, and since G is triangle-free, they cannot be adjacent to u_2. Moreover, by planarity, at least one of the w_i's is not adjacent to v. This implies that H^* has at most $m-15$ edges. Graph H^* has $n-7$ vertices. Adding vertices u_0, u_1, and u_3 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (7, 15, 3)$, $3 < \frac{9}{11}7 - \frac{2}{11}15$, a contradiction.
- Suppose that one of the w_i's, say w_0, is adjacent to u_3 and that the other one (w_1) is not adjacent to u_3. Let w_2 be the neighbour of u_3 distinct from u_0, u_2, and w_0.
  - Suppose w_1 or w_2, say w_1, is a 3-vertex in $G - \{u_0, u_1, u_2, u_3, v, w_0\}$. Suppose w_2 is a 3-vertex in $G - \{u_0, u_1, u_2, u_3, v, w_0, w_1\}$ (note that this implies that $w_1w_2 \notin E$, since otherwise w_2 would have degree greater than 4 in G). Let $H^* = G - \{u_0, u_1, u_2, u_3, v, w_0, w_1, w_2\}$. Graph H^* has $n-8$ vertices and at most $m-20$ edges. Adding vertices u_0, u_1, and u_3 to any induced linear forest of H^* leads to an induced linear forest of G. By Observation 4 applied to $(\alpha, \beta, \gamma) = (8, 20, 3)$, $3 < \frac{9}{11}8 - \frac{2}{11}20$, a contradiction.
    Now w_2 is a 2^--vertex in $G - \{u_0, u_1, u_2, u_3, v, w_0, w_1\}$, and thus $(\{u_0, u_1, u_3\}, \{u_2, v, w_0, w_1\}, u_3, w_2)$ is a chain of G. By Lemma 7, $3 < \frac{9}{11}7 - \frac{2}{11}(17 - \frac{1}{2})$, a contradiction.
  - Suppose w_1 and w_2 are 2^--vertex in $G - \{u_0, u_1, u_2, u_3, v, w_0\}$. We have a double chain $(\{u_0, u_1, u_3\}, \{u_2, v, w_0\}, u_1, u_3, w_1, w_2)$. By Lemma 8, $3 < \frac{9}{11}6 - \frac{2}{11}(14-3)$, a contradiction.
- Suppose the w_i's are not adjacent to u_3. Let us prove by contradiction that the w_i's are 2^--vertices in $G - \{u_0, u_1, u_2, v\}$.
  - Suppose the w_i's are 3-vertices in $G - \{u_0, u_1, u_2, v\}$. Then we have the following chain: $(\{u_0, u_1\}, \{u_2, v, w_0, w_1\}, u_0, u_3)$. By Lemma 7, $2 < \frac{9}{11}6 - \frac{2}{11}(18 - \frac{1}{2})$, a contradiction.
  - Now one of the w_i's, say w_0, is a 2^--vertex in $G - \{u_0, u_1, u_2, v\}$. Suppose w_1 is a 3-vertex in $G - \{u_0, u_1, u_2, v\}$. Then we have the double chain: $(\{u_0, u_1\}, \{u_2, v, w_1\}, u_0, u_1, u_3, w_0)$. By Lemma 8, $2 < \frac{9}{11}5 - \frac{2}{11}(15-3)$, a contradiction.

Now the w_i's are 2^--vertices in $G - \{u_0, u_1, u_2, v\}$. The w_i's are 3-vertices or 4-vertices in G, they are not adjacent to u_0 and u_2 since G is triangle-free, and by Lemma 14, if for some i, w_i is adjacent to v, then w_i is a 4-vertex. Therefore each of the w_i's is either a 3-vertex non-adjacent to v or a 4-vertex adjacent to v, and thus the w_i's are 2-vertices in $G - \{u_0, u_1, u_2, v\}$.
Let x_0 and x_1 be the two neighbours of w_0 in $G - \{u_0, u_1, u_2, u_3, v\}$. Let d be the sum of the degrees of x_0 and x_1 in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$.

- Suppose $d \geq 4$. We have a simple chain $(\{u_0, u_1, w_0\}, \{u_2, v, w_1, x_0, x_1\}, u_0, u_3)$. By Lemma 7, $3 < \frac{9}{11}8 - \frac{2}{11}(20 - \frac{1}{2})$, a contradiction.
- Suppose $d \leq 3$. Suppose one of the x_i, say x_0, is a 0-vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$. We have a simple chain $(\{u_0, u_1, w_0, x_0\}, \{u_2, v, w_1, x_1\}, u_0, u_3)$. By Lemma 7, $4 < \frac{9}{11}8 - \frac{2}{11}(16 - \frac{1}{2})$, a contradiction.
  Thus we can assume one of the x_i, say x_0, is a 1-vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$ and the other one (x_1) is a 1-vertex or a 2-vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$.
  Let us prove by contradiction that x_0 is not adjacent to u_3. Suppose x_0 is adjacent to u_3.
  Suppose $x_0w_1 \in E$. By Lemma 14, at least one of the w_i's, w_0 say, is a 4-vertex, and thus is adjacent to v. By planarity, w_1 is not adjacent to v, and thus w_1 is a 3-vertex in G. In this case, x_0 is adjacent exactly to w_0, w_1 and u_3 since it is a 1-vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$ and G is triangle-free. Then x_0 and w_1 are adjacent 3-vertices in G, which contradicts Lemma 13.
  Now $x_0w_1 \notin E$. We have a simple chain $(\{u_0, u_1, x_0\}, \{u_2, u_3, v, w_0\}, u_1, w_1)$. By Lemma 7, $3 < \frac{9}{11}7 - \frac{2}{11}(16 - \frac{1}{2})$, a contradiction.
  Therefore we know that x_0 is not adjacent to u_3. Let y be the neighbour of x_0 in $G - \{u_0, u_1, u_2, v, w_0, w_1\}$. Suppose y is a 3-vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1, x_0, x_1\}$. Now the following quadruplet is a simple chain: $(\{u_0, u_1, w_0, x_0\}, \{u_2, v, w_1, x_1, y\}, u_0, u_3)$. By Lemma 7, $4 < \frac{9}{11}9 - \frac{2}{11}(21 - \frac{1}{2})$, a contradiction. Now y is a 2^--vertex in $G - \{u_0, u_1, u_2, v, w_0, w_1, x_0, x_1\}$. Then $(\{u_0, u_1, w_0, x_0\}, \{u_2, v, w_1, x_1\}, u_0, x_0, u_3, y)$ is a double chain. By Lemma 8, $4 < \frac{9}{11}8 - \frac{2}{11}(18 - 3)$, a contradiction. □

For every face f of G, let $l(f)$ denote the length of the boundary of f, and let $c_4(f)$ denote the number of 4-vertices in f. For every vertex v, let $d(v)$ be the degree of v. Let k be the number of faces of G, and for every $3 \leq d \leq 4$ and every $4 \leq l$, let k_l be the number of l-faces and n_d the number of d-vertices in G.

Each 4-vertex is in the boundary of at most four faces. Therefore the sum of the $c_4(f)$ over all the 4-faces and 5-faces is $\sum_{f, 4 \leq l(f) \leq 5} c_4(f) \leq 4n_4$. Now, by Lemmas 10, 14, and 15, every 4-face of G has only 4-vertices in its boundary, so for each 4-face f, $c_4(f) = 4$. By Lemma 13, every 5-face of G has at least three 4-vertices, so for each 5-face f we have $c_4(f) \geq 3 \geq 2$. Thus $\sum_{f, l(f)=4} c_4(f) + \sum_{f, l(f)=5} c_4(f) \geq 4k_4 + 2k_5$. Thus $4n_4 \geq 4k_4 + 2k_5$, and thus $2n_4 \geq 2k_4 + k_5$. By Euler's formula, we have:

$$\begin{aligned}
-12 &= 6m - 6n - 6k \\
&= 2\sum_{v \in V} d(v) + \sum_{f \in F(G)} l(f) - 6n - 6k \\
&= \sum_{d \geq 3} (2d - 6)n_d + \sum_{l \geq 4} (l - 6)k_l \\
&\geq 2n_4 - 2k_4 - k_5 \\
&\geq 0
\end{aligned}$$

That contradiction ends the proof of Theorem 1.

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References

[1] J. Akiyama, M. Watanabe, Maximum induced forests of planar graphs, Graphs Combin. 3 (1987) 201–202.
[2] M.O. Albertson, D.M. Berman, A conjecture on planar graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, 1979.
[3] M. Albertson, R. Haas, A problem raised at the DIMACS Graph Coloring Week, New Jersey, 1998.
[4] O.V. Borodin, A proof of Grünbaum's conjecture on the acyclic 5-colorability of planar graphs, Dokl. Akad. Nauk SSSR 231 (1) (1976) 18–20, (in Russian).
[5] G.G. Chappell, M.J. Pelsmajer, Maximum induced forests in graphs of bounded treewidth, Electron. J. Combin. 20 (4) (2013) P8.
[6] F. Dross, M. Montassier, A. Pinlou, Large induced forests in planar graphs with girth 4 or 5, 2014, arXiv:1409.1348.
[7] K. Hosono, Induced forests in trees and outerplanar graphs, Proc. Fac. Sci. Tokai Univ. 25 (1990) 27–29.
[8] M.J. Pelsmajer, Maximum induced linear forests in outerplanar graphs, Graphs Combin. 20 (1) (2004) 121–129.
[9] K.S. Poh, On the linear vertex-arboricity of a plane graph, J. Graph Theory 14 (1) (1990) 73–75.