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Acyclic improper choosability of graphs

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Abstract

We consider *improper* colorings (sometimes called *generalized*, *defective* or *relaxed* colorings) in which every color class has a bounded degree. We propose a natural extension of improper colorings: *acyclic improper choosability*. We prove that subcubic graphs are acyclically $(3,1)^*$-choosable (i.e. they are acyclically 3-choosable with color classes of maximum degree one). Using a linear time algorithm, we also prove that outerplanar graphs are acyclically $(2,5)^*$-choosable (i.e. they are acyclically 2-choosable with color classes of maximum degree five). Both results are optimal. We finally prove that acyclic choosability and acyclic improper choosability of planar graphs are equivalent notions.

Keywords: Improper coloring; Acyclic coloring; Choosability; Cubic graphs; Outerplanar graphs.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph and let $V(G)$ and $E(G)$ be its vertex set and its edge set, respectively.

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Many variations and extensions of graph colorings have been considered. In particular, *improper* colorings (sometimes called *generalized*, *defective* or *relaxed* colorings) have been extensively studied. A *t-improper k-coloring* of G, or simply a $(k,t)^*$-coloring, is a partition of $V(G)$ into k color classes $V_1, V_2, \ldots, V_k$ such that each V_i induces a graph with maximum degree t; in other words, each vertex has at most t neighbors of the same color as itself. The *t-improper chromatic number* of G is therefore defined as the smallest integer k such that G is $(k,t)^*$-colorable. Notice that 0-improper coloring corresponds to the usual notion of proper coloring: a $(k,0)^*$-coloring of G is a proper k-coloring of G, and the 0-improper chromatic number of G is the chromatic number of G.

Improper colorings were introduced by Cowel *et al.* [5]. They proved that every planar graph is $(3,2)^*$-colorable and every outerplanar graph is $(2,2)^*$-colorable. They also showed, without using the Four Color Theorem, that every planar graph is $(4,1)^*$-colorable. In the last past years, several authors studied this coloring and the problem of bounding the t-improper chromatic number has been investigated for various classes of graph (see e.g. [6,14,15]).

A graph G is *L-colorable* if for a given list-assignment $L = \{L(v) : v \in V(G)\}$, there exists a proper coloring f of G such that $f(v) \in L(v)$ for every $v \in V(G)$. If G is L-colorable for any list-assignment L with $|L(v)| \geq l$ for every v, then we say that G is *l-choosable*. The *list chromatic number* is then defined as the smallest integer l such that G is l-choosable. Notice that a graph which is l-choosable is obviously l-colorable. Thomassen [12] proved that every planar graph is 5-choosable and Voigt [13] showed the tightness of this bound.

Eaton and Hull [7] generalized the notion of choosability to *improper* choosability: a graph G is *t-improper l-choosable*, or simply $(l,t)^*$-choosable, if for any list-assignment L such that $|L(v)| \geq l$ for every v, there exists a t-improper coloring f of G such that $f(v) \in L(v)$ for every v. Eaton and Hull [7], and independently Škrekovski [11], proved that every planar graph is $(3,2)^*$-choosable, which extends the above-mentioned Cowel *et al.*'s result. This result is sharp in a certain way since there exist planar graphs which are not $(3,1)^*$-colorable and planar graphs which are not $(2,t)^*$-colorable for every t. Moreover, Eaton and Hull, and Škrekovski, both conjectured that every planar graph is $(4,1)^*$-choosable.

Recall that an *acyclic coloring* of G is a coloring f of G such that for any two distinct colors i and j, the edges uv such that $f(u) = i$ and $f(v) = j$ induce a forest. A cycle is said *alternating* if it is properly colored with two colors. Notice that a coloring of G is acyclic if and only G does not contain any alternating cycle. We can also note that improper bicolored cycles are not necessarily alternating cycles.

Acyclic choosability was recently introduced by Borodin *et al.* in [4]. A graph is acyclically l-choosable if for any list-assignment L such that $|L(v)| \geq l$ for every

v, there exists an acyclic coloring f of G such that $f(v) \in L(v)$. Borodin *et al.* [4] proved that every planar graph is acyclically 7-choosable. They also conjectured that every planar graph is acyclically 5-choosable. Acyclic choosability of graph with bounded degree was also investigated, and Gonçalves and Montassier [10] showed that every subcubic graphs (graphs with maximum degree three) is acyclically 4-choosable.

Boiron *et al.* [2] extended in a natural way the notion of acyclic coloring to the notion of acyclic improper coloring as follows. An *acyclic t-improper k-coloring*, or simply an acyclic $(k,t)^*$-coloring, of G is a $(k,t)^*$-coloring which is acyclic, that is G contains no alternating cycle. The main motivation in the study of acyclic improper coloring is the link with oriented coloring (see [2] for more details). Boiron *et al.* [1] proved that every subcubic graph is acyclically $(3,1)^*$-colorable and conjectured that every subcubic graph is acyclically $(2,2)^*$-colorable. Moreover, they constructed subcubic which are not acyclically $(2,1)^*$-colorable. They also proved that every outerplanar graph is acyclically $(2,5)^*$-colorable and constructed outerplanar graphs which are not acyclically $(2,4)^*$-colorable. They also proved that for every $k \geq 0$, there exist planar graphs which are not acyclically $(4,k)^*$-colorable.

This paper is devoted to introduce and study the *acyclic improper choosability* for some classes of graphs.

In a natural way, one can define *acyclic improper choosability* of graphs: a graph G is *acyclically t-improper L-colorable* if for a given list-assignment $L = \{L(v) : v \in V(G)\}$, there exists an acyclic t-improper coloring f such that $f(v) \in L(v)$ for every v. If G is acyclically t-improper L-colorable for any list-assignment L with $|L(v)| \geq l$ for every v, then we say that G is *acyclically t-improper l-choosable*, or simply *acyclically $(l,t)^*$-choosable*. The *acyclic t-improper list chromatic number* of G is therefore defined as the smallest integer l such that G is acyclically $(l,t)^*$-choosable.

Our first result concerns subcubic graphs (graphs with maximum degree three) and extends the above-mentioned result of Boiron *et al.* [1].

Theorem 1.1 *Every subcubic graph is acyclically $(3,1)^*$-choosable.*

Note that the authors recently studied with Colin McDiarmid the behavior of the acyclic t-improper chromatic number of graphs with bounded maximum degree [8]:

Theorem 1.2 *There exists a constant* $\eta > 0$ *such that if* $t \leq \eta \left(\frac{n}{\log n}\right)^{3/4}$*, then*

$$\chi_a^t(d) = \Omega\left(\frac{d^{4/3}}{(\log d)^{1/3}}\right).$$

Our second result concerns outerplanar graphs (see [9] for a detailed proof):

Theorem 1.3 *Every outerplanar graph is acyclically* $(2,5)^*$*-choosable.*

Notice that this theorem extends the above-mentioned result of Boiron *et al.* [2].

This paper is organized as follows. We give a sketch of proof of Theorem 1.1 in Section 2 and a sketch of proof of Theorem 1.3 in Section 3. Finally, in Section 4, we make some final remarks about the acyclic improper choosability of planar graphs.

2 Acyclic $(3,1)^*$-choosability of subcubic graphs

In this section, we give the main ideas of the proof of Theorem 1.1.

Proof of Theorem 1.1 (Sketch) Let H be a counter-example to Theorem 1.1 with minimum order, and L be a list-assignment, with $|L(v)| \geq 3$ for every $v \in V(H)$, such that H is not acyclically 1-improper L-colorable.

First, the graph H is a 2-connected cubic graph. Then, recall that Boiron *et al.* [2] proved that every subcubic graph is acyclically $(3,1)$-colorable. So, we can assume that H contains two adjacent vertices u^* and v^* such that $L(u^*) \neq L(v^*)$.

We can order the vertices $x_1, x_2, \ldots, x_n$ of H such that $x_1 = u^*$, $x_n = v^*$ and for every i, $1 \leq i < n$, the vertex x_i is adjacent to some vertex x_j with $j > i$.

We then define a sequence of graphs $H_1, H_2, \ldots, H_n$ such that $H_i = H \setminus \{x_{i+1}, x_{i+2}, \ldots, x_n\}$ for $1 \leq i \leq n$.

We now describe an algorithm which colors H. At Step 1, we set $f(x_1) = c \in L(x_1) \setminus L(x_n)$ (recall that we assumed that $L(u^*) \neq L(v^*)$, $x_1 = u^*$ and $x_n = v^*$) and therefore f is an acyclic 1-improper L-coloring of H_1. Suppose that at Step $i-1$, f is an acyclic 1-improper L-coloring of H_{i-1} such that x_1 remains colored with color c. We can then extend f to H_i (i.e. color the vertex x_i) without changing the color of x_1. At Step n, the vertex x_n is the only vertex of H which remains uncolored. The vertex x_n is adjacent to x_1 and $f(x_1) = c \notin L(x_n)$. We can then extend f to $H_n = H$. The graph H is therefore acyclically 1-improper L-colorable, which is a contradiction. □

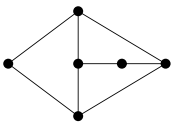

Fig. 1. A subcubic graph which is not acyclically 3-colorable.

The result of Theorem 1.1 is optimal. Indeed, some graphs with maximum degree Δ are not acyclically $(\Delta-1,t)^*$-choosable, for any $t \geq 0$. Therefore, there exist subcubic graphs which are not acyclically $(2,t)$-choosable for any $t \geq 0$. Moreover, the graph depicted on Fig. 1 is clearly not acyclically 3-colorable and therefore not acyclically $(3,0)$-choosable.

3 Acyclic $(2,5)^*$-choosability of outerplanar graphs

Let T be a rooted tree and let $v_1, v_2, \dots, v_n$ be its vertices ordered according to some depth-first search walk in T. Let $\phi: V(T) \to V(T)$ be the function defined as follows:

$$\phi: v_i \mapsto \begin{cases} \phi(v_{i-1}), \text{ if } v_{i-1} \text{ is } v_i\text{'s father,} \\ v_j \text{ a brother of } v_i \text{ with } j \text{ the maximum index smaller than } i, \text{ otherwise.} \end{cases}$$

Observe that the function ϕ is not defined for the vertices $v_1, v_2, \dots, v_{k-1}$, where k is the smallest integer such that v_{k-1} is not v_k's father: we denote this set of vertices by W.

In [3], Bonichon *et al.* proved that for any outerplanar graph G, we can find an order $v_1, \dots, v_n$ on the vertices of G and a rooted spanning tree T_G of G such that

- the order $v_1, \dots, v_n$ is a depth-first search order in T_G,
- let ϕ be defined as above by the rooted tree T_G and the order $v_1, \dots, v_n$. The graph H obtained from T_G by adding the set of *transversal edges* $M = \{v\phi(v), v \in V(T_G)\}$ is a near-triangulated outerplanar graph such that $V(G) = V(H)$ and $G \subseteq H$.

Fig. 2 shows an example of decomposition of G. The transversal edges are dashed for more clarity. Observe that in this example, $W = \{v_1, v_2, v_3\}$.

Observe that for every i, there is at most one integer $j < i$ such that $v_i v_j$ is a transversal edge. This means that in any greedy coloring algorithm according to the order $v_1, \dots, v_n$, at each step i, the vertex v_i will be adjacent to at most two already colored vertices: its father and possibly $\phi(v_i)$.

Let p^k be the function defined as $p^k(v) = v$ if $k = 0$, and $p^k(v)$ equals to

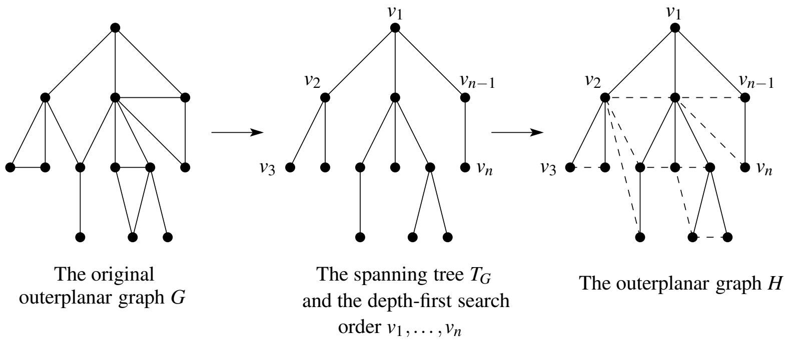

Fig. 2. The Bonichon *et al.*'s outerplanar graph decomposition.

$p^{k-1}(v)$'s father otherwise. In other words, $p^k(v)$ is the ancestor of v at distance k in T_G. Again, p^k is not defined for every vertex of V and every k.

Let v_i be a vertex of G with at least k children in T_G. For $k \geq 1$, the k-th child of v_i, denoted by $s_k(v_i)$, is v_{i+1} if $k = 1$ and is v_j otherwise with j being the smallest integer such that $\phi(v_j)$ is v_i's $(k-1)$-th child.

Proof of Theorem 1.3 (Sketch) Let G be an outerplanar graph. Let L be a list-assignment for the vertices of G such that $|L(v)| \geq 2$ for every v. We decompose G as described above: we obtain an order $v_1, \ldots, v_n$, a rooted spanning tree T_G, a set of transversal edges $M = \{v\phi(v), v \in V(G)\}$ and as a result a near-triangulated outerplanar graph H.

Let $H_1, H_2, \ldots, H_n$ be the sequence of graphs defined as $H_i = H \setminus \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ for $1 \leq i \leq n$.

We color H greedily following the order $v_1, v_2, \ldots, v_n$. At Step 1, we color the graph H_1 by assigning any color from $L(v_1)$ to its unique vertex v_1. At Step $2 \leq i \leq n$, let f be the coloring of H_{i-1}, which is also a partial coloring of H_i. We use the following coloring rules to extend f to H_i ("$f(v_i) \in X$" means that we choose any color $c \in X$ and set $f(v_i) = c$).

Coloring rules:

R0 - If $v_i \in W$: $f(v_i) \in L(v_i)$.

R1 - If v_i is the first child of $p^1(v_i)$ (in other words, if $p^1(v_i) = v_{i-1}$):
- (a) if $f(\phi(v_i)) \neq f(p^1(v_i))$: $f(v_i) \in L(v_i) \setminus \{f(\phi(v_i))\}$;
- (b) if $f(\phi(v_i)) = f(p^1(v_i)) = a$:
  - i. if $f(p^2(v_i)) = a$:
    - A. if $p^2(v_i)$ and $p^1(\phi(v_i))$ are the same vertex:
      - α. if $\phi(v_i)$ is a leaf in T_G and belongs to W: $f(v_i) \in L(v_i)$;

β. if $\phi(v_i)$ is a leaf in T_G or $f(\phi^2(v_i)) \neq f(p^1(v_i))$: $f(v_i) \in L(v_i) \setminus \{f(\phi^2(v_i))\}$;

γ. otherwise: $f(v_i) \in L(v_i) \setminus \{f(s_1(\phi(v_i)))\}$;

B. if $p^2(v_i)$ and $p^1(\phi(v_i))$ are distinct vertices: $f(v_i) \in L(v_i) \setminus \{f(p^3(v_i))\}$;

ii. if $f(p^2(v_i)) \neq a$: $f(v_i) \in L(v_i) \setminus \{f(p^2(v_i))\}$.

R2 - If v_i is the second child of $p^1(v_i)$:

(a) if $p^1(v_i) \in W$ or $f(\phi(v_i)) \neq f(p^1(v_i))$: $f(v_i) \in L(v_i) \setminus \{f(p^1(v_i))\}$;

(b) if $f(\phi(v_i)) = f(p^1(v_i))$:

i. if $\phi(v_i)$ is a leaf in T_G or $f(\phi^2(v_i)) \neq f(p^1(v_i))$ or $f(p^2(v_i)) \neq f(p^1(v_i))$: $f(v_i) \in L(v_i) \setminus \{f(\phi(p^1(v_i)))\}$;

ii. otherwise: $f(v_i) \in L(v_i) \setminus \{f(s_1(\phi(v_i)))\}$.

R3 - If v_i is the k-th child of $p^1(v_i)$ with $k \geq 3$: $f(v_i) \in L(v_i) \setminus \{f(p^1(v_i))\}$.

These rules ensure that the coloring f is a 5-improper L-coloring of H and that the graph H does not contain any alternating cycle. □

Since the Bonichon *et al.* decomposition can be computed in linear time [3], this proof provides a linear time algorithm for finding a 5-improper coloring of any outerplanar graph given lists of size at least two.

The result of Theorem 1.3 is optimal. Indeed, it is clear that outerplanar graphs are not $(1,t)^*$-choosable for every $t \geq 0$ and therefore are not acyclically $(1,t)^*$-choosable. Moreover, Boiron *et al.* [2] constructed outerplanar graphs which are not acyclically $(2,4)^*$-colorable and therefore not acyclically $(2,4)^*$-choosable.

4 Concluding remarks

As noted in Introduction, Borodin *et al.* [4] conjectured that every planar graph is acyclically 5-choosable. We prove that acyclic choosability and acyclic improper choosability of planar graphs are equivalent notions.

Proposition 4.1 *If for some $t \geq 0$, every planar graph is acyclically $(l,t)^*$-choosable, then every planar graph is acyclically l-choosable.*

As a consequence, proving that for some $t \geq 0$, every planar graph is acyclically $(5,t)^*$-choosable is equivalent to proving Borodin *et al.*'s conjecture.

Since there exist planar graphs which are not acyclically 4-choosable [13], Proposition 4.1 also implies that planar graphs are not acyclically $(4,t)^*$-choosable for all $t \geq 0$ (which also follows from the results of Boiron *et al.* [2]).

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