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Spanning galaxies in digraphs [1]

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Abstract

A *star* is an arborescence in which the root dominates all the other vertices. A *galaxy* is a vertex-disjoint union of stars. The *directed star arboricity* of a digraph D, denoted by $dst(D)$, is the minimum number of galaxies needed to cover $A(D)$. In this paper, we show that $dst(D) \leq \Delta(D) + 1$ and that if D is acyclic then $dst(D) \leq \Delta(D)$. These results are proved by considering the existence of spanning galaxies in digraphs. Thus, we study the problem of deciding whether a digraph D has a spanning galaxy or not. We show that it is NP-complete (even when restricted to acyclic digraphs) but that it becomes polynomial-time solvable when restricted to strongly connected digraphs.

Keywords: directed graph, spanning star forest, even subgraph, directed star arboricity.

1   Introduction

All digraphs considered here are finite and loopless. An *arborescence* is a connected digraph in which every vertex has indegree 1 except the *root*, which has indegree 0. A *diforest* is a disjoint union of arborescences. A *star* is an arborescence with at least one arc, in which the root dominates all the other vertices. A *galaxy* is a diforest of stars.

Algor and Alon [1] introduced the notion of *directed star arboricity* of a digraph D, defined as the minimum number of galaxies needed to cover $A(D)$, and denoted by $dst(D)$. A *k-directed-star-colouring* of D is then an arc-colouring of D such that each colour class induces a galaxy. This notion is related to WDM (Wavelength Division Multiplexing) in star optical networks (See [2]). Guiduli [3] showed that if every vertex has indegree and outdegree both less than k then the directed star arboricity is at most $k + 20 \log k + 84$. Since the multigraph underlying a digraph has maximum multiplicity at most two, Vizing's theorem implies that for any digraph D, $dst(D) \leq \Delta + 2$. Amini et al. [2] conjectured the following.

Conjecture 1.1 *[2] Every digraph D with maximum degree $\Delta \geq 3$ satisfies $dst(D) \leq \Delta$.*

The condition $\Delta \geq 3$ is necessary since odd circuits have maximum degree 2 and directed star arboricity 3. This conjecture would be tight since every digraph with a vertex with indegree Δ has directed star arboricity at least Δ. In [2], Amini et al. proved that Conjecture 1.1 holds when $\Delta = 3$.

A galaxy S in a digraph D is *spanning* if $V(S) = V(D)$. The following would be sufficient to prove Conjecture 1.1 for all $\Delta \geq 3$.

Conjecture 1.2 *Every digraph with maximum degree $\Delta \geq 4$ has a galaxy spanning all the vertices of maximum degree.*

Using the notion of spanning galaxy, we prove Conjecture 1.2 for acyclic digraphs in Section 2, which implies Conjecture 1.1 for acyclic digraphs. We also prove that every digraph has a galaxy spanning the vertices with indegree at least two and derive that $dst(D) \leq \Delta(D) + 1$ for every digraph D. This is detailed in Section 2.

We then focus on the SPANNING GALAXY PROBLEM of deciding whether a given digraph has a spanning galaxy. We show that it is linear-time solvable for arborescences. Moreover, we prove that every arborescence T contains a galaxy spanning every vertex except possibly the root. Then we establish that the SPANNING GALAXY PROBLEM is NP-complete, even if restricted to digraphs which are acyclic, planar, bipartite, subcubic, with arbitrary girth, and with maximum outde-

gree 2. The proof is a reduction from PLANAR $(3,^{\leq}4)$-SAT which has been shown to be NP-complete [4].

Finally, we prove that in a strongly connected digraph *D*, the SPANNING GALAXY PROBLEM is equivalent to the one of deciding whether *D* has a strongly connected subgraph with an even number of vertices. It has been proved that deciding whether a strongly connected digraph contains an even circuit is polynomial-time solvable [5,6], while it is NP-complete to decide whether a strongly connected digraph has an even circuit going through a given arc [7]. We prove that the SPANNING GALAXY PROBLEM restricted to strongly connected digraphs has the same characteristics: deciding whether a given strongly connected digraph has a spanning galaxy is polynomial-time solvable while it is NP-complete to decide whether a strong digraph has a spanning galaxy using a prescribed arc. The first of these two results is detailed in Section 3.

2 Directed Star Arboricity

We settle Conjecture 1.2 for acyclic digraphs and derive that Conjecture 1.1 holds for acyclic digraphs.

Theorem 2.1 *Every acyclic digraph has a galaxy spanning all the vertices of maximum degree.*

Proof (Sketch) Let *D* be an acyclic digraph. We show that *D* has a subdigraph D' spanning the vertices of maximum degree which is the orientation of a union of odd cycles and a matching. Since *D* is acyclic the oriented odd cycles are not circuits, hence they have a spanning galaxy. Therefore D' has a spanning galaxy, which spans the vertices of maximum degree of *D*. □

One can clearly deduce the following corollary from Theorem 2.1 and the result of Amini et al. [2] which states that $dst(D) \leq \Delta$ for all digraphs *D* with $\Delta \leq 3$.

Corollary 2.2 *If D is an acyclic digraph then $dst(D) \leq \Delta$.*

We also get an upper bound for the directed star arboricity of all digraphs. To do so, we need the following:

Theorem 2.3 *Every digraph D has a galaxy spanning all the vertices with indegree at least* 2.

We prove the result by induction on the number of vertices. A key-remark is that we can restrict ourselves to digraphs with vertices of indegree 2 or 0. Observe that Theorem 2.3 implies Amini et al. result [2] showing that every 2-diregular digraph has a spanning galaxy.

Theorem 2.4 *Let D be a digraph with maximum degree $\Delta \geq 2$. Then $dst(D) \leq \Delta + 1$.*

Proof Set $D_0 = D$ and for every i from 1 to $\Delta - 2$, let F_i be a galaxy spanning all the vertices of indegree at least 2 in D_{i-1} and $D_i = D_{i-1} \setminus E(F_i)$. One easily sees that a vertex of $D' = D_{\Delta-2}$ has either indegree at most one or indegree 2 and outdegree 0. Hence D' can be obtained from a functional digraph (i.e. digraph where every vertex has outdegree exactly 1) by deleting the loops, by reversing the arcs and by identifying disjoint pairs of outleaves (vertices with outdegree 0). Then $dst(D') \leq 3$ as one can colour the arcs of the circuits in D' with at most three colours and then extend it greedily to all the arcs. □

3   Spanning galaxies in strongly connected digraphs

Given a strongly connected digraph D, a *handle* h of D is a directed path $(s, v_1, \ldots, v_\ell, t)$ from s to t (where s and t may be identical, and the handle possibly restricted to the arc st) such that:

- the vertices v_i satisfy $d^-(v_i) = d^+(v_i) = 1$, for every $1 \leq i \leq \ell$, and
- the digraph $D \setminus h$ obtained by *suppressing* the arcs and internal vertices of h is strongly connected.

A *handle decomposition* of D starting at $v \in V(D)$ is a triplet $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$, where $(D_i)_{0 \leq i \leq p}$ is a sequence of strongly connected digraphs and $(h_i)_{1 \leq i \leq p}$ is a sequence of handles such that:

- $V(D_0) = \{v\}$,
- h_i is a handle of D_i, for $1 \leq i \leq p$ and D_i is the edge disjoint union of D_{i-1} and h_i,
- $D = D_p$.

    A handle decomposition is uniquely determined by v and either $(h_i)_{1 \leq i \leq p}$, or $(D_i)_{0 \leq i \leq p}$. The number of handles p in any handle decomposition of D is exactly $|A(D)| - |V(D)| + 1$. The value p is also called the *cyclomatic number* of D. Observe that $p = 0$ when D is a singleton and $p = 1$ when D is a circuit. A digraph D with cyclomatic number two is called a *theta*.

    A handle is *even* if its length is even. A handle decomposition is *even* if one of its handles is even. A strongly connected digraph is *even* if it has an even number of vertices. Handles, handle decompositions and strongly connected digraphs are *odd* when they are not even.

    One of our main theorem is the following one.

Theorem 3.1 *Given a strongly connected digraph D, the following are equivalent:*

  (i) *D has a spanning galaxy.*

 (ii) *D has an even handle decomposition.*

(iii) *D contains an even circuit or an even theta.*

(iv) *D contains an even strongly connected subgraph.*

Thi result allows us to prove the following one.

Theorem 3.2 *The* SPANNING GALAXY PROBLEM *is polynomial-time solvable when restricted to strongly connected digraphs.*

Proof We provide a polynomial-time algorithm to decide whether a strongly connected digraph contains an even strongly connected subdigraph (ESS for short). This algorithm performs as follows.

We compute in polynomial time a handle decomposition $(v, (h_i)_{1 \leq i \leq p}, (D_i)_{0 \leq i \leq p})$ where $h_q = (x_0, x_1, \ldots, x_\ell)$ is the last non-trivial handle of a digraph D and we return "YES" if it has an even handle, accoridng to Theorem 3.1.

If not, we consider the handle h_q which is assumed to be odd. Let D' is the digraph obtained from D_{q-1} by adding all the arcs between $N_D^-(x_1)$ and $N_D^+(x_{\ell-1})$. Let $S = \{x_1, x_2, \ldots, x_{\ell-1}\}$ be the set of inner vertices of h_q. Let $D'' = D[S]$.

The key-fact is that D has an ESS if and only if D' or D'' has an ESS. □

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