The chromatic number and switching chromatic number of 2-edge-colored graphs of bounded degree

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Extended Abstract

The notion of homomorphisms of 2-edge-colored graphs has already been studied as a way of extending classical results in graph coloring such as Hadwiger's conjecture. Guenin [5] introduced the notion of switching homomorphisms for its relation with a well-known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [6] as it captures a number of well-known conjectures that can be reformulated using the definition of switching homomorphisms. In this extended abstract, we study homomorphisms of 2-edge colored graphs and switching homomorphisms of bounded degree graphs.

A *2-edge-colored graph* $G = (V, E, s)$ is a simple graph (V, E) with two kinds of edges: positive and negative edges. The signature $s : E(G) \to \{-1, +1\}$ assigns to each edge its sign. In the sequel, $\mathcal{D}_k$ (resp. $\mathcal{D}_k^c$) denotes the class of 2-edge-colored graphs (resp. connected 2-edge-colored graphs) with maximum degree k.

Given two 2-edge-colored graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *homomorphism* if φ maps every edge of G to an edge of H with the same sign. This can be seen as coloring the graph G by using the vertices of H as colors. The target graph H gives us the rules that this coloring must obey. If vertices 1 and 2 in H are connected with a positive (resp. negative) edge, then every pair of adjacent vertices in G colored with 1 and 2 must be connected with a positive (resp. negative) edge. The *chromatic number* $\chi_2(G)$ of a 2-edge-colored graph G is the order of a smallest 2-edge-colored graph H such that G admits a homomorphism to H. The chromatic number $\chi_2(\mathcal{C})$ of a class of 2-edge-colored graphs $\mathcal{C}$ is the maximum of the chromatic numbers of the graphs in the class. This number can be infinite.

2-edge-colored graphs are, in some sense, similar to oriented graphs since a pair of vertices can be adjacent in two different ways in both kinds of graphs: with a positive or a negative edge in the case of 2-edge-colored graphs, with a toward or a backward arc in the oriented case.

The notion of homomorphism of oriented graphs has been introduced by Courcell [3] in 1994 and has been widely studied since then. Due to the similarity above-mentioned, we try to adapt techniques used to study the homomorphisms of oriented graphs of bounded degree to 2-edge-colored graphs of bounded degree. We also study switching homomorphisms of 2-edge-colored graphs in order to obtain results on signed graphs.

Switching a vertex v of a 2-edge-colored graph corresponds to reversing the signs of all edges incident to v.

Two 2-edge-colored graphs G and G' are *switching equivalent* if it is possible to turn G into G' after some number of switches. We call the classes created by this equivalence relation *switching classes* (note that switching classes are equivalent to the notion of signed graphs).

Given two 2-edge-colored graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *switching homomorphism* if there is a graph G' switching equivalent to G such that φ maps every edge of G' to an edge of H with the same sign. The *switching chromatic number* $\chi_s(G)$ of a 2-edge-colored graph G is the order of a smallest 2-edge-colored graph H such that G admits a switching homomorphism to H.

Table 1 summarizes results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

The first two lines of Table 1 are more or less folklore. Let us explain in the following the difference that exists between the connected case and the non-connected case for the

	$\chi_2(\mathcal{D}_k)$	$\chi_2(\mathcal{D}_k^c)$	$\chi_s(\mathcal{D}_k)$	$\chi_s(\mathcal{D}_k^c)$
$k = 1$	$\chi_2(\mathcal{D}_k) = 3$	$\chi_2(\mathcal{D}_k^c) = 2$	$\chi_s(\mathcal{D}_k) = \chi_s(\mathcal{D}_k^c) = 2$	
$k = 2$	$\chi_2(\mathcal{D}_k) = 6$	$\chi_2(\mathcal{D}_k^c) = 5$	$\chi_s(\mathcal{D}_k) = \chi_s(\mathcal{D}_k^c) = 4$	
$k = 3$	$8 \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 11$		$6 \leq \chi_s(\mathcal{D}_k) \leq 7$ [1]	$\chi_s(\mathcal{D}_k^c) = 6$ [1]
$k = 4$	$12 \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 31$		$10 \leq \chi_s(\mathcal{D}_k^c) \leq \chi_s(\mathcal{D}_k) \leq 16$	
$k \geq 5$	$2^{\frac{k}{2}} \leq \chi_2(\mathcal{D}_k^c) \leq \chi_2(\mathcal{D}_k) \leq 2^{k+1}(k-1)^2$ [4]		$\chi_s(\mathcal{D}_k^c) \leq \chi_s(\mathcal{D}_k) \leq 2^{k+1}(k-1)^2$ [4]	

Table 1: Results on the chromatic number and switching chromatic number of the classes of (connected) 2-edge-colored graphs of bounded degree.

chromatic number of 2-edge-colored graphs with maximum degree 1 or 2. An edge of a 2-edge-colored graph has chromatic number 2 and thus $\chi_2(\mathcal{D}_1^c) = 2$; however, a 2-edge-colored graph with two non-adjacent edges, one positive and one negative, has chromatic number 3 (the target graph needs a positive and a negative edge, hence at least three vertices) and thus $\chi_2(\mathcal{D}_1) = 3$. We therefore have a difference between the chromatic numbers of connected and non-connected 2-edge-colored graphs with maximum degree 1. This difference does not exist for switching homomorphisms since a negative edge can be changed into a positive one after a switch. This difference (and lack thereof for switching homomorphisms) appears also in graphs with maximum degree 2. We have $\chi_2(\mathcal{D}_2) \geq 6$ since there is no 2-edge-colored graph on 5 vertices that can color all the four graphs depicted in Figure 1. However, every connected 2-edge-colored graph with maximum degree 2 admits a homomorphism to one of the two graphs on 5 vertices depicted in Figure 2 and thus $\chi_2(\mathcal{D}_2^c) \leq 5$. In order to color any graph of $\mathcal{D}_2$, we need a target graph that contains both graphs depicted in Figure 2 as subgraphs. This is possible with 6 vertices so $\chi_2(\mathcal{D}_2) = 6$. We do not know yet if this is also the case for graphs with maximum degree at least 3.

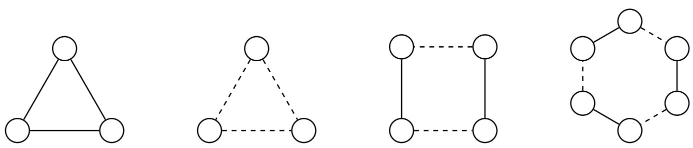

Figure 1: Four examples of 2-edge-colored graphs with chromatic number 5.

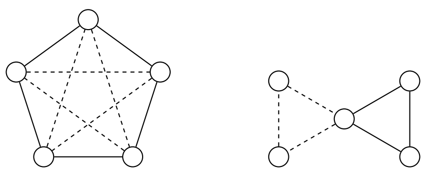

Figure 2: Target graphs for connected 2-edge-colored graphs of maximum degree 2.

The last three lines of Table 1 are dedicated to graphs with maximum degree at least 3. Our main results are the following:

Theorem 1 *We have:*

- $8 \leq \chi_2(\mathcal{D}_3) \leq 11$,
- $12 \leq \chi_2(\mathcal{D}_4) \leq 31$,
- $10 \leq \chi_s(\mathcal{D}_4) \leq 16$.

In order to find an upper bound for a class of graphs, we need to find a target graph that can color every graph in the class. In the case of oriented homomorphisms, oriented graphs that are antiautomorphic, K_n-transitive for some n, or that have Property $P_{n,k}$ for some n and k are good candidates. We analogously define these properties in term of 2-edge-colored graphs.

A 2-edge-colored graph (V, E, s) is said to be *antiautomorphic* if it is isomorphic to $(V, E, -s)$.

A 2-edge-colored graph $G = (V, E, s)$ is said to be *K_n-transitive* if for every pair of cliques $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ in G such that for all $i \neq j$, $s(u_iu_j) = s(v_iv_j)$ there exists an automorphism that maps u_i to v_i for all i. For $n = 1$, 2, or 3, we say that the graph is *vertex*, *edge*, or *triangle-transitive*, respectively.

A 2-edge-colored graph G has *Property $P_{k,n}$* if for every sequence of k distinct vertices $(v_1, v_2, \ldots, v_k)$ that induces a clique in G and for every sign vector $(\alpha_1, \alpha_2, ..., \alpha_k) \in \{-1, +1\}^k$ there exist at least n distinct vertices $\{u_1, u_2, ..., u_n\}$ such that $s(v_iu_j) = \alpha_i$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Given an integer $q \equiv 1 \pmod 4$, we consider the family of complete signed Paley graphs SP_q built from the field of order q which has the above-mentioned properties. The vertices of SP_q are the elements of the field of order q and $s(uv) = +1$ if $u - v$ is a square and $s(uv) = -1$ otherwise.

Lemma 2 ([7]) *Graph SP_q is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1,\frac{q-1}{2}}$ and $P_{2,\frac{q-5}{4}}$.*

Let us consider the following operation. Given a 2-edge-colored graph G, we create the *antitwinned graph* of G denoted by $\rho(G)$ as follows. Let G^{+1}, G^{-1} be two copies of G. The vertex corresponding to $v \in V(G)$ in G^i is denoted by v_i, $V(\rho(G)) = V(G^+) \cup V(G^-)$, $E(\rho(G)) = \{u_iv_j : uv \in E(G),\ i, j \in \{-1, +1\}\}$ and $s_{\rho(G)}(u_iv_j) = i \times j \times s_G(u, v)$.

Lemma 3 ([2]) *Let G and H be two 2-edge-colored graphs. The graph G admits a homomorphism to $\rho(H)$ if and only if it admits a switching homomorphism to H.*

In other words, if a 2-edge-colored graph admits a homomorphism to an antitwinned target graph on n vertices, then it also admits a switching homomorphism to a target graph on $\frac{n}{2}$ vertices. The family $\rho(SP_q)$ also are interesting target graphs (especially for bounding the switching chromatic number since they are antitwinned).

Lemma 4 ([7]) *The graph $\rho(SP_q)$ is vertex-transitive, antiautomorphic, and has properties $P_{1,q-1}$, $P_{2,\frac{q-3}{2}}$, and $P_{3,max(\frac{q-9}{4},0)}$.*

One last family of interesting target graphs are the Tromp-Paley graphs (this construction due to Tromp (unpublished) has been widely used in the case of oriented homomorphisms). Let SP_q^+ be SP_q with an additional vertex that is connected to every other vertex with a positive edge. The Tromp-Paley graph $TR(SP_q)$ corresponds to $\rho(SP_q^+)$. This construction improves the properties of $\rho(SP_q)$ at the cost of having two more vertices. Since Tromp-Paley graphs are antitwinned, they are interesting for bounding the switching chromatic number.

Lemma 5 ([7]) *$TR(SP_q)$ is vertex-transitive, edge-transitive, antiautomorphic, and has properties $P_{1,q}$, $P_{2,\frac{q-1}{2}}$, and $P_{3,\frac{q-5}{4}}$.*

Bensmail et al. [1] recently proved that every 2-edge-colored graph with maximum degree 3 except the all positive and all negative K_4 admits a homomorphism to $TR(SP_5)$, hence $\chi_2(\mathcal{D}_3^c) \leq 12$, and $\chi_s(\mathcal{D}_3^c) \leq 6$ by Lemma 3. In the non-connected case, we can easily get $\chi_2(\mathcal{D}_3) \leq 14$ and thus $\chi_s(\mathcal{D}_3) \leq 7$ by Lemma 3 (it is possible to create an all positive K_4 and an all negative K_4 in $TR(SP_5)$ by adding two vertices). Their proof uses a computer to show that a minimal counter-example cannot contain some configurations and then concludes by using the properties of $TR(SP_5)$. Theorem 1 improves the upper bound of 14 to 11.

Let us give a sketch of proof of the first result of Theorem 1, namely $\chi_2(\mathcal{D}_3) \leq 11$.

Consider the graph SP_9^* obtained from SP_9 by adding two new vertices $0'$ and $1'$ as follows. Take the two vertices 0 and 1 of SP_9 (note that $s(01) = +1$), and link $0'$ and $1'$ to the vertices of SP_9 in the same way as 0 and 1 are, respectively; add an edge $0'1'$ with $s(0'1') = -1$; finally we add edges $00'$ and $11'$ with $s(00') = -1$ and $s(11') = +1$. We will prove that every graph from $\mathcal{D}_3$ admits a homomorphism to SP_9^*.

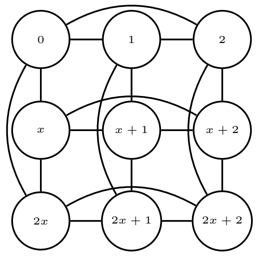

Figure 3: The graph SP_9, non-edges are negative edges

We first show that every connected 2-degenerate 2-edge-colored graph with maximum degree 3 admits a homomorphism to SP_9 by using its structural properties given by Lemma 2 (a unique exception exists and is separately treated).

Let G be a connected 3-regular 2-edge-colored graph. If G is all positive, then we color it using an all positive K_4 that SP_9^* contains as a subgraph. Assume now that G is not all positive. Let uv be a negative edge of G. We remove uv from G to create a new graph G'. Graph G' is 2-degenerate so it admits a homomorphism φ' to SP_9. If $s(\varphi'(u)\varphi'(v)) = -1$, then φ' is also a homomorphism from G to SP_9.

If $s(\varphi'(u)\varphi'(v)) = +1$, then by edge-transitivity of SP_9 we can recolor the vertices of G' such that $\varphi'(u) = 0$ and $\varphi'(v) = 1$. We can then extend φ' to a homomorphism φ of G to SP_9^* by recoloring u and v such that $\varphi(u) = 0'$ and $\varphi(v) = 1'$ since $s(0'1') = -1$.

Finally, if $\varphi'(u) = \varphi'(v)$, then by vertex-transitivity of SP_9 we can recolor the vertices of G' such that $\varphi'(u) = \varphi'(v) = 0$. We can then extend φ' to a homomorphism φ of G to SP_9^* by recoloring v such that $\varphi(v) = 0'$ since $s(00') = -1$.

We have proven that every graph in $\mathcal{D}_3^c$ admits a homomorphism to SP_9^* which means that SP_9^* is universal for $\mathcal{D}_3$. This concludes the proof.

To prove the two other upper bounds of Theorem 1, we use the same method on target graphs SP_{29} and $TR(SP_{13})$.

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