The Chromatic Number of Signed Graphs with Bounded Maximum Average Degree

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Abstract. A signed graph is a simple graph with two types of edges: positive and negative. A homomorphism from a signed graph G to another signed graph H is a mapping $\varphi : V(G) \to V(H)$ that preserves vertex adjacencies and balance of closed walks (the balance is the parity of the number of negative edges). The chromatic number $\chi_s(G)$ of a signed graph G is the order of a smallest signed graph H such that there is a homomorphism from G to H.

The maximum average degree $\mathrm{mad}(G)$ of a graph G is the maximum of the average degrees of all the subgraphs of G.

The girth $g(G)$ of a graph G is the length of a shortest cycle of G.

In this paper, we consider signed graphs with bounded maximum average degree and we prove that:

- If $\mathrm{mad}(G) < \frac{20}{7}$ and $g(G) \leq 7$ then $\chi_s(G) \leq 5$.
- If $\mathrm{mad}(G) < \frac{17}{5}$ then $\chi_s(G) \leq 10$.
- If $\mathrm{mad}(G) < 4 - \frac{8}{q+3}$ then $\chi_s(G) \leq q + 1$ where q is a prime power congruent to 1 modulo 4.

The first result implies that the chromatic number of planar signed graphs of girth at least 7 is at most 5.

Keywords: Signed graph · Chromatic number · Homomorphism · Maximum average degree · Planar graph

1 Introduction

There exist several notions of colorings of signed graphs which are all natural extensions and generalizations of colorings of simple graphs. It is well-known that a (classical) k-coloring of a graph is no more than a homomorphism to the complete graph on k vertices. Using the notion of homomorphism of signed graphs introduced by Guenin [9] in 2005, a corresponding notion of coloring of signed graphs can be defined. This has attracted a lot of attention since then and the general question of knowing whether every signed graph of a given family admits a homomorphism to some H has been extensively studied. We can for example cite the papers by Naserasr et al. [11,12] in which they develop many aspects of this notion.

Coloring planar graphs has become a famous problem in the middle of the 19^{th} century thanks to the Four Color Theorem, that states that four colors are

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enough to color any simple planar graph. Various branches of this topic then arose, one of which being devoted to the coloring of *sparse* planar graphs. A way to measure the sparseness of a planar graph is to consider its girth (i.e. the length of a shortest cycle): the higher the girth is, the sparser the graph is. Colorings of signed sparse planar graphs have already been considered in the last decade (see e.g. [1,4,10,11,13,14]).

A way to get results on sparse planar graphs is to consider graphs (not necessarily planar) with bounded maximum average degree thanks to the well-known relation that links the maximum average degree and the girth of a planar graph: Every planar graph of girth at least g has maximum average degree less than $\frac{2g}{g-2}$.

In this paper, we consider homomorphisms of signed graphs with bounded maximum average degree.

Signed Graphs. A *signed graph* $G = (V, E, s)$ is a simple graph (V, E) (we do not allow parallel edges nor loops) with two kinds of edges: positive and negative edges. The signature $s : E(G) \to \{-1, +1\}$ assigns to each edge its sign. *Switching* a vertex v of a signed graph corresponds to reversing the signs of all the edges that are incident to v. Two signed graphs G and G' are *switching equivalent* if it is possible to turn G into G' after some number of switches. The *balance* of a closed walk of a signed graph is the parity of its number of negative edges; a closed walk is said to be *balanced* (resp. *unbalanced*) if it has an even (resp. odd) number of negative edges. We can note that a switch does not alter the balance of any closed walk since a switch reverses the sign of an even number of edges of a closed walk. Therefore, Zaslavsky [16] showed the following:

Theorem 1 (Zaslavsky [16]). *Two signed graphs are switching equivalent if and only if they have the same underlying graph and the same set of balanced cycles.*

Homomorphisms of Signed Graphs. Given two signed graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *homomorphism* if φ preserves vertex adjacencies (i.e. $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$) and the balance of closed walks (i.e. the closed walk $\varphi(v_1)\varphi(v_2)\ldots\varphi(v_k)$ in H has the same balance as the closed walk $v_1v_2\ldots v_k$ in G). In that case we write $G \to H$. This type of homomorphism was introduced by Guenin [9] in 2005 and arises naturally from the fact that the balance of closed walks is central in the field of signed graphs.

There exists an alternate way to define homomorphisms of signed graphs using the notion of *sign-preserving homomorphims*. Given two signed graphs G and H, the mapping $\varphi : V(G) \to V(H)$ is a *sign-preserving homomorphism* (sp-homomorphism for short) if φ preserves vertex adjacencies and the signs of edges. In that case we write $G \xrightarrow{sp} H$. Naserasr et al. [12] showed that, given two signed graphs G and H, we have $G \to H$ if and only if there exists a signed graph G' switching equivalent to G such that $G' \xrightarrow{sp} H$.

The *chromatic number* $\chi_s(G)$ (resp. *sign-preserving chromatic number* $\chi_{sp}(G)$) of a signed graph G is the order of a smallest graph H such that $G \to H$

(resp. $G \xrightarrow{sp} H$). The (sign-preserving) chromatic number $\chi_{s/sp}(\mathcal{C})$ of a class of signed graphs $\mathcal{C}$ is the maximum of the (sign-preserving) chromatic numbers of the graphs in the class. Clearly, an sp-homomorphism is a homomorphism and thus $\chi_s(G) \leq \chi_{sp}(G)$ for any signed graph G.

If G admits a (sp-)homomorphism φ to H, we say that G is *H(-sp)-colorable* and that φ is an *H(-sp)-coloring* of G.

Target Graphs. Let q be a prime power with $q \equiv 1 \pmod 4$. Let $\mathbb{F}_q$ be the finite field of order q. The *signed Paley graph* SP_q has vertex set $V(SP_q) = \mathbb{F}_q$. Two vertices u and $v \in V(SP_q)$, $u \neq v$, are connected with a positive edge if $u - v$ is a square in $\mathbb{F}_q$ and with a negative edge otherwise. See Fig. 1 for a picture of the signed Paley graph on five vertices.

Notice that this definition is consistent since $q \equiv 1 \pmod 4$ ensures that -1 is always a square in $\mathbb{F}_q$ and if $u - v$ is a square then $v - u$ is also a square.

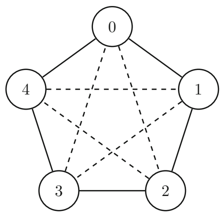

Fig. 1. SP_5, the signed Paley graph on 5 vertices.

Given a signed graph SP_q, we denote by SP_q^- the graph obtained from SP_q by removing any vertex (SP_q is vertex-transitive) and by SP_q^+ the graph obtained from SP_q by adding a vertex that is connected with a positive edge to every other vertex.

Such graphs SP_q, SP_q^+ and SP_q^- have remarkable structural properties but due to lack of space, we will not list them (see [14] for more details). We use these target graphs to obtain our results.

2 State of the Art and Results

Let us denote by $\mathcal{P}_g$ the class of planar signed graphs of girth at least g and by $\mathcal{M}_d$ the class of signed graphs with maximum average degree less than d.

Table 1. Known results on the chromatic number of signed planar graphs with given girth and signed graphs with bounded maximum average degree.

Graph families	χ_s	Remarks	Refs
$\mathcal{P}_3$	$10 \leq \chi_s \leq 40$		[10,13]
$\mathcal{P}_4$	$6 \leq \chi_s \leq 25$		[14]
$\mathcal{M}_{\frac{10}{3}}$	$\chi_s \leq 10$	$\mathcal{P}_5 \subset \mathcal{M}_{\frac{10}{3}}$	[10]
$\mathcal{M}_3$	$\chi_s \leq 6$	$\mathcal{P}_6 \subset \mathcal{M}_3$	[10]
$\mathcal{M}_{\frac{18}{7}}$	$\chi_s = 4$	$\mathcal{P}_9 \subset \mathcal{M}_{\frac{18}{7}}$	[6]

Note first that for planar graphs, the gap between the lower and upper bounds is huge ($10 \leq \chi_s(\mathcal{P}_3) \leq 40$) and in 2020, Bensmail et al. [2] conjectured that

$\chi_s(\mathcal{P}_3) = 10$. Recently, Bensmail et al. [1] proved that if this conjecture is true, then the target graph is necessarily SP_9^+. This question remains widely open.

Finally, note that for maximum average degree less than $\frac{10}{3}$, 3, and $\frac{18}{7}$ (lines 4–6 of Table 1), this gives bounds for planar graphs of girth at least 5, 6, and 9. Note that since unbalanced even cycles have chromatic number 4 (see [8]), the bound for maximum average degree $\frac{18}{7}$ is tight.

In this paper, we prove the following theorem, improving several above-mentioned results:

Theorem 2. *Let G be a signed graph.*

(1) If $G \in \mathcal{M}_{4-\frac{8}{q+3}}$, then $G \to SP_q^+$. Thus $\chi_s(G) \leq q+1$, with $q \equiv 1 \pmod 4$ and q is a prime power.
(2) If $G \in \mathcal{M}_{\frac{17}{5}}$, then $G \to SP_9^+$. Thus $\chi_s(G) \leq 10$.
(3) If $G \in \mathcal{M}_{\frac{20}{7}}$ and $g(G) \geq 7$, then $G \to SP_5$. Thus $\chi_s(G) \leq 5$.

It is not hard to see that signed cliques in which each edge is subdivided once have a maximum average degree that tends to 4 as the number of vertices grows. Such signed graphs have unbounded chromatic number and Theorem 2(1) gives an upper bound on the chromatic number of signed graphs of maximum average degree $4-\varepsilon$ in function of ε. Theorem 2(2) improves the previous known result of Montejano et al. [10] saying that $\chi_s(\mathcal{M}_{\frac{10}{3}}) \leq 10$ by reaching the same upper bound for a superclass of graphs ($\mathcal{M}_{\frac{10}{3}} \subset \mathcal{M}_{\frac{17}{5}}$). Theorem 2(3) gives, as a corollary, that $\chi_s(\mathcal{P}_7) \leq 5$ since $\mathcal{P}_7 \subset \mathcal{M}_{\frac{20}{7}}$, which are new results that contribute to the above-mentioned collection of known results.

3 Proof Techniques

To prove our results, let us first introduce what we call antitwinned graphs. Given a signed graph G of signature s_G, we can create the signed graph $\rho(G)$ as follows: We take two copies G^{+1}, G^{-1} of G, hence $V(\rho(G)) = V(G^{+1}) \cup V(G^{-1})$; the edge set is defined as $E(\rho(G)) = \{u^i v^j : uv \in E(G),\ i,j \in \{-1,+1\}\}$ and the signature as $s_{\rho(G)}(u^i v^j) = i \times j \times s_G(u,v)$. A signed graph G is said to be antitwinned if there exists a signed graph H such that $G = \rho(H)$.

Antitwinned signed graphs play a central role for our proofs thanks to the following lemma:

Lemma 1 ([5]). *Given two signed graphs G and H, G admits an sp-homomorphism to $\rho(H)$ if and only if G admits a homomorphism to H.*

Therefore, Theorem 2 will be proved by showing that:

(1) If $G \in \mathcal{M}_{4-\frac{8}{q+3}}$, then $G \xrightarrow{sp} \rho(SP_q^+)$.
(2) If $G \in \mathcal{M}_{\frac{17}{5}}$, then $G \xrightarrow{sp} \rho(SP_9^+)$.
(3) If $G \in \mathcal{M}_{\frac{20}{7}}$, then $G \xrightarrow{sp} \rho(SP_5)$.

We prove these results by contradiction, by assuming that they have counterexamples. Among all of these counterexamples, we take a graph G with the fewest number of vertices. Our goal is to prove that G satisfies structural properties incompatible with having a maximum average degree smaller than a certain value, hence the conclusion.

For each theorem, we start by introducing sets of so-called forbidden configurations, which by minimality G cannot contain. We then strive to reach a contradiction with the bounded maximum average degree. To do so, we use the discharging method. This means that we give some initial weight to vertices of G, we then redistribute those weights and obtain a contradiction by double counting the total weight. We present appropriate collections of discharging rules, and argue that every vertex of G ends up with non-negative weight while the total initial weight was negative.

The discharging method was introduced more than a century ago to study the Four-Color Conjecture [15], now a theorem. It is especially well-suited for studying sparse graphs, and leads to many results, as shown in two recent surveys [3,7].

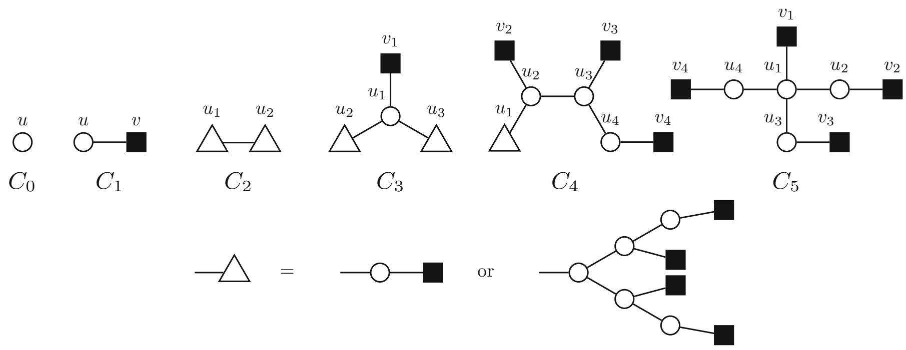

Fig. 2. Forbidden configurations. Every edge incident to round vertices is represented. Square vertices can be of any degree. Triangle vertices are replaced by one of the two represented structures.

Due to lack of space, let us just give the sketch of the proof of Theorem 2(3). To prove this theorem, we prove that every signed graph of maximum average degree less than $\frac{20}{7}$ and girth at least 7 admits a sp-homomorphism to $\rho(SP_5)$ which implies the theorem by Lemma 1.

Let G be a smallest signed graph with $\text{mad}(G) < \frac{20}{7}$ and girth at least 7 admitting no sp-homomorphism to $\rho(SP_5)$. We start by proving that the configurations depicted in Fig. 2 cannot appear in G.

We then define the weighting $\omega(v) = d(v) - \frac{20}{7}$ for each vertex v of degree $d(v)$. By construction, the sum of all the weights $\sum_{v \in V(G)} \omega(v)$ is negative since the maximum average degree of G (and therefore its average degree) is strictly smaller than $\frac{20}{7}$. We say that a *k-vertex* (resp. k^+*-vertex*) is a vertex of degree k (resp. at least k). A 3-vertex is said to be 3-*worse* if its adjacent to a 2-vertex, 3-*bad* if it is adjacent to two 3-worse vertices or 3-*good* otherwise. We then introduce the following discharging rules:

(R_1) Every 3^+-vertex gives $\frac{3}{7}$ to each of its 2-neighbors.
(R_2) Every 3-good, 3-bad or 4^+-vertex gives $\frac{1}{7}$ to each of its 3-worse-neighbors.
(R_3) Every 3-good or 4^+-vertex gives $\frac{1}{7}$ to each of its 3-bad-neighbors.

Finally, we show that every vertex has a positive final weight by using the fact that the configurations of Fig. 2 cannot appear in G, a contradiction.

References

1. Bensmail, J., Das, S., Nandi, S., Pierron, T., Sen, S., Sopena, É.: On the signed chromatic number of some classes of graphs (2020). http://arxiv.org/abs/2009.12059
2. Bensmail, J., Nandi, S., Roy, M., Sen, S.: Classification of edge-critical underlying absolute planar cliques for signed graphs. Australas. J. Comb. **77**(1), 117–135 (2020)
3. Borodin, O.V.: Colorings of plane graphs: a survey. Discrete Math. **313**(4), 517–539 (2013)
4. Borodin, O.V., Kim, S.J., Kostochka, A.V., West, D.B.: Homomorphisms from sparse graphs with large girth. J. Comb. Theory Ser. B **90**(1), 147–159 (2004)
5. Brewster, R.C., Graves, T.: Edge-switching homomorphisms of edge-coloured graphs. Discrete Math. **309**(18), 5540–5546 (2009)
6. Charpentier, C., Naserasr, R., Sopena, É.: Homomorphisms of sparse signed graphs. Electr. J. Comb. **27**(3), 1–20 (2020)
7. Cranston, D.W., West, D.B.: An introduction to the discharging method via graph coloring. Discrete Math. **340**(4), 766–793 (2017)
8. Duffy, C., Jacques, F., Montassier, M., Pinlou, A.: The chromatic number of 2-edge-colored and signed graphs of bounded maximum degree (2021). (To be published)
9. Guenin, B.: Packing odd circuit covers: a conjecture. Manuscript (2005)
10. Montejano, A., Ochem, P., Pinlou, A., Raspaud, A., Sopena, É.: Homomorphisms of 2-edge-colored graphs. Discrete Appl. Math. **158**(12), 1365–1379 (2010)
11. Naserasr, R., Rollová, E., Sopena, É.: Homomorphisms of signed graphs. J. Graph Theory **79**(3), 178–212 (2015)
12. Naserasr, R., Sopena, É., Zaslavsky, T.: Homomorphisms of signed graphs: an update. Euro. J. Comb. **91**, 103–222 (2021)
13. Nešetřil, J., Raspaud, A.: Colored homomorphisms of colored mixed graphs. J. Comb. Theory Ser. B **80**(1), 147–155 (2000)
14. Ochem, P., Pinlou, A., Sen, S.: Homomorphisms of 2-edge-colored triangle-free planar graphs. J. Graph Theory **85**(1), 258–277 (2017)
15. Wernicke, P.: Über den kartographischen vierfarbensatz. Math. Annal. **58**(3), 413–426 (1904)
16. Zaslavsky, T.: Signed graphs. Discrete Appl. Math. **4**(1), 47–74 (1982)