On a conjecture on k-Thue sequences

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Extended Abstract

A *repetition* in a sequence S is a subsequence $\xi_1 \dots \xi_t \xi_{t+1} \dots \xi_{2t}$ of consecutive terms of S such that $\xi_i = \xi_{t+i}$ for every $i = 1, \dots, t$. A sequence is called *nonrepetitive* or *Thue* if it does not contain a repetition of any length.

In 1906 Thue [6] showed that using only three symbols one can construct an arbitrarily long sequence without a repetition. His famous work attracted a considerable attention and later many applications have been found in various fields of science.

Let us consider the following generalization of Thue sequences introduced by Currie and Simpson [1]: a (possibly infinite) sequence S is *k-Thue* (or *nonrepetitive up to mod k*) if every j-subsequence of S is Thue, for $1 \leq j \leq k$. Here, a *j-subsequence* of S is a subsequence $\xi_i \xi_{i+j} \xi_{i+2j} \dots$, for any i. Notice that a 1-Thue sequence is simply a Thue sequence. As an example, consider a sequence $a\,b\,d\,c\,b\,c$, which is Thue, but not 2-Thue, since the 2-subsequence $b\,c\,c$ is not Thue. On the other hand, $a\,b\,c\,a\,d\,b$ is 2-Thue, but not 3-Thue.

Currie and Simpson [1] introduced this notion in connection with *nonrepetitive tilings*, i.e. assignments of symbols to the lattice points of the plane such that all lines in prescribed directions are nonrepetitive.

A natural question arises what is the minimum number of symbols required to construct an arbitrarily long k-Thue sequence. In [1], the authors showed that four symbols are enough to create 2-Thue sequences and five symbols suffice for 3-Thue sequences of arbitrary lengths. The lower bound on the number of symbols needed to construct a k-Thue sequence of an arbitrary length is obvious; for a positive integer k, at least $k + 2$ symbols are required to construct such sequences.

In 2002 Grytczuk conjectured that, in fact, the upper bound is equal to the lower bound for any k.

Conjecture 1 (Grytczuk, 2002 [3]) *For any k, $k + 2$ symbols suffices to construct a k-Thue sequence.*

This conjecture is hence true for $k = 2, 3$, and as shown in [2] also for $k = 5$. However, the upper bound for general k is still far from the conjectured. The bound of $e^{33}k$ established in [3] was substantially improved to $2k + O(\sqrt{k})$ in [4]. The authors in [5] presented a construction of arbitrarily long k-Thue sequences using at most $2k$ symbols which improves the previous bounds and currently it is the best known upper bound.

Theorem 2 (Kranjc et al., 2015 [5]) *For an arbitrary $k \geq 3$, there is an arbitrarily long k-Thue sequence using at most $2k$ symbols.*

Notice that this proof is constructive and provides a k-Thue sequence of a given length.

Recently, we solved Conjecture 1 for the cases $k = 4$ and 6 in two ways.

Theorem 3 (Lužar, Mockovčiaková, Ochem, Pinlou, Soták, 2016) *There is an arbitrarily long k-Thue sequence using $k + 2$ symbols for $k = 4$ and 6.*

In this talk we present constructions of 4- and 6-Thue sequences using 6 and 8 symbols, respectively.

References

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