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Strong oriented chromatic number of planar graphs without cycles of specific lengths

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Abstract

A *s*trong oriented k-coloring of an oriented graph G is a homomorphism φ from G to H having k vertices labelled by the k elements of an abelian additive group M, such that for any pairs of arcs $\overrightarrow{uv}$ and $\overrightarrow{zt}$ of G, we have $\varphi(v)-\varphi(u) \neq -(\varphi(t)-\varphi(z))$. The *s*trong oriented chromatic number $\chi_s(G)$ is the smallest k such that G admits a strong oriented k-coloring. In this paper, we consider the following problem: Let $i \geq 4$ be an integer. Let G be an oriented planar graph without cycles of lengths 4 to i. What is the strong oriented chromatic number of G?

1 Introduction

Oriented graphs are directed graphs without loops nor opposite arcs. Let G be an oriented graph. We denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. An *oriented k-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of k colors such that (1) $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv}$

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is an arc in G, and (2) $\varphi(u) \neq \varphi(x)$ whenever $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in G with $\varphi(v) = \varphi(w)$. The *oriented chromatic number* of an oriented graph, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-coloring.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ such that: $\overrightarrow{xy} \in A(G) \Rightarrow \overrightarrow{\varphi(x)\varphi(y)} \in A(H)$

An oriented k-coloring of G can be equivalently defined as a homomorphism from G to H, where H is an oriented graph of order k. Then, the *oriented chromatic number* $\chi_o(G)$ of G can be defined as the smallest order of an oriented graph H such that G admits a homomorphism to H.

The problem of bounding the oriented chromatic number has already been investigated for various graph classes: graphs with bounded maximum average degree [1], graphs with bounded degree [2], graphs with bounded treewidth [7,8], graphs subdivisions [9].

Raspaud and Nešetřil [5] introduced the *strong oriented chromatic number* $\chi_s(G)$. A *strong oriented k-coloring* of an oriented graph G is a homomorphism φ from G to H with k vertices labelled by the k elements of an abelian additive group M of order k, such that for any pair of arcs $\overrightarrow{uv}$ and $\overrightarrow{zt}$ of $A(G)$, $\varphi(v) - \varphi(u) \neq -(\varphi(t) - \varphi(z))$. The strong oriented chromatic number $\chi_s(G)$ is the smallest k such that G admits a strong oriented k-coloring.

Therefore, any strong oriented coloring of G is an oriented coloring of G; hence, $\chi_o(G) \leq \chi_s(G)$.

Let M be an additive group and let $S \subset M$ be a set of group elements. The *Cayley digraph* associated with (M, S), denoted by $C_{(M,S)}$, is then defined as follows: $V(C_{(M,S)}) = M$ and $A(C_{(M,S)}) = \{(g, g+s)\ ; g \in M, s \in S\}$. If the set S is a group generator of M, then $C_{(M,S)}$ is connected. Assuming that M is abelian and $S \cap -S = \emptyset$, then $C_{(M,S)}$ is oriented (neither loops nor opposite arcs), and for any pair $(g_1, g_1 + s_1)$ and $(g_2, g_2 + s_2)$ of arcs of $C_{(M,S)}$, $g_1 + s_1 - g_1 \neq -(g_2 + s_2 - g_2)$. Thus, finding a strong oriented k-coloring of an oriented graph G may be viewed as finding a homomorphism from G to an oriented Cayley graph $C_{(M,S)}$ of order k, for some abelian group M with $S \subset M$ and $S \cap -S = \emptyset$.

In the following we will consider the Paley tournament QR_p (where $p \equiv 3 \pmod 4$ is a prime power) that is the Cayley graph $C_{(M,S)}$ with $M = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and $S = \{x^2\ ;\ x \in \mathbb{F}_p \setminus \{0\}\}$.

Strong oriented coloring of planar graphs was recently studied. Sámal [6] proved that every oriented planar graph admits a strong oriented coloring with at most 672 colors. Marshall [3] improved this result and proved the following:

Theorem 1.1 *[3] Let G be an oriented planar graph. Then $\chi_s(G) \leq 271$.*

Borodin *et al.* [1] studied the relationship between the oriented chromatic number and the maximum average degree of a graph, where the maximum average degree, denoted by $Mad(G)$ is: $Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}$. Since they considered homomorphisms to oriented Cayley graphs, they proved that if $Mad(G) < 7/3$ (resp. $8/3$, 3, $10/3$) then $\chi_s(G) \leq 5$ (resp. 7, 11, 19). The *girth* of a graph G is the length of a shortest cycle of G. Since every planar graph G with girth g satisfies $Mad(G) < \frac{2g}{g-2}$, it follows that if G is planar with girth at least 14 (resp. 8, 6, 5), then $\chi_s(G) \leq 5$ (resp. 7, 11, 19).

In this paper, we consider the following problem:

Problem 1.2 *Let $i \geq 4$. Let G be a planar graph without cycles of lengths 4 to i. What is the smallest value k such that $\chi_s(G) \leq k$ for each such G?*

We proved [4] that if G is a planar graph without cycles of lengths 4 to i with $i \geq 5$, then $Mad(G) < 3 + \frac{3}{i-2}$ and that, for any $\epsilon > 0$, there exists a planar graph G without cycles of lengths 4 to i with $3 + \frac{3}{i-2} - \epsilon < Mad(G)$. Consequently, we obtain the following corollary by the above result of Borodin *et al.* [1]:

Corollary 1.3 *Let G be a planar graph without cycles of lengths 4 to 14, $\chi_s(G) \leq 19$.*

A first improvement over Corollary 1.3 is given by the authors [4].

Theorem 1.4 *[4] Every oriented planar graph without cycles of lengths 4 to 11 has a homomorphism to the Cayley graph QR_7.*

In this paper, we continue this study and prove that:

Theorem 1.5 (i) *Every oriented planar graph without cycles of length 4 has a homomorphism to the Cayley graph QR_{43}.*

(ii) *Every oriented planar graph without cycles of lengths 4 and 5 has a homomorphism to the Cayley graph QR_{19}.*

(iii) *Every oriented planar graph without cycles of lengths 4 to 9 has a homomorphism to the Cayley graph QR_{11}.*

In the following, we present a sketch of the proof of Theorem 1.5.(i) based on the *method of reducible configurations and discharging procedure*. Theorems 1.5.(ii) and 1.5.(iii) are based on the same method of proof.

A k-vertex (resp. $^{\geq}k$-vertex, $^{\leq}k$-vertex) is a vertex of degree k (resp. $\geq k$, $\leq k$). The size of a face f, denoted by $d(f)$, is the number of edges on its boundary walk, where each cut-edge is counted twice. A k-face (resp. $^{\geq}k$-face, $^{\leq}k$-face) is a face of size k (resp. $\geq k$, $\leq k$). We say that an edge e is incident to a face f if e belongs to the boundary walk of f.

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Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions holds: either (1) G_1 is a proper subgraph of G_2, or (2) $n_3(G_1) < n_3(G_2)$. Note that this partial order is well-defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \leq n_3(G_2)$. So $\preceq$ is a partial linear extension of the subgraph poset.

Let H be a minimal counterexample to Theorem i according to $\prec$.

2.1 Structural properties of H

Let us begin with some definitions: A *light* 4-vertex is a 4-vertex incident to two 3-faces. A *light* 3-face is a 3-face incident to two light 4-vertices.

Claim 2.1 *The counterexample H does not contain:*

(C1) A k-vertex with $k \in [1, 3]$.

(C2) A 2-vertex incident to a 3-face.

(C3) A k-vertex adjacent to k 2-vertices with $k \leq 42$.

(C4) A k-vertex adjacent to $k-1$ 2-vertices with $2 \leq k \leq 21$.

(C5) A k-vertex adjacent to $k-2$ 2-vertices with $3 \leq k \leq 11$.

(C6) A k-vertex adjacent to $k-3$ 2-vertices with $4 \leq k \leq 5$.

(C7) A 3-face incident to three 4-vertices.

(C8) A 3-face incident to two 4-vertices and to a 5-vertex which is adjacent to a 2-vertex.

2.2 Discharging procedure

Lemma 2.2 *Let H be a connected plane graph with n vertices, m edges and r faces. Then we have the following:*

$$\sum_{v\in V(H)} (3d(v) - 10) + \sum_{f\in F(H)} (2d(f) - 10) = -20 \qquad (1)$$

We define the weight function ω by $\omega(x) = 3 \cdot d(x) - 10$ if $x \in V(H)$ and $\omega(x) = 2 \cdot d(x) - 10$ if $x \in F(H)$. It follows from identity (1) that the total sum of weights is equal to -20. In what follows, we define discharging rules (R1) to (R3) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed by the discharging rules. Nevertheless, we can show that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) \leq \sum_{x \in V(H) \cup F(H)} \omega(x) = -20 < 0$$

Thus no such counterexample exists.

The discharging rules are defined as follows:

(R1) Each ${}^{\geq}6$-vertex gives 2 to each adjacent 2-vertex and to each incident 3-face.

(R2) Each 5-vertex gives 2 to each adjacent 2-vertex, $\frac{3}{2}$ to each incident non light 3-face and 2 to each incident light 3-face.

(R3) Let v be a 4-vertex.
- (R3.1) If v is light, then it gives 1 to each incident 3-face
- (R3.2) If v is not light, then it gives 2 to each incident 3-face.

Now, let us compute the new charges produced after the discharging procedure. Let v be a k-vertex, with $k \notin \{1, 3\}$ by (C1).

If $k = 2$, then $\omega(v) = -4$. Since v is adjacent to ${}^{\geq}5$-vertices by (C1), (C4) and (C6), it receives 2 from each adjacent vertices by (R1) and (R2). So, $\omega^*(v) = 0$.

If $k = 4$, then $\omega(v) = 2$. If v is light, by (R3.1) it gives twice 1 and so, $\omega^*(v) = 0$. If v is not light, then v is incident to at most one 3-face. So, $\omega^*(v) \geq 0$ by (R3.2).

If $k = 5$, then $\omega(v) = 5$. By (C6), v is adjacent to at most one 2-vertex. Moreover, it can be incident to at most two 3-faces. If v is adjacent to a 2-vertex, then it is not incident to a light 3-face by (C8) and so, $\omega^*(v) \geq 5 - 2 \cdot \frac{3}{2} - 2 \geq 0$ by (R2). If v is not adjacent to a 2-vertex, then $\omega^*(v) \geq 5 - 2 \cdot 2 \geq 1$.

Observe that (R1) is equivalent for v to give 2 per edge incident to a 2-vertex and 1 per edge incident to a 3-face. It follows that the worst case of discharging for v appears when v is adjacent to the biggest number of 2-vertices according to (C3)-(C6). If $k = 6$, then $\omega(v) = 8$. By (C5), v is adjacent to at most three 2-vertices. So, $\omega^*(v) \geq 8 - 3 \cdot 2 - 2 \geq 0$. If $k = 7$, then $\omega(v) = 11$. By (C5), v is adjacent to at most four 2-vertices. So, $\omega^*(v) \geq 11 - 4 \cdot 2 - 2 \geq 1$.

If $k = 8$, then $\omega(v) = 14$. By (C5), v is adjacent to at most five 2-vertices. So, $\omega^*(v) \geq 14 - 5 \cdot 2 - 2 \geq 2$. If $k = 9$, then $\omega(v) = 17$. By (C5), v is adjacent to at most six 2-vertices. So, $\omega^*(v) \geq 17 - 6 \cdot 2 - 2 \geq 3$. If $k \geq 10$, then $\omega(v) = 3 \cdot k - 10$ and trivially $\omega^*(v) \geq 3 \cdot k - 10 - 2 \cdot k \geq k - 10 \geq 0$.

Let f be a 3-face; $\omega(f) = -4$. By (C1) and (C2), f is incident to $^{\geq}4$-vertices. By (C7), f is incident to at most two 4-vertices. Let x, y, z be the vertices incident to f. Without loss of generality, we consider that $4 \leq d(x) \leq d(y) \leq d(z)$. If $d(z) = 6$, then by (R1)-(R3), f receives at least $2 + 2 \cdot 1 = 4$ and so $\omega^*(f) \geq 0$. Consider $4 \leq d(x) \leq d(y) \leq d(z) \leq 5$. If $d(y) = 5$, then $\omega^*(f) \geq 2 \cdot \frac{3}{2} + 1 \geq 0$. Now, it remains one case: $d(x) = d(y) = 4, d(z) = 5$. If x (resp. y) is not light, then x (resp. y) gives 2 and $\omega^*(f) \geq 2 + 1 + \frac{3}{2} \geq \frac{1}{2}$. Consider that x and y are light; hence f is light and receives 1 from x, 1 from y by (R3) and 2 from z by (R2) and $\omega^*(f) = -4 + 2 \cdot 1 + 2 = 0$.

That shows that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$. The contradiction with (1) completes the proof.

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