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Oriented vertex and arc colorings of partial 2-trees

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1 Introduction

We consider finite simple *oriented graphs*, that is digraphs with no opposite arcs. For an oriented graph G, we denote by $V(G)$ its *set of vertices* and by $A(G)$ its *set of arcs*. The number of vertices of G is the *order* of G. The *girth* of a graph G is the size of a smallest cycle in G. We denote by $\mathcal{T}_g$ the class of partial 2-trees (also known as series-parallel graphs) with girth at least g.

The notion of oriented vertex-coloring was introduced by Courcelle [2] as follows: an *oriented k-vertex-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of k colors such that (i) $\varphi(u) \neq \varphi(v)$ whenever $uv \in A(G)$ and (ii) $\varphi(v) \neq \varphi(x)$ whenever $uv, xy \in A(G)$ and $\varphi(u) = \varphi(y)$. The *oriented chromatic number* of G, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-vertex-coloring. The oriented chromatic number $\chi_o(\mathcal{F})$ of a class of oriented graphs $\mathcal{F}$ is defined as the maximum of $\chi_o(G)$ taken over all graphs G in $\mathcal{F}$.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping φ from $V(G)$ to $V(H)$ that preserves the arcs: $\varphi(u)\varphi(v) \in A(H)$ whenever

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$uv \in A(G)$. An oriented k-vertex-coloring of an oriented graph G can be equivalently defined as a homomorphism φ from G to H, where H is an oriented graph of order k. The oriented chromatic number of G can then be viewed as the smallest order of an oriented graph H such that G admits a homomorphism to H. Links between colorings and homomorphisms are presented in more details in the monograph [3] by Hell and Nešetřil.

Oriented vertex-colorings have been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes (see *e.g.* [1,8,9]).

Concerning partial 2-trees, Sopena proved [9] that their oriented chromatic number is at most 7 (this bound was shown to be tight). Pinlou and Sopena [8] obtained tight bounds for the oriented chromatic number of outerplanar graphs with given girth (outerplanar graphs form a strict subclass of partial 2-trees). Moreover, they proved that $\chi_o(\mathcal{T}_g) = 7$ for every g, $3 \le g \le 4$. In this paper, we complete the characterization of the oriented chromatic numbers of partial 2-trees with given girth:

Theorem 1.1

(1) $\chi_o(\mathcal{T}_g) = 6$ *for every girth g,* $5 \le g \le 6$*;*

(2) $\chi_o(\mathcal{T}_g) = 5$ *for every girth g,* $g \ge 7$*;*

One can define oriented arc-colorings of oriented graphs in a natural way by saying that, as in the undirected case, an *oriented arc-coloring* of an oriented graph G is an oriented vertex-coloring of its line digraph $LD(G)$ (recall that $LD(G)$ is given by $V(LD(G)) = A(G)$ and $ab \in A(LD(G))$ whenever $a = uv$ and $b = vw$). Therefore, an oriented arc-coloring φ of G must satisfy (i) $\varphi(uv) \neq \varphi(vw)$ whenever uv and vw are two consecutive arcs in G, and (ii) $\varphi(vw) \neq \varphi(xy)$ whenever $uv, vw, xy, yz \in A(G)$ with $\varphi(uv) = \varphi(yz)$. The *oriented chromatic index* of G, denoted by $\chi'_o(G)$, is defined as the smallest order of an oriented graph H such that $LD(G)$ admits a homomorphism to H. The oriented chromatic index $\chi'_o(\mathcal{F})$ of a class of oriented graphs $\mathcal{F}$ is defined as the maximum of $\chi'_o(G)$ taken over all graphs G in $\mathcal{F}$.

The oriented chromatic index of oriented graphs was recently studied and several upper and lower bounds are known (see [6,7,8]).

Upper bounds for the oriented chromatic index can be easily derivated from oriented chromatic number:

Claim 1.2 *[6] For every oriented graph G,* $\chi'_o(G) \le \chi_o(G)$.

Our second result gives estimates of the oriented chromatic indexes of partial 2-trees with girth 4, 5 and 6, and a characterization for all other girths:

Theorem 1.3

(1) $\chi'_o(\mathcal{T}_3) = 7$*;*

(2) $6 \leq \chi'_o(\mathcal{T}_4) \leq 7$*;*

(3) $5 \leq \chi'_o(\mathcal{T}_g) \leq 6$ *for every girth g,* $5 \leq g \leq 6$*;*

(4) $\chi'_o(\mathcal{T}_g) = 5$ *for every girth g,* $7 \leq g \leq 17$*;*

(5) $\chi'_o(\mathcal{T}_g) = 4$ *for every girth g,* $g \geq 18$*;*

In the rest of the paper, we will use the following notation. A vertex of degree k will be called a *k-vertex*. We denote by $\delta(G)$ the minimum degree of the graph G.

A *k-path* in a graph G is a path $P = [u, v_1, v_2, \ldots, v_{k-1}, w]$ of length k (*i.e.* a path with k arcs) ; the vertices u and w are the *endpoints* of P. Note that a 1-path is an arc. A (k,d)-path is a k-path such that all internal vertices v_i have degree d.

A *2-vertex contraction* is the contraction of an edge incident to a 2-vertex.

2 Sketches of proof

The proofs of Theorems 1.1 and 1.3 use some structural properties on partial 2-trees with given girth and on graph classes closed under 2-vertex contraction. These properties are given in the two following lemmas.

Lemma 2.1 *Let* $\mathcal{C}$ *be a graph class closed under 2-vertex contraction such that every non-empty graph* $G \in \mathcal{C}$ *with girth at least g contains either a 1-vertex or a* $(k,2)$*-path, for some* $k \geq 2$*. Then, for every* $n \geq 0$*, every non-empty graph* $G' \in \mathcal{C}$ *with girth at least* $g + n\left\lfloor \frac{g-1}{k-1} \right\rfloor$ *contains either a 1-vertex or a* $(k+n,2)$*-path.*

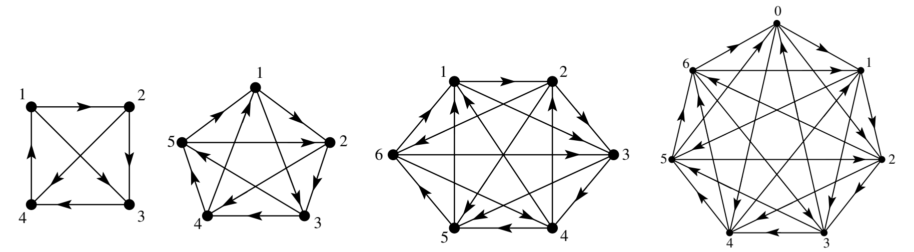
For a graph G with girth at least g and a vertex $v \in V(G)$, we denote:
$D_g^G(v) = |\{u \in V(G),\ d(u) \geq 3$ such that there exists a unique path of 2-vertices linking u and v or u and v are the endpoints of at least a $(\lceil \frac{g}{2} \rceil, 2)$-path$\}|$.

Lemma 2.2 *Let G be a partial 2-tree with girth g such that* $\delta(G) \geq 2$*. Then, either there exists a* $(\lceil \frac{g}{2} \rceil + 1, 2)$*-path, or there exists a* $^{\geq}3$*-vertex v such that* $D_g^G(v) \leq 2$*.*

Note that this lemma generalizes Lemma 2 p. 305 of Lih et al. [4] which characterizes partial 2-trees with girth 3.

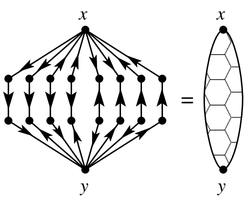
Upper bounds

Thanks to the above lemmas, the upper bounds of Theorems 1.1 and 1.3 are obtained by showing that the considered partial 2-trees admit a homomorphism to one of the tournaments T_4, T_5, T_6, and T_7 depicted on Fig. 1.

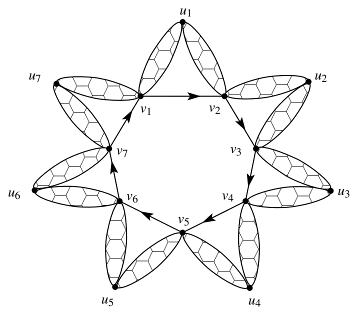


(a) (b) (c) (d)

Fig. 1. The four target tournaments.

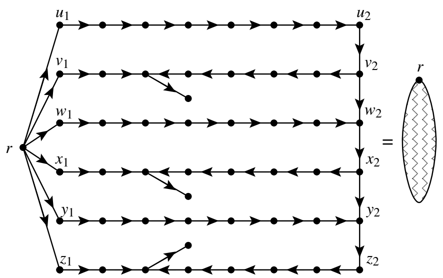


(a) The graph H

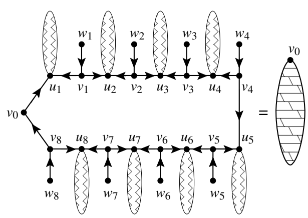


(b) The graph G_6

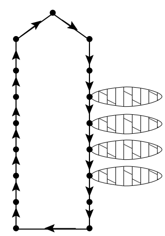
Fig. 2. An oriented partial 2-tree with girth 6 and oriented chromatic number 6.



(a) The graph H_1.



(b) The graph H_2.



(c) The graph G_{17}.

Fig. 3. An oriented partial 2-tree with girth 17 and oriented chromatic index 5.

Lower bounds

Finally, to get the lower bounds of Theorems 1.1 and 1.3, we construct partial 2-trees with the required girth which need the specified number of colors. More fully :

- The graph G_6 depicted in Fig. 2(b) is a partial 2-tree with girth 6 such that

$\chi_o(G_6) = 6$. Therefore, $\chi_o(\mathcal{T}_g) \geq 6$ for every $g \leq 6$.

- Nešetřil *et al.* [5] constructed for every $g \geq 3$, an oriented outerplanar graph with girth g which has oriented chromatic number 5. Therefore, $\chi_o(\mathcal{T}_g) \geq 5$ for every $g \geq 7$.
- The first three assumptions of Theorem 1.3 directly follow from Claim 1.2, Theorem 1.1(1) and some results of Pinlou and Sopena [8], namely $\chi_o(\mathcal{T}_3) = 7$, $\chi'_o(\mathcal{O}_4) = 6$, and $\chi'_o(\mathcal{O}_6) = 5$.
- The graph G_{17} depicted in Fig. 3(c) is a partial 2-tree with girth 17 such that $\chi'_o(G_{17}) = 5$. Therefore, $\chi'_o(\mathcal{T}_g) \geq 5$ for every $g \leq 17$.
- It not difficult to check that, for every $g \geq 3$, the partial 2-tree G obtained from two vertex-disjoint circuits, the first one of size g and the second one of size $k \geq g$ with $k \not\equiv 0 \pmod 3$ has girth g and $\chi'_o(G) = 4$. Therefore $\chi'_o(\mathcal{T}_g) \geq 4$ for every $g \geq 18$.

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