



Oriented colorings of partial 2-trees

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ABSTRACT

A homomorphism from an oriented graph G to an oriented graph H is an arc-preserving mapping f from $V(G)$ to $V(H)$, that is $f(x)f(y)$ is an arc in H whenever xy is an arc in G . The oriented chromatic number of G is the minimum order of an oriented graph H such that G has a homomorphism to H . In this paper, we determine the oriented chromatic number of the class of partial 2-trees for every girth $g \geq 3$. We also give an upper bound for the oriented chromatic number of planar graphs with girth at least 11.

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1. Introduction

We consider finite simple *oriented graphs*, that is digraphs without opposite arcs nor loops. For an oriented graph G , we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. The number of vertices of G is the *order* of G . The *girth* of a graph G is the size of a smallest cycle in G . We denote by \mathcal{O}_g , \mathcal{T}_g , and \mathcal{P}_g , the class of outerplanar graphs with girth at least g , the class of partial 2-trees with girth at least g , and the class of planar graphs with girth at least g , respectively.

The notion of oriented coloring was introduced by Courcelle [5] as follows: an *oriented k -coloring* of an oriented graph G is a mapping f from $V(G)$ to a set of k colors such that

- (i) $f(u) \neq f(v)$ whenever $uv \in A(G)$ and
- (ii) $f(v) \neq f(x)$ whenever $uv, xy \in A(G)$ and $f(u) = f(y)$.

In other words, an oriented k -coloring of G is a partition of the vertices of G into k stable sets S_1, S_2, \dots, S_k such that all the arcs between any pair of stable sets S_i and S_j have the same direction (either from S_i to S_j , or from S_j to S_i). The *oriented chromatic number* of G , denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k -coloring. The oriented chromatic number $\chi_o(\mathcal{F})$ of a class of oriented graphs \mathcal{F} is defined as the maximum of $\chi_o(G)$ taken over all graphs G in \mathcal{F} .

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ that preserves the arcs: $f(u)f(v) \in A(H)$ whenever $uv \in A(G)$. An oriented k -coloring of an oriented graph G can be equivalently defined as a homomorphism f from G to H , where H is an oriented graph of order k ; such a homomorphism is called a *H -coloring* of G or simply an *oriented coloring* of G .

The existence of such a homomorphism from G to H is denoted by $G \rightarrow H$. The vertices of H are called *colors*, and we say that G is *H -colorable*. The oriented chromatic number of G can then be equivalently defined as the smallest order of an oriented graph H such that $G \rightarrow H$. Links between colorings and homomorphisms are presented in more details in the monograph [6] by Hell and Nešetřil.

Oriented colorings have been studied by several authors in the last decade and the problem of bounding the ori-

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ented chromatic number has been investigated for various graph classes (see, e.g., [3,4,12–14]).

A very challenging question is to determine the oriented chromatic number of planar graphs. Raspaud and Sopena [13] proved in 1994 that their oriented chromatic number is at most 80. Recently, Marshall [8] proved that there exist planar graphs with an oriented chromatic number at least 17. The gap between the lower and the upper bound is very large, but it seems very hard to reduce.

Some authors then studied the oriented chromatic number of planar graphs with given girth to get some hints on the behavior of this invariant. The following bounds have been obtained:

Theorem 1. (See [1–4,9–11].)

- (1) $11 \leq \chi_o(\mathcal{P}_4) \leq 47$ [2,10],
- (2) $5 \leq \chi_o(\mathcal{P}_5) \leq 16$ [9,11],
- (3) $5 \leq \chi_o(\mathcal{P}_6) \leq 11$ [4,9],
- (4) $5 \leq \chi_o(\mathcal{P}_g) \leq 7$ for every girth $7 \leq g \leq 11$ [1,9],
- (5) $\chi_o(\mathcal{P}_g) = 5$ for every girth $g \geq 12$ [3,9].

We can remark that the oriented chromatic number of planar graphs with girth at least 12 have been characterized. However, for girths 4 to 11, only estimates are known.

Sopena [14], and Pinlou and Sopena [12] considered the oriented chromatic number of the class of outerplanar graphs (which is a graph class strictly included in the class of planar graphs). They obtained exact bounds for every girth:

Theorem 2. (See [12,14].)

- (1) $\chi_o(\mathcal{O}_3) = 7$ [14],
- (2) $\chi_o(\mathcal{O}_4) = 6$ [12],
- (3) $\chi_o(\mathcal{O}_g) = 5$ for every $g \geq 5$ [12].

It is then natural to study the behavior of the oriented chromatic number of the class of partial 2-trees (also known as series-parallel graphs or K_4 -minor free graphs) since $\mathcal{O}_g \subset \mathcal{T}_g \subset \mathcal{P}_g$.

Sopena [14] proved that $\chi_o(\mathcal{T}_3) = 7$, and Pinlou and Sopena [12] showed that $\chi_o(\mathcal{T}_4) = 7$. In this paper, we complete the characterization of the oriented chromatic numbers of partial 2-trees with given girth:

Theorem 3.

- (1) $\chi_o(\mathcal{T}_g) = 6$ for every girth $g, 5 \leq g \leq 6$;
- (2) $\chi_o(\mathcal{T}_g) = 5$ for every girth $g \geq 7$.

This paper is organized as follows. We give in the next section some preliminary results which are used in Section 3 to prove Theorem 3. In Section 4, we show that the proof techniques we use in this paper allow us to improve the upper bound of the oriented chromatic number of planar graphs with girth at least 11.

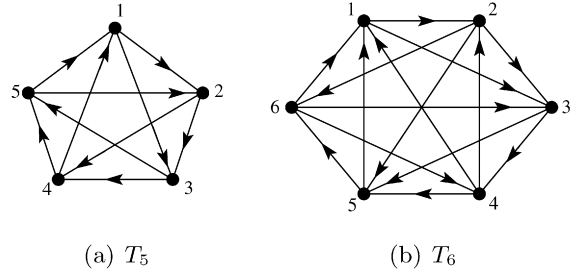


Fig. 1. The two target tournaments.

2. Notation and preliminary results

In the remainder, we will use the following notation. For a graph G and a vertex v , we denote by $d_G(v)$ the degree of v . A vertex of degree k (resp., at least k) will be called a k -vertex (resp., $\geq k$ -vertex). We denote by $\delta(G)$ (resp., $\Delta(G)$) the minimum (resp., maximum) degree of the graph G . If uv is an arc, u is a predecessor of v and v is a successor of u . A vertex will be called a source if it has no predecessors and a sink if it has no successors.

A k -path in a graph G is a path $P = [u, v_1, v_2, \dots, v_{k-1}, w]$ of length k (i.e., a path with k arcs). The vertices u and w are the endpoints of P . Note that a 1-path is an arc. A (k, d) -path is a k -path such that all internal vertices v_i have degree d .

The upper bounds of Theorem 3 will be obtained by proving that the considered partial 2-trees admit a T -coloring, for some tournament T . We will use the tournaments T_5 and T_6 depicted on Fig. 1, whose properties, given below, have already been used in the literature to bound oriented chromatic number and oriented chromatic index of graphs.

The tournament T_5 is a circular tournament and thus is vertex-transitive.

Proposition 4. (See [4].) For every pair of (not necessarily distinct) vertices $u, v \in V(T_5)$, there exists an oriented 4-path connecting u with v for any of the 16 possible orientations of such an oriented 4-path.

Proposition 5. (See [12].) For every pair of (not necessarily distinct) vertices $u, v \in V(T_6)$, there exists an oriented 3-path connecting u with v for any of the 8 possible orientations of such an oriented 3-path.

Our proof techniques to get upper bounds for the oriented chromatic number are based on the well-known method of reducible configurations. We suppose that there exists a hypothetical minimal counterexample H to the considered theorem and we prove that H does not contain some configurations. Then, thanks to structural properties of partial 2-trees with given girth, we show that H necessarily contains one of the forbidden configurations, otherwise H would not be a partial 2-tree. This contradiction allow us to conclude.

In the remainder of this section, we state a structural property of partial 2-trees due to Lih et al. [7] and generalize it to partial 2-trees with given girth.

For a given undirected graph G and a vertex $v \in V(G)$, we denote:

$$D^G(v) = \left| \left\{ u \in V(G), d(u) \geq 3, \text{ such that} \right. \right. \\ \left. \left. \begin{array}{l} \text{either } uv \in A(G) \\ \text{or } \exists w \in V(G), d_G(w) = 2, uw, wv \in E(G) \end{array} \right\} \right|.$$

Lih, Wang and Zhu [7] proved the following structural lemma for partial 2-trees:

Lemma 6. (See [7].) *Let G be a partial 2-tree such that $\delta(G) \geq 2$. Then, one the following holds:*

- (1) *there exists a $(3, 2)$ -path (i.e., two adjacent 2-vertices);*
- (2) *there exists a ≥ 3 -vertex v such that $D^G(v) \leq 2$.*

We generalize the previous lemma to partial 2-trees with given girth. For a given undirected graph G with girth at least g and a vertex $v \in V(G)$, we denote:

$$S_g^G(v) = \left\{ u \in V(G), d(u) \geq 3, \text{ such that either} \right. \\ \left. \begin{array}{l} \exists \text{ a unique } (k, 2)\text{-path linking } u \text{ and } v, k \geq 1, \\ \text{or } \exists \text{ at least one } (\lceil g/2 \rceil, 2)\text{-path linking } u \text{ and } v \end{array} \right\}.$$

We then denote $D_g^G(v) = |S_g^G(v)|$. Note that $D_3^G(v) = D^G(v)$ for every $v \in V(G)$.

Lemma 7. *Let G be a partial 2-tree with girth g such that $\delta(G) \geq 2$. Then, one the following holds:*

- (1) *there exists a $(\lceil g/2 \rceil + 1, 2)$ -path;*
- (2) *there exists a ≥ 3 -vertex v such that $D_g^G(v) \leq 2$.*

Proof. Let $H \in \mathcal{T}_g$ with $\delta(H) \geq 2$ such that it contains no $(\lceil g/2 \rceil + 1, 2)$ -path. Note that in this case H is not a cycle and thus contains ≥ 3 -vertices. Then, consider the graph H' obtained from H by removing all the 2-vertices and adding an arc between every pair of the remaining vertices which were linked by at least one $(k, 2)$ -path in H , for some k . Since H contains ≥ 3 -vertices, H' is not reduced to a unique vertex.

Let v be any vertex of H' and let $N_{H'}(v)$ be the set of v 's neighbors in H' . By construction, there exists, for every $w \in N_{H'}(v)$, at least one $(k, 2)$ -path linking v and w in H for some $k \geq 1$. In addition, if there exists more than one $(k, 2)$ -path linking v and w in H , then at most one of these paths is a $(\lfloor g/2 \rfloor, 2)$ -path and the others are $(\lceil g/2 \rceil, 2)$ -paths since H has girth g . This shows that for every $v \in H'$, we have $d_{H'}(v) = D_g^H(v)$.

Since the class of partial 2-trees is closed under edge-contraction, H' is clearly a partial 2-tree. Since partial 2-trees are 2-degenerate, H' contains a vertex v of degree at most 2, and therefore $D_g^H(v) \leq 2$ (note that $d_H(v) \geq 3$ since this vertex remains in H'). That completes the proof. \square

Corollary 8. *Every partial 2-tree with girth $g \geq 3$ contains either a 1-vertex or a $(\lceil g/2 \rceil, 2)$ -path.*

Proof. Let $H \in \mathcal{T}_g$ with $\delta(H) \geq 2$ having no $(\lceil g/2 \rceil + 1, 2)$ -path. By Lemma 7, H contains a ≥ 3 -vertex v such that $D_g^H(v) \leq 2$. Therefore, by definition of D_g^H , this means that v has degree at least 3 in H , but degree at most 2 in H' (H' is the graph obtained from H by removing all the 2-vertices and adding an arc between every pair of the remaining vertices which were linked by at least one $(k, 2)$ -path in H , for some k). That implies that H contains a vertex w such that v and w are the endpoints of at least two $(k, 2)$ -paths, and at least one of them must be a $(\lceil g/2 \rceil, 2)$ -path. \square

3. The oriented chromatic number of partial 2-trees

In this section, we prove Theorem 3.

Proof of Theorem 3(1). We first prove that $\chi_0(\mathcal{T}_g) \leq 6$ for every girth $g, 5 \leq g \leq 6$. Note that it is sufficient to consider the case $g = 5$; we therefore prove that every partial 2-tree with girth at least 5 admits a homomorphism to the tournament T_6 depicted in Fig. 1(b). Let H be a minimal (with respect to the number of vertices) partial 2-tree with girth at least 5 having no homomorphism to T_6 . We show that H contains neither a 1-vertex nor a $(3, 2)$ -path.

- (1) Suppose that H contains a 1-vertex u . Then, due to the minimality of H , the partial 2-tree $H' = H \setminus u$ has girth at least 5, and thus admits a T_6 -coloring f . Since every vertex of T_6 has at least two successors and at least two predecessors, f can be easily extended to H .
- (2) Suppose now that H contains a $(3, 2)$ -path $[u, v_1, v_2, w]$. Then, due to the minimality of H , the partial 2-tree $H' = H \setminus \{v_1, v_2\}$ admits a T_6 -coloring f . By Proposition 5, f can be extended to H .

By Corollary 8, H contains either a 1-vertex, or a $(3, 2)$ -path. This leads us to a contradiction: H does not exist.

To complete this proof, we have to construct a partial 2-tree with girth 6 and oriented chromatic number 6. Let us consider the graph Q depicted in Fig. 2(a) obtained from two vertices x and y linked by the eight possible oriented 3-paths. Then, consider G_6 obtained from a circuit $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1]$ of length seven and fourteen copies of the graph Q arranged as depicted on Fig. 2(b). We can easily see that G_6 is a partial 2-tree with girth 6.

Suppose first that $\chi_0(G_6) \leq 4$. Therefore, there exists a homomorphism $f: G_6 \rightarrow T$, where T is a tournament on 4 vertices. Since G_6 contains a circuit of length $7 \not\equiv 0 \pmod{3}$, T must contain a circuit of length 4. There exist four nonisomorphic tournaments on four vertices, but only one contains a circuit of length 4: the tournament T_4 depicted in Fig. 3. However, we can check that $Q \not\rightarrow T_4$ since there does not exist a pair of colors u and v in $V(T_4)$ which color the vertices x and y (i.e., such that u and v are the endpoints of the eight possible oriented 3-paths in T_4). Thus, $G_6 \not\rightarrow T_4$.

Hence, $\chi_0(G_6) \geq 5$. Suppose that $\chi_0(G_6) = 5$. There exist twelve nonisomorphic tournaments on 5 vertices. We will prove that none of these tournaments allows us to color G_6 . We can first omit those containing a source or a sink. The six remaining tournaments are depicted in Fig. 4.

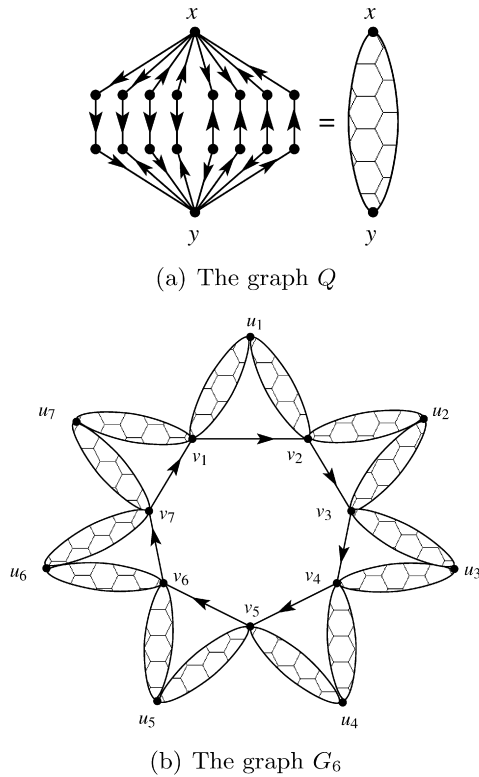


Fig. 2. Construction of an oriented partial 2-tree with girth 6 and oriented chromatic number 6.

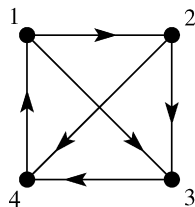


Fig. 3. The tournament T_4 .

A case study shows that every T -coloring f of Q , where T is one of the five tournaments in Figs. 4(a), 4(b), 4(c), 4(d), 4(e), implies that $f(x) = f(y)$. Hence, this would mean that $f(v_1) = f(u_1) = f(v_2)$ in G_6 , which is forbidden. It is then clear that if $\chi_o(G_6) = 5$, then $G_6 \rightarrow T$ where T is the tournament depicted in Fig. 4(f). A case study shows that every T -coloring f of Q is such that $f(x) \in \{1, 2, 5\}$ and $f(y) \in \{1, 2, 5\}$. However, the circuit $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1]$ in G_6 needs four colors, that is a contradiction. Thus, the graph G_6 has oriented chromatic number 6.

Proof of Theorem 3(2). We first prove that $\chi_o(\mathcal{T}_g) \leq 6$ for every girth $g \geq 7$. Note that it is sufficient to consider the case $g = 7$; we therefore prove that every partial 2-tree with girth at least 7 admits a T_5 -coloring, where T_5 is the tournament depicted in Fig. 1(a). Let H be a minimal (with respect to the number of vertices) partial 2-tree with girth at least 7 having no homomorphism to T_5 . The proof techniques are the same than those in the proof of Theo-

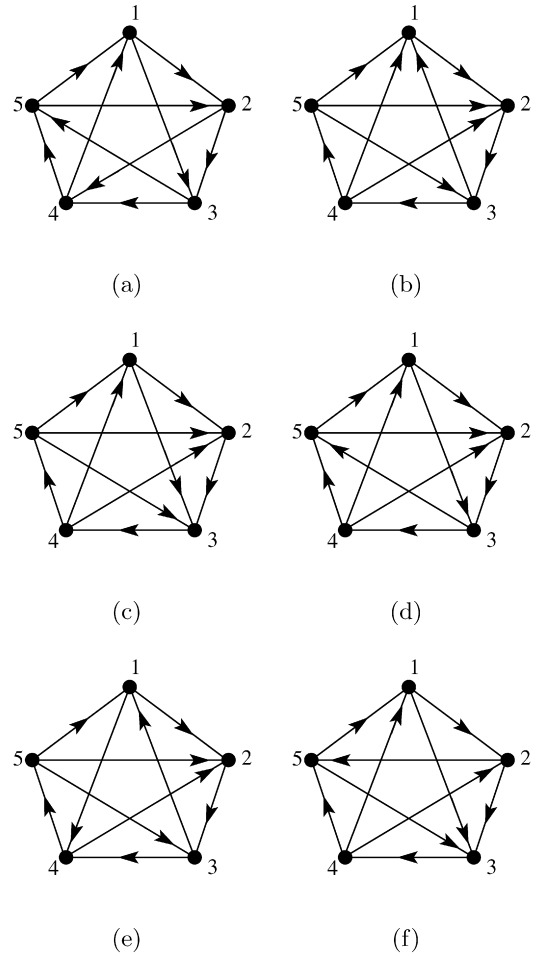


Fig. 4. The six nonisomorphic tournaments on five vertices without source nor sink.

rem 3(1). By minimality and Proposition 4, we prove that H contains neither a 1-vertex nor a $(4, 2)$ -path. We thus get a contradiction thanks to Corollary 8.

To complete this proof, we have to construct, for all girths $g \geq 7$, a partial 2-tree with girth g and oriented chromatic number 5. Nešetřil et al. [9] constructed for every $g, g \geq 3$, an oriented outerplanar graph with girth at least g which has oriented chromatic number 5. The class of outerplanar graphs is strictly included in the class of partial 2-trees: that completes the proof. \square

4. Concluding remarks

In this paper, we characterized the oriented chromatic number of partial 2-trees for every girth $g \geq 3$. Note that our results improve the previously known lower bounds for the oriented chromatic number of planar graphs with girths 5 and 6 (see Theorems 1(2) and 1(3)): $\chi_o(\mathcal{P}_g) \geq 6$ for every $g \in [5, 6]$. Moreover, we show in the remainder that our proof techniques can be used to improve the upper bound of the oriented chromatic number of planar graphs with girth at least 11.

Remind that Theorem 1 gives $\chi_o(\mathcal{P}_{11}) \leq 7$. It is well known that a planar graph with minimum degree 2 and girth at least $g = 5k + 1$ necessarily contains a $(k + 1, 2)$ -path [9]. Thus, a planar graph with girth at least 11 contains a $(3, 2)$ -path (i.e., two adjacent 2-vertices). This allows us to prove the following new upper bound:

Theorem 9. $\chi_o(\mathcal{P}_{11}) \leq 6$.

The proof techniques are the same than those in the proof of Theorem 3(1). Actually, we can prove that every planar graph with girth at least 11 admits a homomorphism to the tournament T_6 depicted in Fig. 1(b). To get this result, we consider a minimal (with respect to the number of vertices) planar graph with girth at least 11 having no homomorphism to T_6 . We can prove that this graph contains neither a 1-vertex, nor a $(3, 2)$ -path thanks to Proposition 5. This contradicts the above-mentioned remark: such a planar graph does not exist.

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