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Oriented coloring of triangle-free planar graphs and 2-outerplanar graphs ⋆

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Abstract

A graph is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar (i.e. with all its vertices on the external face). An oriented k-coloring of an oriented graph G is a homomorphism from G to an oriented graph H of order k. We prove that (1) every oriented triangle-free planar graph has an oriented chromatic number at most 40, and (2) every oriented 2-outerplanar graph has an oriented chromatic number at most 40, that improves the previous known bounds of 47 and 67, respectively.

Keywords: Oriented coloring; Planar graph; Girth; 2-outerplanar graph; Discharging procedure.

1 Introduction

Oriented graphs are directed graphs without loops nor opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs.

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For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v. The number of vertices of G is the *order* of G.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs: $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An oriented k-coloring of G can be defined as a homomorphism from G to H, where H is an oriented graph of order k. In other words, that corresponds to a partition of the vertices of G into k stable sets $S_1, S_2, \dots, S_k$ such that all the arcs between any pair of stable sets S_i and S_j have the same direction (either from S_i to S_j, or from S_j to S_i). The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The *oriented chromatic number* of an oriented graph G, denoted by $\chi_o(G)$, is defined as the smallest order of an oriented graph H such that $G \to H$. Links between colorings and homomorphisms are presented in more details in the monograph [7] by Hell and Nešetřil.

A graph is *planar* if it can be embedded on the plane without edge-crossing. The *girth* of a graph is the length of a shortest cycle.

The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given minimum girth) [10,12], 2-outerplanar graphs [6], planar graphs (with given minimum girth) [1,2,3,4,9,11], graphs with bounded maximum average degree [3,4], and graphs with bounded treewidth [8,12,13].

Theorem 1.1 gives the current best known bounds on oriented chromatic number of planar graphs.

Theorem 1.1 *[1,2,3,4,8,9,11] Let G be a planar graph.*

(i) *If G has girth at least 12, then* $\chi_o(G) \leq 5$ *[3] (this bound is tight).*

(ii) *If G has girth at least 11, then* $\chi_o(G) \leq 6$ *[8].*

(iii) *If G has girth at least 7, then* $\chi_o(G) \leq 7$ *[1].*

(iv) *If G has girth at least 6, then* $\chi_o(G) \leq 11$ *[4].*

(v) *If G has girth at least 5, then* $\chi_o(G) \leq 16$ *[9].*

(vi) *If G has girth at least 4, then* $\chi_o(G) \leq 47$ *[2].*

(vii) *If G has no girth restriction, then* $\chi_o(G) \leq 80$ *[11].*

A graph is *2-outerplanar* if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar (i.e. with all its vertices on the external face).

In 2007, Esperet and Ochem [6] studied the oriented chromatic number of 2-outerplanar graphs and they proved the following:

Theorem 1.2 *[6] Let G be a 2-outerplanar graph. Then* $\chi_o(G) \leq 67$.

One way to bound the oriented chromatic number of a graph family $\mathcal{F}$ is to find a universal target graph H such that, for every graph $G \in \mathcal{F}$, we have $G \to H$. Such a result can be obtained if the target graph H has "interesting" structural properties that can be used to prove the existence of the homomorphism; thus an important part of the task is to construct such a target graph. In this paper, we first describe the construction of the graph T_{40} in Section 2, an oriented graph on 40 vertices which has very useful properties for oriented coloring of planar graphs.

These structural properties of T_{40} allow us to prove that every oriented triangle-free planar graph admits a homomorphism to T_{40}; this gives the following theorem, which improves Theorem 1.1(vi).

Theorem 1.3 *Let G be a triangle-free planar graph. Then* $\chi_o(G) \leq 40$.

We also show that every oriented 2-outerplanar graph admits a homomorphism to T_{40}; this allows us to improves Theorem 1.2.

Theorem 1.4 *Let G be a 2-outerplanar graph. Then* $\chi_o(G) \leq 40$.

In the remainder of this paper, we use the following notions. The set of vertices (resp. arcs, faces) of a graph G is denoted by $V(G)$ (resp. $A(G)$, $F(G)$). For a vertex v of a graph G, we denote by $d_G(v)$ its degree. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. $^{\geq}k$*-vertex*, $^{\leq}k$*-vertex*). If a vertex u is adjacent to a k-vertex v, then v is a *k-neighbor* of u. A path of length k (i.e. formed by k edges) is called a *k-path*. The length of a face f of a graph G is denoted by $d_G(f)$. If $d_G(f) = k$ (resp. $d_G(f) \leq k$, $d_G(f) \geq k$), then f is called a *k-face* (resp. $^{\leq}k$*-face*, $^{\geq}k$*-face*).

The paper is organised as follows. The next section is devoted to the target graph T_{40}. We prove Theorem 1.3 in Section 3. Due to lack of space, we do no give the proof of Theorem 1.4 (the proof technique of this theorem is similar to that of Theorem 1.3).

2 The Tromp graph T_{40}

Tromp [14] proposed the following construction. Let G be an oriented graph and G' be an isomorphic copy of G. The Tromp graph $Tr(G)$ has $2|V(G)|+2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$

- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$

In the remainder, we focus on the specific graph family obtained by applying the Tromp's construction to Paley tournaments. For a prime power $p \equiv 3 \pmod 4$, the *Paley tournament* QR_p is defined as the oriented graph whose vertices are the integers modulo p and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of p. For instance, the Paley tournament QR_{19} has vertex set $V(QR_{19}) = \{0, 1, \ldots, 18\}$ and $\overrightarrow{uv} \in A(QR_{19})$ whenever $v - u \equiv r \pmod{19}$ for $r \in \{1, 4, 5, 6, 7, 9, 11, 16, 17\}$. Let $T_{40} = Tr(QR_{19})$ be the Tromp graph on 40 vertices obtained from QR_{19}.

3 Proof of Theorem 1.3

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of ${}^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) \leq n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the induced subgraph poset.

In the following, H is an hypothetical minimal counterexample to Theorem 1.3 according to $\prec$, i.e a triangle-fre planar graph (given with its embedding in the plane) which does not admit a homomorphism to T_{40}. A *weak* 7*-vertex* u in H is a 7-vertex adjacent to four 2-vertices $v_1, \ldots, v_4$ and three ${}^{\geq}3$-vertices w_1, w_2, w_3 in such a way that the sequence of neighbors of v appear as $v_1, w_1, v_2, w_2, v_3, w_3, v_4$ (clockwise or counterclockwise).

Lemma 3.1 *The graph H does not contain the following configurations:*

(C1) a ${}^{\leq}1$-vertex;

(C2) a k-vertex adjacent to k 2-vertices for $2 \leq k \leq 39$;

(C3) a k-vertex adjacent to $(k-1)$ 2-vertices for $2 \leq k \leq 19$;

(C4) a k-vertex adjacent to $(k-2)$ 2-vertices for $3 \leq k \leq 10$;

(C5) a 3-vertex;

(C6) a k-vertex adjacent to $(k-3)$ 2-vertices for $3 \leq k \leq 6$;

(C7) two vertices u and v linked by three distinct 2-paths, two of which have internal vertex of degree 2;

(C8) *two vertices u and v linked by two distinct* 2*-paths, both paths having a* 2*-vertex as internal vertex;*

(C9) *a* 4*-face wxyz such that x is 2-vertex, w and y are weak* 7*-vertices, and z is a k-vertex adjacent to* $(k-3)$ *2-vertices for* $3 \leq k \leq 8$*;*

(C10) *a* 4*-face wxyz such that x is 2-vertex, w and y are weak* 7*-vertices, and z is a k-vertex adjacent to* $(k-4)$ *2-vertices for* $4 \leq k \leq 7$*;*

To complete the proof of Theorem 1.3, we use a discharging procedure. We define the weight function ω by $\omega(x) = d(x) - 4$ for every $x \in V(H) \cup F(H)$. Since H is a planar graph, we have by Euler formula ($|V(H)| - |A(H)| + |F(H)| = 2$):

$$\sum_{v \in V(H)} \omega(v) + \sum_{f \in F(H)} \omega(f) = \sum_{v \in V(H)} (d(v) - 4) + \sum_{f \in F(H)} (d(f) - 4) = -8 < 0.$$

Let us define the discharging rules (R1), (R2), and (R3).

(R1) Each $^{\geq}4$-vertex gives 1 to each its 2-neighbors.

(R2) Each $^{\geq}5$-face $\ldots axb \ldots$ such that a and b are 2-vertices gives 1 (resp. $\frac{1}{2}$) to x if x is a weak 7-vertex (resp. is not a weak 7-vertex).

(R3) Each $^{\geq}5$-face $f = \ldots awxyb \ldots$, such that a, b, x are 2-vertices and w, y are weak 7-vertices, either receives $\frac{1}{2}$ from the vertex z if $wxyz$ is a 4-face, or receives 1 from the $^{\geq}5$-face $f' = \ldots cwxyd \ldots$ if c, d are $^{\geq}4$-vertices.

We redistribute weights accordingly to the previous three discharging rules. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is fixed by the discharging rules. Nevertheless, we can show that $\omega^*(v) \geq 0$ for every $x \in V(H) \cup F(H)$ by means of Lemma 3.1. This leads to the following obvious contradiction:

$$0 \leq \sum_{v \in V(H)} \omega^*(v) + \sum_{f \in F(H)} \omega^*(f) = \sum_{v \in V(H)} \omega(v) + \sum_{f \in F(H)} \omega(f) < 0.$$

Therefore, no such counterexample H exists.

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