Homomorphisms of 2-Edge-Colored Triangle-Free Planar Graphs

Pascal Ochem,[1] Alexandre Pinlou,[2] and Sagnik Sen[3]

[1] UNIVERSITÉ DE MONTPELLIER
LIRMM, CNRS, FRANCE
E-mail: Pascal.Ochem@lirmm.fr

[2] UNIVERSITÉ PAUL-VALERY MONTPELLIER 3
LIRMM, CNRS, FRANCE
E-mail: Alexandre.Pinlou@lirmm.fr

[3] INDIAN STATISTICAL INSTITUTE
BANGALORE, INDIA
E-mail: sen007isi@gmail.com

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Abstract: In this article, we introduce and study the properties of some target graphs for 2-edge-colored homomorphism. Using these properties, we obtain in particular that the 2-edge-colored chromatic number of the class of triangle-free planar graphs is at most 50. We also show that it is at least 12. © 2016 Wiley Periodicals, Inc. J. Graph Theory 85: 258–277, 2017

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1. INTRODUCTION

Graph homomorphisms and proper graph colorings are closely related. Indeed, the homomorphisms $\varphi : G \to K_k$ are the k-colorings of G. This explains that homomorphisms of G to H are usually called *H-colorings* of G (we also say G is *H-colorable*) and that the vertices of such graphs H are called *colors*. The *chromatic number* of G is defined as the minimum number of colors k such that G admits a k-coloring; it can be equivalently defined as the minimum order of a graph H such that $G \to H$.

In this article, we propose to study homomorphisms of 2-edge-colored graphs. They have been already studied as a way of extending classical results in graph coloring such as Hadwigers conjecture. Guenin [5] introduced the notion of switching homomorphism for its relation with a well-known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [8] as this theory captures a number of well-known conjectures that can be reformulated using the definition of switching homomorphism. In this article, we study 2-edge-colored graph homomorphism and switching homomorphism for themselves.

A *2-edge-colored graph* (G, s) is a simple graph G with a *signature* $s : E(G) \to \{-1, +1\}$ assigning a negative or positive sign to every edge of G. In the figures, negative edges are drawn with dashed edges. Figure 1a gives an example of 2-edge-colored graph. *Switching* a vertex v of a 2-edge-colored graph (G, s) corresponds to giving the opposite sign to the edges incident to v. Given a 2-edge-colored graph (G, s) and a set of vertices $X \subseteq V(G)$, the graph obtained from (G, s) by switching every vertex in X is denoted by $(G, s^{(X)})$. Let $\partial(X)$ denote the edge cut between X and $G \setminus X$. Notice that $(G, s^{(X)})$ is also obtained by changing the sign of all the edges in $\partial(X)$.

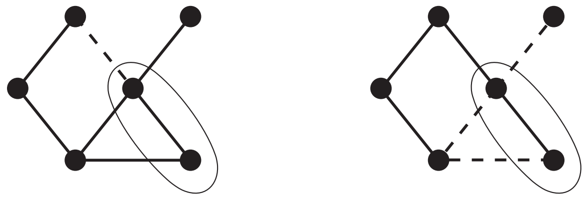
Two 2-edge-colored graphs (G, s_1) and (G, s_2) are *switching equivalent* if we can obtain (G, s_1) from (G, s_2) by switching a set of vertices of (G, s_2), that is, $s_2 = s_1^{(X)}$ with $X \subseteq V(G)$. If (G, s_1) and (G, s_2) are switching equivalent, we write $(G, s_1) \sim (G, s_2)$. Figure 1 gives an example of switching equivalent 2-edge-colored graphs where the surrounded vertices belong to X. We use the notation (G) for a 2-edge-colored graph when its signature is not relevant or is clear from the context, whereas G refers to the underlying simple graph of (G).

Given two graphs (G, s) and (H, t), φ is a *2-edge-colored homomorphism* of (G, s) to (H, t) if $\varphi : V(G) \to V(H)$ is a mapping such that every edge of (G, s) is mapped to an edge of the same sign in (H, t). Given two graphs (G, s_1) and (H, t_1), we say that there is a *switching homomorphism* φ of (G, s_1) to (H, t_1) if there exist $(G, s_2) \sim (G, s_1)$ and $(H, t_2) \sim (H, t_1)$ such that φ is a 2-edge-colored homomorphism of (G, s_2) to (H, t_2).

Lemma 1.1. *If (G, s) admits a switching homomorphism to (H, t), then there exists $(G, s') \sim (G, s)$ such that (G, s') admits a 2-edge-colored homomorphism to (H, t).*

Proof. Since (G, s) admits a switching homomorphism to (H, t), this implies that there exist $(G, s'') \sim (G, s)$, $(H, t') \sim (H, t)$, and a 2-edge-colored homomorphism φ of (G, s'') to (H, t'). Let $X \subseteq V(H)$ be such that $(H, t') = (H, t^{(X)})$. Let $Y = \{v \in V(G) \mid \varphi(v) \in X\}$. Let $(G, s') = (G, s''^{(Y)})$. For every $uv \in E(G)$ we have

- $s'(uv) = -s''(uv) \iff uv \in \partial(Y)$ by the switching of Y.
- $t(\varphi(u)\varphi(v)) = -t'(\varphi(u)\varphi(v)) \iff \varphi(u)\varphi(v) \in \partial(X)$ by the switching of X.
- $uv \in \partial(Y) \iff \varphi(u)\varphi(v) \in \partial(X)$ by the definition of Y.
- $s''(uv) = t'(\varphi(u)\varphi(v))$ by the definition of φ.



(a) (b)

FIGURE 1. Two switching equivalent 2-edge-colored graphs.

Then we deduce that $s'(uv) = t(\varphi(u)\varphi(v))$, which means that φ is a 2-edge-colored homomorphism of (G, s') to (H, t). ■

The above lemma implies that, in order to prove the existence of a switching homomorphism of (G, s) to (H, t), there is no need to switch (H, t).

The *2-edge-colored chromatic number* $\chi_2(G, s)$ of the graph (G, s) is the minimum order (number of vertices) of a graph (H, t) such that (G, s) admits a 2-edge-colored homomorphism to (H, t). Similarly, we define $\chi_{sw}(G, s)$ as the minimum order of a graph (H, t) such that (G, s) admits a switching homomorphism to (H, t). Equivalently, $\chi_{sw}(G, s) = \min\{\chi_2(G, s') \mid (G, s') \sim (G, s)\}$.

The *2-edge-colored chromatic number* $\chi_2(G)$ of a simple graph G is defined as the maximum $\chi_2(G) = \max\{\chi_2(G, s)\}$ over all signatures. The *2-edge-colored chromatic number* $\chi_2(\mathcal{C})$ of a class of simple graphs $\mathcal{C}$ is defined as $\chi_2(\mathcal{C}) = \max\{\chi_2(G) \mid G \in \mathcal{C}\}$. Similarly, $\chi_{sw}(G) = \max\{\chi_{sw}(G, s)\}$ and $\chi_{sw}(\mathcal{C}) = \max\{\chi_{sw}(G) \mid G \in \mathcal{C}\}$.

We use the following notations. The set of positive (resp. negative) neighbors of a vertex v in a 2-edge-colored graph is denoted by $N^+(v)$ (resp. $N^-(v)$). A vertex of degree k (resp. at least k, at most k) is called a k-vertex (resp. $^{\geqslant}k$-vertex, $^{\leqslant}k$-vertex). A path (resp. a cycle) containing k edges is called a k-path (resp. a k-cycle).

In this article, we study 2-edge-colored and switching homomorphisms of outerplanar and planar graphs of given girth. The *girth* of a graph is the length of a shortest cycle. We denote by $\mathcal{P}_g$ (resp. $\mathcal{O}_g$) the class of planar graphs (resp. outerplanar graphs) with girth at least g. So $\mathcal{P}_3$ is simply the class of planar graphs.

In Section 2, we introduce and study the properties of several families of target graphs, namely the *antitwinned graph* $AT(G, s)$, the *2-edge-colored Zielonka graph* SZ_k, the *2-edge-colored Paley graph* SP_q, and the *2-edge-colored Tromp–Paley graph* $Tr(SP_q)$. We consider 2-edge-colored homomorphisms of planar graphs and outerplanar graphs in Section 3 and we provide lower and upper bounds on the 2-edge-colored chromatic number. In particular, we prove that $12 \leqslant \chi_2(\mathcal{P}_4) \leqslant 50$. This improves the previous known upper bound of 80 that holds for planar graphs. Table I summarizes the current knowledge of lower and upper bounds for the 2-edge-colored chromatic number of planar graphs with given girth, including the results in this article.

We obtain results on switching homomorphisms of planar graphs and outerplanar graphs of given girth in Section 4. We finally conclude in Section 5.

TABLE I. Bounds on the 2-edge-colored chromatic number of planar graphs

Girth	Lower bound	Upper bound	Target	References
$g = 3$	20	80	SZ_5	[1, 7]
$g = 4$	12	50	$AT(SP_{25})$	Th. 3.6, Th. 3.11
$g = 5$	8	20	$Tr(SP_9)$	Th. 3.12, [7]
$6 \leqslant g \leqslant 7$	8	12	$Tr(SP_5)$	Th. 3.12, [7]
$8 \leqslant g \leqslant 12$	5	8	$SP_9 \setminus \{0\}$	[7]
$g \geqslant 13$	5	5	SP_5	[3]

2. TARGET GRAPHS

Given a class of simple graphs $\mathcal{C}$, we say that a 2-edge-colored graph (H) is *$\mathcal{C}$-universal* if every 2-edge-colored graph (G) such that $G \in \mathcal{C}$ admits a 2-edge-colored homomorphism to (H).

In this section, our goal is not only to find target graphs that will give the required upper bounds of our results in Sections 3 and 4. We describe several families of target graphs that may be $\mathcal{C}$-universal, for some classes $\mathcal{C}$, and we determine their properties. We consider below *antitwinned graphs*, the *2-edge-colored Zielonka graph SZ_k*, the *2-edge-colored Paley graph*, and the *2-edge-colored Tromp–Paley graph*.

We say that a 2-edge-colored graph (G, s) is

- *vertex-transitive* if for every two vertices u and v, there exists a 2-edge-colored automorphism mapping u to v.
- *arc-transitive* if for every vertices u_1, u_2, v_1, and v_2 such that u_1u_2 and v_1v_2 are edges of the same sign, there exists a 2-edge-colored automorphism mapping u_1 to v_1 and u_2 to v_2.
- *triangle-transitive* if for every vertices u_1, u_2, u_3, v_1, v_2, and v_3 such that $u_1u_2u_3$ and $v_1v_2v_3$ are triangles satisfying $s(u_1u_2) = s(v_1v_2), s(u_2u_3) = s(v_2v_3)$, and $s(u_3u_1) = s(v_3v_1)$, there exists a 2-edge-colored automorphism mapping u_1 to v_1, u_2 to v_2, and u_3 to v_3.

A. Antitwinned Graphs

In a 2-edge-colored graph, two distinct vertices u and v are *twins* if $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. Also, u and v are *antitwins* if $N^+(u) = N^-(v)$ and $N^-(u) = N^+(v)$. Note that twins (resp. antitwins) are necessarily nonadjacent. Moreover, if u has two antitwins v_1 and v_2, then v_1 and v_2 are twins. A 2-edge-colored graph is *twin-free* if it contains neither a pair of twins nor a pair of antitwins.

Let (G, s) be a 2-edge-colored graph and let (G^{+1}) and (G^{-1}) be two copies of (G). The vertex corresponding to $u \in V(G)$ in (G^i) is denoted by u_i. We define the graph $AT(G, s) = (H, t)$ on $2|V(G)|$ vertices as follows:

- $V(H) = V(G^{+1}) \cup V(G^{-1})$
- $E(H) = \{u_iv_j : uv \in E(G),\ i \in \{-1, +1\},\ j \in \{-1, +1\}\}$
- $t(u_iv_j) = i \times j \times s(uv)$

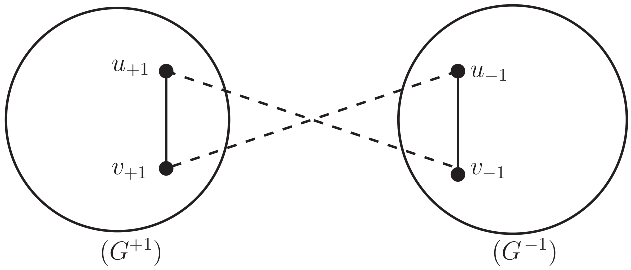


FIGURE 2. The graph $AT(G, s)$.

Figure 2 illustrates the construction of the graph $AT(G, s)$.

We say that a 2-edge-colored graph H is *antitwinned* if and only if every vertex has a unique antitwin.

Observation 2.1. *A 2-edge-colored graph H is antitwinned if and only if H is isomorphic to $AT(G, s)$ such that (G, s) is a twin-free 2-edge-colored graph.*

Given a twin-free 2-edge-colored graph (G, s), we denote by $\text{atw} : V(AT(G, s)) \to V(AT(G, s))$ the fixed-point-free involution that maps every vertex u_i to its unique antitwin, that is, $\text{atw}(u_i) = u_{-i}$. Note that atw is an automorphism of $AT(G, s)$.

Lemma 2.2. *A graph (G, s) admits a switching homomorphism to a twin-free graph (H, t) if and only if (G, s) admits a 2-edge-colored homomorphism to the antitwinned graph $AT(H, t)$.*

Proof. Brewster and Graves [4], Theorem 12] obtained a general result on m-edge-colored graphs, which gives the following in the case $m = 2$: a graph (G, s) admits a switching homomorphism to a graph (H, t) if and only if $AT(G, s)$ admits a 2-edge-colored homomorphism to $AT(H, t)$.

Notice that $AT(G, s)$ admits a 2-edge-colored homomorphism to $AT(H, t)$ if and only if (G, s) admits a 2-edge-colored homomorphism to $AT(H, t)$. This implies that (G, s) admits a switching homomorphism to a graph (H, t) if and only if (G, s) admits a 2-edge-colored homomorphism to $AT(H, t)$.

Finally, we can conclude by using Observation 2.1. ∎

Corollary 2.3. *If (G, s) admits a 2-edge-colored homomorphism to an antitwinned graph T, then (G, s') admits a 2-edge-colored homomorphism to T for every $(G, s') \sim (G, s)$.*

B. The 2-Edge-Colored Zielonka Graph SZ_k

The Zielonka graph Z_k is an oriented graph introduced by Zielonka [14] in the theory of bounded timestamp systems. Raspaud and Sopena [11] have used Z_k in the context of oriented homomorphism. Alon and Marshall [1] have adapted this construction to m-edge-colored graphs in order to obtain bounds on the m-edge-colored chromatic number of graphs having an acyclic k-coloring. Let us describe the construction of the *2-edge-colored Zielonka graph SZ_k* corresponding to the case $m = 2$. Every vertex is of the form $(i; \alpha_1, \alpha_2, \ldots, \alpha_k)$, where $1 \leqslant i \leqslant k$, $\alpha_j \in \{-1, +1\}$ for $j \neq i$, and $\alpha_i = 0$. Thus,

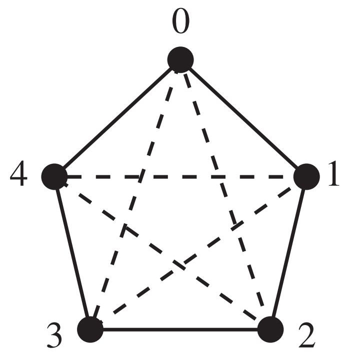


FIGURE 3. The 2-edge-colored graph SP_5.

$|V(SZ_k)| = k \cdot 2^{k-1}$. For $i \neq j$, there is an edge between the vertices $(i; \alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(j; \beta_1, \beta_2, \ldots, \beta_k)$ and the sign of this edge is the product $\alpha_j \times \beta_i$.

Proposition 2.4. *The graph SZ_k is antitwinned.*

Proof. We denote by s the signature of SZ_k. By definition of an antitwinned graph, we have to show that every vertex of SZ_k has a unique antitwin. To prove that the antitwin of the vertex $u = (i; \alpha_1, \alpha_2, \ldots, \alpha_k)$ is the vertex $u' = (i; -\alpha_1, -\alpha_2, \ldots, -\alpha_k)$, we check that for every edge uv, the edge $u'v$ exists and that $s(u'v) = -s(uv)$. If uv is an edge, then $v = (j; \beta_1, \beta_2, \ldots, \beta_k)$ for some $j \neq i$ and thus $s(u'v) = (-\alpha_j) \times \beta_i = -(\alpha_j \times \beta_i) = -s(uv)$. ■

C. The 2-Edge-Colored Paley Graph SP_q

In the remainder of this section, q is any prime power such that $q \equiv 1 \pmod 4$. We denote by $\mathbb{F}_q$ the unique (up to isomorphism) finite field of order q. Let g be a generator of the multiplicative group $\mathbb{F}_q^*$ and let $\mathrm{sq} : \mathbb{F}_q^* \to \{-1, +1\}$ be the function *square* defined as $\mathrm{sq}(v) = +1$ if and only if v is a square of $\mathbb{F}_q$. Note that g is necessarily a nonsquare, so that

$$\mathrm{sq}(g^t) = (-1)^t. \tag{1}$$

The *Paley graph P_q* is the undirected graph with vertex set $V(P_q) = \mathbb{F}_q$ and edge set $E(P_q) = \{xy \mid \mathrm{sq}(y - x) = +1\}$. Since -1 is a square in $\mathbb{F}_q$, $\mathrm{sq}(x - y) = \mathrm{sq}(y - x)$ and therefore the definition of an edge is consistent. A Paley graph is vertex-transitive, arc-transitive, and self-complementary [12], that is, it is isomorphic to its complement.

A *strongly regular* graph with parameters (n, k, λ, μ) is a k-regular graph G with n vertices such that (1) every two adjacent vertices have λ common neighbors and (2) every two nonadjacent vertices have μ common neighbors. Paley graphs P_q are known to be strongly regular graphs with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.

We define the *2-edge-colored Paley graph $SP_q = (K_q, s)$* as the complete graph on q vertices such that $V(SP_q) = \mathbb{F}_q$ and $s(uv) = \mathrm{sq}(u - v)$. That is, SP_q is obtained from the Paley graph P_q by replacing nonedges by negative edges. Figure 3 represents the 2-edge-colored Paley graph SP_5.

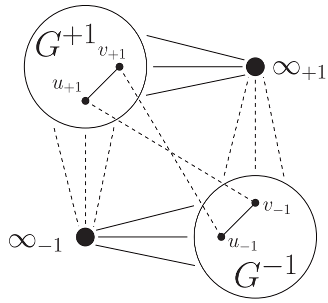


FIGURE 4. The 2-edge-colored graph $Tr(G)$.

An *antiautomorphism* of a 2-edge-colored graph (G, s) is a permutation ρ of $V(G)$ such that for every edge uv, $\rho(u)\rho(v)$ is an edge and $s(\rho(u)\rho(v)) = -s(uv)$. Then (G) is *antiautomorphic* if it admits an antiautomorphism.

Since P_q is vertex-transitive, arc-transitive, and self-complementary, SP_q is vertex-transitive, arc-transitive, and antiautomorphic.

D. The 2-Edge-Colored Tromp–Paley Graph $Tr(SP_q)$

Given an oriented graph $\overrightarrow{G}$, Tromp [13] proposed a construction of an oriented graph $Tr(G)$ called the *Tromp graph*. We adapt this construction to 2-edge-colored graphs as follows.

Given a 2-edge-colored graph (G), we denote by $(G^\bullet)$ the graph obtained from (G) by adding a universal vertex, denoted by ∞, that is positively linked to all the vertices of (G).

The *Tromp 2-edge-colored graph* $Tr(G)$ of (G) is then defined as the 2-edge-colored graph $AT(G^\bullet)$. By construction, $Tr(G)$ is obtained from two copies (G^{+1}) and (G^{-1}) of (G) and the vertices ∞_{+1} and ∞_{-1} (see Figure 4).

Given a 2-edge-colored graph (G, s), let $Tr(G)$ be the Tromp graph of (G) with signature s'. Let $u_i v_j \in Tr(G)$ such that $u \neq \infty$. If $v \neq \infty$, then

$$s'(u_i v_j) = i \times j \times s(uv). \tag{2}$$

Otherwise

$$s'(u_i \infty_j) = i \times j. \tag{3}$$

Lemma 2.5. *If (G) is antiautomorphic, then $AT(G)$ and $Tr(G)$ are antiautomorphic.*

Proof. Let s be the signature of G and s' be the signature of $Tr(G)$. Let ρ be an antiautomorphism of (G). We define the mapping $\gamma : V(Tr(G)) \to V(Tr(G))$ as:

$$\gamma : u_i \to \begin{cases} \infty_{-i} & \text{if } u = \infty \\ (\rho(u))_i & \text{if } u \neq \infty. \end{cases}$$

Let us check that γ maps every edge $u_i v_j \in E(Tr(G))$ to an edge of opposite sign. If $u, v \neq \infty$, then γ maps $u_i v_j$ to $\rho(u)_i \rho(v)_j$ and we have by (2):

$$s'(\rho(u)_i \rho(v)_j) = i \times j \times s(\rho(u)\rho(v)) = i \times j \times (-s(uv)) = -s'(u_i v_j).$$

If $u \neq \infty$ and $v = \infty$, then γ maps $u_i\infty_j$ to $\rho(u)_i\infty_{-j}$ and we have by (3):

$$s'(\rho(u)_i\infty_{-j}) = i \times (-j) = -(i \times j) = -s'(u_i\infty_j).$$

Since the restriction of γ to $V(AT(G))$ is well defined, the same proof applies to $AT(G)$. ■

In the remainder of this section, we focus on the Tromp–Paley graph $Tr(SP_q)$ obtained by applying the Tromp construction to the 2-edge-colored Paley graph SP_q. It has $2q + 2$ vertices denoted u_i such that $u \in \{0, 1 = g^0, g, g^2, \ldots, g^{q-2}, \infty\}$ and $i \in \{-1, +1\}$ (recall that g is a generator of the multiplicative group $\mathbb{F}_q^*$). We denote by s the signature of SP_q and by s' the signature of $Tr(SP_q)$. Note that if $u_iv_j \in Tr(SP_q)$, $u \neq \infty$, and $v \neq \infty$, then $s'(u_iv_j) = i \times j \times \text{sq}(u - v)$ by (2) since $s(uv) = \text{sq}(u - v)$.

The graph $Tr(SP_q)$ has remarkable symmetries and some useful properties given below.

Lemma 2.6. *The 2-edge-colored graph $Tr(SP_q)$ is vertex-transitive.*

Proof. To prove that $Tr(SP_q)$ is vertex-transitive, we show that every vertex u can be mapped to ∞_{+1}. Recall that SP_q is vertex-transitive and arc-transitive. Moreover, for every vertex $w_i \in V(Tr(SP_q))$, either $w = \infty$, $w = 0$, or $w = g^t$ for some t.

If φ is an automorphism of SP_q, we define the corresponding automorphism γ_φ of $Tr(SP_q)$ as:

$$\gamma_\varphi : u_i \to \begin{cases} u_i & \text{if } u = \infty \\ (\varphi(u))_i & \text{if } u \neq \infty. \end{cases}$$

We also define the mapping $\gamma_\infty : V(Tr(SP_q)) \to V(Tr(SP_q))$ as:

$$\gamma_\infty : u_i \to \begin{cases} \infty_i & \text{if } u = 0 \\ 0_i & \text{if } u = \infty \\ g^{-t}_{i\times(-1)^t} & \text{if } u = g^t. \end{cases}$$

Let us check that γ_∞ is an automorphism of $Tr(SP_q)$, that is, that for every edge u_iv_j, we have $\gamma_\infty(u_i)\gamma_\infty(v_j) \in E(Tr(SP_q))$ and $s'(\gamma_\infty(u_i)\gamma_\infty(v_j)) = s'(u_iv_j)$.

If $u = g^t$ and $v = g^r$, then

$$\begin{aligned} s'(\gamma_\infty(u_i)\gamma_\infty(v_j)) &= s'\big(g^{-t}_{i\times(-1)^t} g^{-r}_{j\times(-1)^r}\big) \\ &= i \times (-1)^t \times j \times (-1)^r \times \text{sq}(g^{-t} - g^{-r}) && \text{by (2)} \\ &= i \times j \times \text{sq}(g^{t+r}) \times \text{sq}(g^{-t} - g^{-r}) && \text{by (1)} \\ &= i \times j \times \text{sq}(g^{t+r}(g^{-t} - g^{-r})) \\ &= i \times j \times \text{sq}(g^r - g^t) \\ &= i \times j \times \text{sq}(g^t - g^r) \\ &= s'(g^t_i g^r_j) && \text{by (2)} \\ &= s'(u_iv_j). \end{aligned}$$

If $u = g^t$ and $v = 0$, then

$$\begin{aligned}
s'(\gamma_\infty(u_i)\gamma_\infty(v_j)) &= s'\big(g^{-t}_{i\times(-1)^t}\infty_j\big) \\
&= i \times (-1)^t \times j && \text{by (3)} \\
&= i \times j \times \mathrm{sq}(g^t) && \text{by (1)} \\
&= i \times j \times \mathrm{sq}(g^t - 0) \\
&= s'(g^t_i 0_j) && \text{by (2)} \\
&= s'(u_i v_j).
\end{aligned}$$

If $u = g^t$ and $v = \infty$, then

$$\begin{aligned}
s'(\gamma_\infty(u_i)\gamma_\infty(v_j)) &= s'\big(g^{-t}_{i\times(-1)^t}0_j\big) \\
&= i \times (-1)^t \times j \times \mathrm{sq}(g^{-t} - 0) && \text{by (2)} \\
&= i \times (-1)^t \times j \times (-1)^{-t} && \text{by (1)} \\
&= i \times j \\
&= s'\big(g^t_i \infty_j\big) && \text{by (3)} \\
&= s'(u_i v_j).
\end{aligned}$$

If $u = 0$ and $v = \infty$, then

$$\begin{aligned}
s'(\gamma_\infty(u_i)\gamma_\infty(v_j)) &= s'(\infty_i 0_j) \\
&= i \times j && \text{by (3)} \\
&= s'(0_i \infty_j) && \text{by (3)} \\
&= s'(u_i v_j).
\end{aligned}$$

Given a vertex $u \in SP_q$, we can choose φ so that $\varphi(u) = 0$. Thus, $\gamma_\varphi(u_i) = 0_i$ for every $u \neq \infty$. Also, $\gamma_\infty(0_i) = \infty_i$ and $\mathrm{atw}(\infty_{-1}) = \infty_{+1}$. Hence, we can map every vertex of $Tr(SP_q)$ to ∞_{+1} by combining the automorphisms γ_φ, γ_∞, and atw. So $Tr(SP_q)$ is vertex-transitive. ■

Lemma 2.7. *$Tr(SP_q)$ is triangle-transitive.*

Proof. There are four types of triangles (a, b, c) according to the number of positive edges in their signature. Without loss of generality, the edges incident to c have the same sign β and we denote by α the sign of ab. Thus, every type of triangle is characterized by a couple $(\alpha, \beta) \in \{-1, +1\}^2$. We have to prove that for every two triangles of the same type, there exists an automorphism of $Tr(SP_q)$ that maps one triangle to the other.

We first prove that every triangle (a, b, c) of type $(+1, \beta)$ maps to the triangle $0_{+1}1_{+1}\infty_\beta$. By Lemma 2.6, $Tr(SP_q)$ is vertex-transitive, so there exists an automorphism φ that maps c to ∞_β. Every edge of sign β incident to ∞_β has its other extremity in SP_q^{+1}. Thus, φ maps the edge ab to a positive edge $u_{+1}v_{+1}$ in SP_q^{+1}. Since SP_q is arc-transitive, we can finally map $u_{+1}v_{+1}$ to $0_{+1}1_{+1}$.

Let $\overline{Tr(SP_q)}$ be obtained from $Tr(SP_q)$ by changing the sign of every edge. Since SP_q is antiautomorphic, $Tr(SP_q)$ is also antiautomorphic by Lemma 2.5. So $\overline{Tr(SP_q)}$ is isomorphic to $Tr(SP_q)$. Let us fix $\beta \in \{-1, +1\}$ and let T_1 and T_2 be two triangles of type $(-1, \beta)$ in $Tr(SP_q)$. So T_1 and T_2 are triangles of type $(+1, -\beta)$ in $\overline{Tr(SP_q)}$. By the previous case, there exists an automorphism that maps T_1 to T_2 in $\overline{Tr(SP_q)}$. This automorphism also maps T_1 to T_2 in $Tr(SP_q)$. ■

E. Coloring Properties of Target Graphs

A *sign vector* of size k is a k-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{-1, +1\}^k$. The *negation* of α is the sign vector $-\alpha = (-\alpha_1, -\alpha_2, \ldots, -\alpha_k)$. Given a sequence of k distinct vertices $X_k = (v_1, v_2, \ldots, v_k)$ of a 2-edge-colored graph (G, s) that induces a clique in G, an *α-neighbor* of X_k is a vertex $u \in V(G)$ such that $s(v_i u) = \alpha_i$ for $1 \leqslant i \leqslant k$. The set of α-neighbors of X_k is denoted by $N^\alpha(X_k)$. Thus, the notion of α-neighbor generalizes to sequences of vertices the notions of positive and negative neighbors of a vertex.

Consider the 2-edge-colored graph SP_5 depicted in Figure 3. For example, given $\alpha = (+1, -1)$ and $X = (0, 3)$, the vertex 1 is an α-neighbor of X, the vertex 2 is a $(-\alpha)$-neighbor of X, and thus $N^\alpha(X) = \{1\}$ and $N^{-\alpha}(X) = \{2\}$.

A 2-edge-colored graph (G) has property $P_{k,l}$ if $|N^\alpha(X_k)| \geqslant l$ for every sequence X_k of k distinct vertices inducing a clique in G and for every sign vector α of size k.

Lemma 2.8. *If SP_q has property $P_{n-1,k}$, then $Tr(SP_q)$ has property $P_{n,k}$.*

Proof. Suppose that SP_q has property $P_{n-1,k}$ and let $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$ be a given sign vector. Let $X = (x_1, \ldots, x_{n-1}, x_n)$ be a sequence of n distinct vertices inducing a clique of $Tr(SP_q)$. We have to prove that X admits $k\,\alpha$-neighbors. By Lemma 2.6, $Tr(SP_q)$ is vertex-transitive and we assume without loss of generality that $x_n = \infty_{+1}$. Notice that $N^{-\alpha}(X) = \{atw(v) \parallel v \in N^{-\alpha}(X)\}$. So $|N^{-\alpha}(X)| = |N^\alpha(X)|$ and we also assume without loss of generality that $\alpha_n = +1$. Since X is a clique, we have $x_i \notin \{\infty_{+1}, \infty_{-1}\}$ for $1 \leqslant i \leqslant n-1$. We define $Y = (y_1, \ldots, y_{n-1})$ such that $y_i = x_i$ if $x_i \in SP_q^{+1}$ and $y_i = \mathrm{atw}(x_i)$ if $x_i \in SP_q^{-1}$. Hence, the vertices in Y belong to SP_q^{+1}. We define $\beta = (\beta_1, \ldots, \beta_{n-1})$ such that, for $1 \leqslant i \leqslant n-1$, $\beta_i = \alpha_i$ if and only if $x_i = y_i$. By Property $P_{n-1,k}$ of SP_q, there exist $k\,\beta$-neighbors $v_1, v_2, \ldots, v_k$ of Y in SP_q^{+1}. The v_i's are positive neighbors of ∞_{+1}, so they are $(\beta_1, \ldots, \beta_{n-1}, +1)$-neighbors of $(y_1, \ldots, y_{n-1}, \infty_{+1})$. Hence X has $k\,\alpha$-neighbors. ■

Lemma 2.9. *If (G) is a 2-edge-colored graph and $Tr(G)$ has property $P_{n,k}$, then $AT(G)$ has property $P_{n,k-1}$.*

Proof. Recall that $AT(G)$ is obtained from two isomorphic copies of (G) and then $Tr(G)$ is obtained from $AT(G)$ by adding the antitwin vertices ∞_{+1} and ∞_{-1}. Let X be a sequence of n distinct vertices inducing a clique in $AT(G)$. Since $Tr(G)$ has property $P_{n,k}$, then $N^\alpha(X)$ contains k vertices of $Tr(G)$ for every sign vector α of length n. However, $N^\alpha(X)$ cannot contain both ∞_{+1} and ∞_{-1}, since they are antitwins. So X has at least $k-1\alpha$-neighbors in $AT(G)$, which means that $AT(G)$ has property $P_{n,k-1}$. ■

Lemma 2.10.

(1) *SP_q has properties $P_{1,(q-1)/2}$ and $P_{2,(q-5)/4}$.*
(2) *$Tr(SP_q)$ has properties $P_{1,q}$, $P_{2,(q-1)/2}$, and $P_{3,(q-5)/4}$.*
(3) *$AT(SP_q)$ has properties $P_{1,q-1}$, $P_{2,(q-3)/2}$, and $P_{3,\max(0,(q-9)/4)}$.*

Proof.

(1) These properties follow from the fact that the 2-edge-colored Paley graph SP_q is built from the Paley graph P_q, which is self-complementary, arc-transitive, and strongly regular with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.

(2) $Tr(SP_q)$ has property $P_{1,q}$ since it is vertex-transitive by Lemma 2.6 and the vertex ∞_{+1} has q positive and q negative neighbors. The other properties follow from Lemma 2.10.(2.10) and Lemma 2.8. ■
(3) These properties follow from Lemma 2.10.(2.10) and Lemma 2.9.

3. RESULTS ON 2-EDGE-COLORED HOMOMORPHISMS

This section is devoted to 2-edge-colored homomorphisms of planar graphs and outerplanar graphs. Recall that a graph (H) is $\mathcal{C}$-universal if every 2-edge-colored graph (G) of the class $\mathcal{C}$ admits a 2-edge-colored homomorphism to (H).

An *acyclic k-coloring* is a proper vertex-coloring such that every cycle has at least three colors. In other words, the graph induced by every two color classes is a forest. Let $\mathcal{A}_k$ be the class of graphs that admit a k-acyclic coloring.

In 1998, Alon and Marshall [1] proved the following (the 2-edge-colored graph SZ_k has $k \cdot 2^{k-1}$ vertices and has been considered in Section 2.2):

Theorem 3.1 ([1]). *SZ_k is $\mathcal{A}_k$-universal. Thus $\chi_2(\mathcal{A}_k) \leqslant k \cdot 2^{k-1}$.*

Huemer et al. [6] then proved that this bound cannot be improved:

Theorem 3.2 ([6]). *For every $k \geqslant 1$, there exists a graph $G_k \in \mathcal{A}_k$ such that $\chi_2(G_k) = k \cdot 2^{k-1}$. Thus $\chi_2(\mathcal{A}_k) = k \cdot 2^{k-1}$.*

Borodin [2] proved that every planar graph admits an acyclic 5-coloring. Thus, Alon and Marshall deduced the following from Theorem 3.1:

Corollary 3.3 ([1]). *SZ_5 is $\mathcal{P}_3$-universal. Thus $\chi_2(\mathcal{P}_3) \leqslant 80$.*

In this same context, Borodin et al. [3] and Montejano et al. [7] obtained the following results:

Theorem 3.4 ([3, 7]).

(1) *$Tr(SP_9)$ is $\mathcal{P}_5$-universal. Thus $\chi_2(\mathcal{P}_5) \leqslant 20$ [7].*
(2) *$Tr(SP_5)$ is $\mathcal{P}_6$-universal. Thus $\chi_2(\mathcal{P}_6) \leqslant 12$ [7].*
(3) *$SP_9 \setminus \{0\}$ is $\mathcal{P}_8$-universal. Thus $\chi_2(\mathcal{P}_8) \leqslant 8$ [7].*
(4) *SP_5 is $\mathcal{P}_{13}$-universal. Thus $\chi_2(\mathcal{P}_{13}) \leqslant 5$ [3].*

In this section, we obtain antitwinned target graphs for triangle-free outerplanar graphs (Theorem 3.5) and triangle-free planar graphs (Theorem 3.6). The latter result gives a new upper bound on the 2-edge-colored chromatic number. Then, we give properties that must be satisfied by the target graphs for outerplanar graphs (Theorem 3.8) and planar graphs (Theorem 3.10). Finally, we obtain new lower bounds on the 2-edge-colored chromatic number of triangle-free planar graphs (Theorem 3.11) and planar graphs with girth at least 7 (Theorem 3.12).

Theorem 3.5. *$AT(SP_5 \setminus \{0\})$ is $\mathcal{O}_4$-universal.*

Proof. Assume by contradiction that (H) is a minimal counterexample to the result, that is, (H) is in $\mathcal{O}_4$, (H) does not map to $AT(SP_5 \setminus \{0\})$, and every proper subgraph of (H) maps to $AT(SP_5 \setminus \{0\})$.

Suppose that H contains a $^{\leqslant}1$-vertex u. By minimality, the graph $(H') = (H \setminus \{u\})$ admits a 2-edge-colored homomorphism to $AT(SP_5 \setminus \{0\})$. Since every vertex of

$AT(SP_5 \setminus \{0\})$ is incident to a positive and a negative edge, we can extend the 2-edge-colored homomorphism to (H), a contradiction.

Suppose that H contains two adjacent 2-vertices u and v. By minimality, the graph $(H') = (H \setminus \{u, v\})$ admits a 2-edge-colored homomorphism to $AT(SP_5 \setminus \{0\})$. We have checked that for every pair of (not necessarily distinct) vertices x and y of $AT(SP_5 \setminus \{0\})$, the eight possible 2-edge-colored 3-paths exist. Therefore, we can extend the 2-edge-colored homomorphism to (H), a contradiction.

Pinlou and Sopena [10] have shown that every outerplanar graph with girth at least k and minimum degree at least 2 contains a face of length $l \geqslant k$ with at least $(l-2)$ consecutive 2-vertices. Therefore, H is not a triangle-free outerplanar graph. This contradiction completes the proof. ■

Theorem 3.6. *$AT(SP_{25})$ is $\mathcal{P}_4$-universal. Thus $\chi_2(\mathcal{P}_4) \leqslant 50$.*

Let $n_3(G)$ be the number of $^{\geqslant}3$-vertices in the graph G. Let us define the partial order . Given two graphs G_1 and G_2, we have $G_1 G_2$ if and only if one of the following conditions holds:

- $n_3(G_1) < n_3(G_2)$.
- $n_3(G_1) = n_3(G_2)$ and $|V(G_1)| + |E(G_1)| < |V(G_2)| + |E(G_2)|$.

Note that the partial order is well defined and is an extension of the partial ordering by minors.

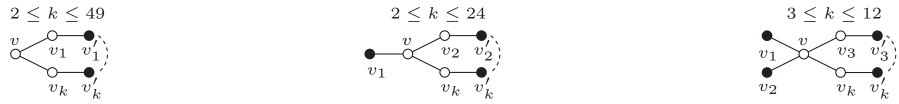
Let (H) be a 2-edge-colored graph that does not admit a homomorphism to the 2-edge-colored graph $AT(SP_{25})$ and such that its underlying graph H is a triangle-free planar graph that is minimal with respect to . In the following, H is given with its embedding in the plane. A *weak 7-vertex u* in H is a 7-vertex adjacent to four 2-vertices $v_1, \ldots, v_4$ and three $^{\geqslant}3$-vertices w_1, w_2, w_3 such that $v_1, w_1, v_2, w_2, v_3, w_3$, and v_4 are clockwise consecutive.

Lemma 3.7. *The graph H does not contain the following configurations:*

- (C1) *a $^{\leqslant}1$-vertex;*
- (C2) *a k-vertex adjacent to k 2-vertices for* $2 \leqslant k \leqslant 49$*;*
- (C3) *a k-vertex adjacent to $(k-1)$ 2-vertices for* $2 \leqslant k \leqslant 24$*;*
- (C4) *a k-vertex adjacent to $(k-2)$ 2-vertices for* $3 \leqslant k \leqslant 12$*;*
- (C5) *a 3-vertex;*
- (C6) *a k-vertex adjacent to $(k-3)$ 2-vertices for* $4 \leqslant k \leqslant 6$*;*
- (C7) *two vertices u and v linked by two distinct 2-paths, both paths having a 2-vertex as internal vertex;*
- (C8) *a 4-face $wxyz$ such that x is 2-vertex, w and y are weak 7-vertices, and z is a k-vertex adjacent to $(k-4)$ 2-vertices for* $4 \leqslant k \leqslant 9$.

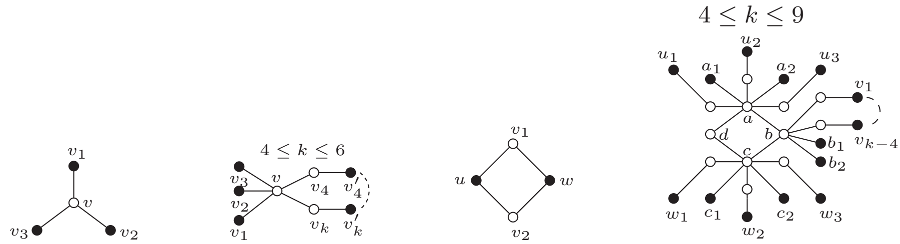
Proof. Configurations C2–C8 are depicted in Figures 5 and 6. The drawing conventions for a configuration Ck contained in a graph H are as follows. The neighbors of a white vertex in H are exactly its neighbors in Ck, whereas a black vertex may have other neighbors in H. Two or more black vertices in Ck may coincide in a single vertex in H, provided they do not share a common white neighbor.

For each configuration, we suppose that H contains the configuration and we consider a 2-edge-colored triangle-free planar graph (H') such that . We only argue that for configuration C5. For every other configuration, H' is a minor of H and thus . By minimality of



(a) C2 (b) C3 (c) C4

FIGURE 5. Configurations C2–C4.



(a) C5 (b) C6 (c) C7 (d) C8

FIGURE 6. Configurations C5–C8.

H, (H') admits a 2-edge-colored homomorphism f to $AT(SP_{25})$. We modify and extend f to obtain a 2-edge-colored homomorphism of (H) to $AT(SP_{25})$, contradicting the fact that (H) is a counterexample.

By Lemma 2.10, $AT(SP_{25})$ satisfies $P_{1,24}$, $P_{2,11}$, and $P_{3,4}$.

Proof of configuration C1. Trivial.

Proof of configuration C2. Suppose that H contains the configuration depicted in Figure 5a and f is a 2-edge-colored homomorphism of $(H') = (H) \setminus \{v, v_1, \ldots, v_k\}$ to $AT(SP_{25})$. For every i, if the edges vv_i and v_iv_i' have the same sign (resp. different signs), then v must get a color distinct from $\text{atw}(f(v_i'))$ (resp. $f(v_i')$). So, each v_i' forbids at most one color for v. Thus there remains an available color for v. Then we extend f to the vertices v_i using property $P_{2,11}$ if $f(v_i') \neq f(v)$ or $P_{1,24}$ if $f(v_i') = f(v)$.

Proof of configuration C3. Suppose that H contains the configuration depicted in Figure 5 b and f is a 2-edge-colored homomorphism of $(H') = (H) \setminus \{v, v_2, \ldots, v_k\}$ to $AT(SP_{25})$. As shown in the proof of Configuration C2, each v_i' forbids at most one color for v. So, we have at most 23 forbidden colors for v and by property $P_{1,24}$, there remains at least one available color for v. Then we extend f to the vertices v_i $(2 \leqslant i \leqslant k)$ using property $P_{2,11}$.

Proof of configuration C4. Suppose that H contains the configuration depicted in Figure 5c and f is a 2-edge-colored homomorphism of $(H') = (H) \setminus \{v_3, \ldots, v_k\}$ to $AT(SP_{25})$. As shown in the proof of Configuration C2, each v_i' forbids at most one color for v. So, we have at most 10 forbidden colors for v and by property $P_{2,11}$ applied to $f(v_1)$, $f(v_2)$ this remains at least one available color in order to recolor v. Then we extend f to the vertices v_i $(3 \leqslant i \leqslant k)$ using property $P_{2,11}$.

Proof of configuration C5. Suppose that H contains the configuration depicted in Figure 6 a. Let (H') be the graph obtained from (H) by deleting the vertex v and by adding, for every $1 \leqslant i < j \leqslant 3$, a new vertex v_{ij} and the edges v_iv_{ij} and $v_{ij}v_j$. Each of the six edges v_iv_{ij} gets the sign α_i of the edge v_iv in (H). Since configuration C4 is forbidden in

H, v_1, v_2, and v_3 are $\geqslant$3-vertices. We have since $n_3(H') < n_3(H)$. Clearly, H' is triangle-free. Hence, there exists a 2-edge-colored homomorphism f of (H') to $AT(SP_{25})$. By $P_{3,4}$, we can find an α-neighbor u of $(f(v_1), f(v_2), f(v_3))$ in $AT(SP_{25})$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Now fix $f(v) = u$. Note that f restricted to $V(H)$ is a homomorphism of (H) to $AT(SP_{25})$.

Proof of configuration C6. Suppose that H contains the configuration depicted in Figure 6b and f is a 2-edge-colored homomorphism of $(H') = (H) \setminus \{v_4, \dots, v_k\}$ to $AT(SP_{25})$. As shown in the proof of Configuration C2, each v_i' forbids at most one color for v. So, we have at most three forbidden colors for v and by property $P_{3,4}$ applied to $f(v_1)$, $f(v_2)$, $f(v_3)$, there remains at least one available color for v. Then we extend f to the vertices v_i $(4 \leqslant i \leqslant k)$ using property $P_{2,11}$.

Proof of configuration C7. Suppose that H contains the configuration depicted in Figure 6c.

If u and w have no common neighbor other than v_1 and v_2, then we consider the graph (H') obtained from $(H) \setminus \{v_1, v_2\}$ by adding the positive edge uw.

If u and w have at least one other common neighbor v_3, then consider the graph (H') obtained from $(H) \setminus \{v_1, v_2\}$ by adding a vertex v adjacent to u and w such that uv is negative and the sign of vw is the product of the signs of uv_3 and v_3w. Therefore, we have at least two 2-paths linking u and w, one whose both edges have the same sign and one whose edges have different signs.

In both cases, H' is triangle-free, planar, and is a minor of H, so that (H') admits a 2-edge-colored homomorphism f to $AT(SP_{25})$. Also, in both cases, $f(u)$ and $f(w)$ form an edge in $AT(SP_{25})$ since $f(u) \neq f(w)$ and $f(u) \neq f(\text{atw}(w))$. Thus, the homomorphism of $(H) \setminus \{v_1, v_2\}$ induced by f can be extended to (H) using property $P_{2,11}$.

Proof of configuration C8. Suppose that H contains the configuration depicted in Figure 6d. By Corollary 2.3, (H) admits a 2-edge-colored homomorphism to $AT(SP_{25})$ if and only if every 2-edge colored graph that is switching equivalent to (H) admits a 2-edge-colored homomorphism to $AT(SP_{25})$. So, by switching a subset of vertices in $\{a, b, c, c_1, c_2\}$, we can assume that the edges da, ab, bc, cc_1, and cc_2 are positive. Consider a 2-edge-colored homomorphism f of $(H') = (H \setminus \{d\})$ to $AT(SP_{25})$. The edge dc in (H) has to be negative, since otherwise f would be extendable to (H) by setting $f(d) = f(b)$. Also, we must have $f(c) = f(a)$, since otherwise we could color d using property $P_{2,11}$. In the remainder of the proof, we show that we can modify f such that $f(c) \neq f(a)$. For $1 \leqslant i \leqslant 3$, let k_i denote the color that is forbidden for c by w_i, that is, $k_i = f(w_i)$ if the edges of the 2-path linking c and w_i have distinct signs and $k_i = \text{atw}(f(w_i))$ otherwise.

First, we rule out the cases in which $f(c_1)$, $f(c_2)$, and $f(b)$ have at least five common positive neighbors, since this allows to choose for c a color that is not in $\{k_1, k_2, k_3, f(a)\}$. If $f(c_1)$, $f(c_2)$, and $f(b)$ are not pairwise distinct, then they have at least 11 common positive neighbors by property $P_{2,11}$. So we assume that $f(c_1)$, $f(c_2)$, and $f(b)$ are distinct. We define the sequence $X = (f(c_1), f(c_2), f(b))$ and the sign vector $\alpha = (+1, +1, +1)$. Recall that $AT(SP_{25})$ contains two copies SP_{25}^{+1} and SP_{25}^{-1} of SP_{25}. We consider the graph $Tr(SP_{25})$ obtained by adding the antitwin vertices ∞_{+1} and ∞_{-1} to $AT(SP_{25})$. Using the triangle-transitivity of $Tr(SP_{25})$, a quick computer check shows that X has five α-neighbors in $Tr(SP_{25})$ if X induces a triangle with three positive edges and X has six α-neighbors otherwise. Notice that ∞_{+1} and ∞_{-1} cannot be both α-neighbors of X. Thus, if X does not induce three positive edges, then X has at least five α-neighbors in

TABLE II. Sets of the form $(0_{+1}, 1_{+1}, \beta_{+1})$ having exactly four $(+1, +1, +1)$-neighbors in $AT(SP_{25})$

$(0_{+1}, 1_{+1}, 2_{+1})$	3_{+1},	4_{+1},	$(1+2\sqrt{2})_{-1}$,	$(1+3\sqrt{2})_{-1}$
$(0_{+1}, 1_{+1}, 3_{+1})$	2_{+1},	4_{+1},	$(3+\sqrt{2})_{-1}$,	$(3+4\sqrt{2})_{-1}$
$(0_{+1}, 1_{+1}, 4_{+1})$	2_{+1},	3_{+1},	$(2\sqrt{2})_{-1}$,	$(3\sqrt{2})_{-1}$
$(0_{+1}, 1_{+1}, (3+2\sqrt{2})_{+1})$	$(3+\sqrt{2})_{-1}$,	$(3\sqrt{2})_{-1}$,	$(1+3\sqrt{2})_{-1}$,	$(3+4\sqrt{2})_{-1}$
$(0_{+1}, 1_{+1}, (3+3\sqrt{2})_{+1})$	$(3+\sqrt{2})_{-1}$,	$(2\sqrt{2})_{-1}$,	$(1+2\sqrt{2})_{-1}$,	$(3+4\sqrt{2})_{-1}$

$AT(SP_{25})$. Also, if X is not contained in one copy of SP_{25} of the subgraph $AT(SP_{25})$, then neither ∞_{+1} nor ∞_{-1} is an α-neighbor of X and thus X has at least five α-neighbors in $AT(SP_{25})$.

So X is contained in one copy of SP_{25}, say SP_{25}^{+1}. We represent the field $\mathbb{F}_{25}$ by the numbers $x + y\sqrt{2}$, where x and y are integers modulo 5. Without loss of generality, we can assume that $f(c_1) = 0_{+1}$ and $f(c_2) = 1_{+1}$ since $f(c_1)f(c_2)$ is a positive edge and SP_{25} is arc-transitive. Moreover, X induces three positive edges, so $f(b) = \beta_{+1}$, where β is in the set B of positive neighbors of 0 and 1 in SP_{25}. We thus have $\beta \in B = \{2, 3, 4, 3+2\sqrt{2}, 3+3\sqrt{2}\}$. Table II gives the suitable sequences and their four α-neighbors in $AT(SP_{25})$.

We are now ready to modify f. We decolor the vertices a, b, and c. By property $P_{2,11}$, there exist at least two colors for b that are distinct from the colors forbidden by the k vertices $v_1, \ldots, v_{k-4}, a_1, a_2, c_1, c_2$. By previous discussions, these two colors are β_{+1} and β'_{+1} with $\{\beta, \beta'\} \subset B$. Let us first set $f(b) = \beta_{+1}$. By property $P_{3,4}$, we can color a such that $f(a)$ is distinct from the colors forbidden by u_1, u_2, and u_3. Since f is not extendable to c and d, the four α-neighbors of $(0_{+1}, 1_{+1}, \beta_{+1})$ are k_1, k_2, k_3, and $f(a)$. In particular, k_1, k_2, and k_3 are α-neighbors of $(0_{+1}, 1_{+1}, \beta_{+1})$. Now we set $f(b) = \beta'_{+1}$ and obtain that k_1, k_2, and k_3 are α-neighbors of $(0_{+1}, 1_{+1}, \beta'_{+1})$ as well. This is a contradiction, since no two distinct sequences in Table II have three common α-neighbors. ■

Proof of Theorem 3.6. Let (H) be a counterexample that is minimal with respect to . By Lemma 3.7, H does not contain any of the configurations $C1$–$C8$. It remains to show that every triangle-free planar graph contains at least one of these configurations. This has been already done using a discharging procedure in the proof of Theorem 2 in [9], where slightly weaker configurations were used. ■

Montejano et al. [7] proved that SP_9 is $\mathcal{O}_3$-universal. They also construct an outerplanar graph G such that $\chi_2(G) = 9$, so that $\chi_2(\mathcal{O}_3) = 9$. We precise their result as follows:

Theorem 3.8. *The only $\mathcal{O}_3$-universal graph of order 9 is SP_9.*

We say that a 2-edge-colored path or cycle is *alternating* if it has an even number of edges and if every two consecutive edges have different signs. Let (P) be the alternating 6-path.

Observation 3.9. $\chi_2(P) = 4$. *Moreover, if (P) maps to a graph (T_4) with four vertices, then (T_4) contains no monochromatic triangle.*

Indeed, if (P) maps to a graph T_4 with four vertices, then (T_4) must contain an alternating 4-cycle. So (T_4) cannot contain a monochromatic triangle.

Proof of Theorem 3.8. Let (H) be the outerplanar graph consisting in a universal vertex u positively linked to every vertex of a copy (P_p) of (P) and negatively linked to every vertex of a copy (P_n) of (P). So $|V(H)| = 15$. By Observation 3.9, every homomorphic image of (H) uses at least four colors for the vertices of (P_p), at least four other colors for the vertices of (P_n), and an additional color for u. Therefore $\chi_2(H) \geqslant 9$.

Let (G) be the outerplanar graph obtained from 16 copies $(H_0), (H_1), \ldots, (H_{15})$ of (H) as follows: we identify each of the 15 vertices of (H_0) with the vertex u of a copy (H_i). Consider a graph (T_9) with nine vertices such that (G) maps to (T_9). In every 2-edge-colored homomorphism of (G) to (T_9), each of the nine colors appears on a vertex of (H_0) since $\chi_2(H_0) \geqslant 9$. Since a copy of (H) is attached to every vertex of (H_0), every vertex c of (T_9) satisfies $|N^+(c)| = |N^-(c)| = 4$. By Observation 3.9, $N^+(c)$ and $N^-(c)$ contain no monochromatic triangle.

Let s be the signature of (T_9). We denote the vertices of (T_9) by $v_{i,j}$ with $(i, j) \in \mathbb{Z}_3^2$. Since (G) contains positive triangles, (H) necessarily contains a positive triangle. Without loss of generality, the edges of the triangle $v_{0,0}v_{0,1}v_{0,2}$ are positive. Let ∂ denote the edge cut between the vertices $v_{0,j}$ and the vertices $v_{i,j}$ with $i \neq 0$. Since every vertex $v_{0,j}$ satisfies $|N^+(v_{0,j})| = 4$ and $s(v_{0,j}v_{0,j-1}) = s(v_{0,j}v_{0,j+1}) = +1$, every vertex $v_{0,j}$ is incident to exactly two positive edges in ∂. So ∂ contains exactly six positive edges. Now, every vertex $v_{i,j}$ with $i \neq 0$ must be positively linked to at least one vertex $v_{0,j}$, since otherwise $N^-(v_{i,j})$ would contain the monochromatic triangle $v_{0,0}v_{0,1}v_{0,2}$, which is forbidden. Since ∂ contains exactly six positive edges, each of the six vertices $v_{i,j}$ with $i \neq 0$ is incident to exactly one positive edge in ∂. Without loss of generality, we can assume that $s(v_{0,j}v_{1,j}) = s(v_{0,j}v_{2,j}) = +1$ for every $j \in \mathbb{Z}_3$, and thus that every other edge in ∂ is negative. Now, the edge $v_{1,0}v_{2,0}$ must be positive, since otherwise $N^+(v_{0,0})$ would contain the monochromatic triangle $v_{0,1}v_{1,0}v_{2,0}$, which is forbidden . By symmetry, we thus have $s(v_{1,j}v_{2,j}) = +1$ for every $j \in \mathbb{Z}_3$. Therefore, assuming that a vertex v of (T_9) belongs to a positive triangle (e.g., $v_{0,0}$ belongs to $v_{0,0}v_{0,1}v_{0,2}$) implies that v belongs to two positive triangles that intersect only at v (e.g., $v_{0,0}$ also belongs to $v_{0,0}v_{1,0}v_{2,0}$). Since we have proved that every vertex v of (T_9) belongs to a positive triangle, v belongs to two positive triangles that intersect only at v.

The set of edges of (T_9) whose sign is not yet determined induces $K_{2,2,2}$. Since $|N^+(v_{i,j})| = 4$, this $K_{2,2,2}$ must contain six positive edges that induce two triangles. Without loss of generality, these two positive triangles are $v_{1,0}v_{1,1}v_{1,2}$ and $v_{2,0}v_{2,1}v_{2,2}$.

Notice that every edge $v_{i,j}v_{i',j'}$ of (T_9) is positive if and only if $i = i'$ or $j = j'$, so (T_9) is SP_9. ■

Concerning planar graphs, it is known that $\chi_2(\mathcal{P}_3) \geqslant 20$ [7, Corollary 9]. The following result is similar to Theorem 3.8 for planar graphs.

Theorem 3.10. *If there exists an antitwinned $\mathcal{P}_3$-universal graph of order 20, then that graph is $Tr(SP_9)$.*

Proof. Suppose that every 2-edge-colored planar graph maps to an antitwinned graph (H_{20}). Let (G) be the outerplanar graph considered in the proof of Theorem 3.8. Then the planar graph $(G^\bullet)$ maps to (H_{20}) and the subgraph (G) of $(G^\bullet)$ maps to the positive neighborhood of some vertex v of (H_{20}). Since (H_{20}) is antitwinned, every vertex has at most nine positive neighbors. Therefore, by Theorem 3.8, the positive neighborhood of v is isomorphic to SP_9. Then the subgraph of (H_{20}) induced by v and its positive neighborhood is $SP_9^\bullet$. Since $SP_9^\bullet$ is a clique of order 10, it does not contain a

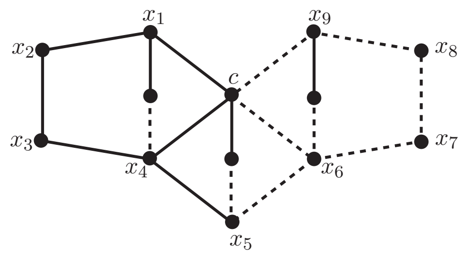


FIGURE 7. The graph (G) of Theorem 3.11.

pair of antitwin vertices and thus (H_{20}) is isomorphic to $AT(SP_9^\bullet)$. By the definition of 2-edge-colored Tromp–Paley graphs, (H_{20}) is isomorphic to $Tr(SP_9)$. ■

Concerning triangle-free planar graphs, we obtain a new lower bound.

Theorem 3.11. *There does not exist a $\mathcal{P}_4$-universal graph of order 11. Thus* $\chi_2(\mathcal{P}_4) \geqslant 12$.

Proof. A vertex u in a 2-edge-colored graph is *good* if u is contained in a positive cycle of odd length, in a negative cycle of odd length, and in a 4-cycle $upvn$ such that pu and pv are positive and nu and nv are negative (these cycles are not necessarily induced). We have checked, both by hand and by computer, that

(i) no graph with at most five vertices contains a good vertex,
(ii) no graph with six vertices contains six good vertices.

We consider the plane graph $(G) \in \mathcal{P}_4$ depicted in Figure 7. If (G) admits a 2-edge-colored homomorphism h to a graph (H) such that the outer face of (G) maps to a subgraph (H') of (H), then $h(c)$ is good in (H'). We construct the plane graph $(G') \in \mathcal{P}_4$ from 10 copies $(G_0), (G_1), \ldots, (G_9)$ of (G) by identifying the vertex c of (G_i) with the vertex x_i of (G_0), for $1 \leqslant i \leqslant 9$. The embedding of (G') is such that the x_j's of each of the 10 copies are on the outer face. Thus, the outer face of (G') contains 91 vertices.

Suppose for contradiction that there exists a 2-edge-colored homomorphism of (G') to a graph (H) such that the outer face of (G') maps to a subgraph (H') of (H) such that (H') has at most six vertices. Since (G') contains (G) as a subgraph, (H') contains a good vertex. So (H') contains exactly six vertices by (i). Moreover, every vertex on the outer face of (G_0) corresponds to the vertex c of some copy (G_i) of (G). Thus, all the six vertices of (H') must be good. This contradicts (ii). Therefore, if the graph (G') maps to some graph (H), then its outer face maps to some subgraph (H') of (H) of order at least 7.

Now, we finish the proof. In a 2-edge-colored graph, two distinct vertices u and v are *friends* if $|N^\alpha(u, v)| \geqslant 2$ for every $\alpha \in \{-1, +1\}^2$. Notice that for every homomorphism h of the graph (F) depicted in Figure 8, the vertices $h(t)$ and $h(b)$ must be friends in the target graph. Consider a 2-edge-colored graph $(J) \in \mathcal{P}_4$ such that $\chi_2(J) = \chi_2(\mathcal{P}_4)$. We construct (J') by adding to every vertex u of (J) a copy (G'_u) of (G') and by connecting each of the 91 vertices on the outer face of (G'_u) to u using a copy of (F). Recall that in every homomorphic image of (G'_u), at least seven distinct colors appear on the outer face. So, if $\chi_2(\mathcal{P}_4) = k$, then (J') must map to a 2-edge-colored clique with k vertices such that every vertex has at least seven friends. We have checked by computer that no

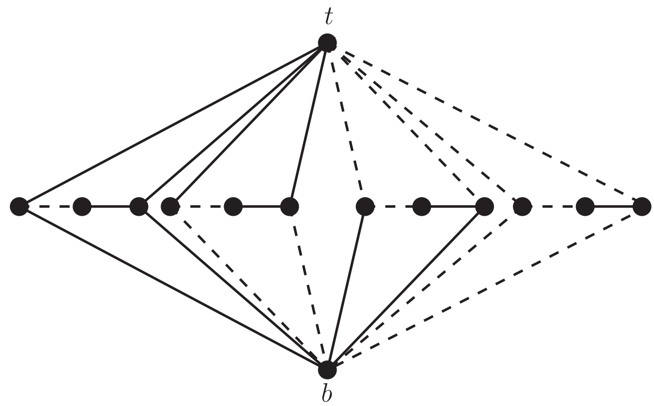


FIGURE 8. The graph (F) of Theorem 3.11.

2-edge-colored clique with at most 11 vertices is such that every vertex has at least seven friends. ■

Finally, concerning planar graphs with girth at least 7, the following theorem gives a new lower bound.

Theorem 3.12. *There does not exist a $\mathcal{P}_7$-universal graph of order 7. Thus $\chi_2(\mathcal{P}_7) \geqslant 8$.*

Proof. Consider a 2-edge-colored graph (J) such that $J \in \mathcal{P}_7$ and $\chi_2(J) = \chi_2(\mathcal{P}_7)$. We construct (J') from (J) as follows. For every vertex u in (J) and for every pair $(s_c, s_p) \in \{-1, +1\}^2$, we add a 7-cycle whose edges have sign s_c and connect u to every vertex of this 7-cycle using a path with three edges of sign s_p.

If $\chi_2(\mathcal{P}_7) = k$, then (J') must map to a 2-edge-colored clique with k vertices such that for every vertex u and every $s_p \in \{-1, +1\}$, the graph induced by the vertices reachable by a walk of three edges of sign s_p starting from u contains both a positive and a negative cycle of odd length. We have checked by computer that no 2-edge-colored clique with at most seven vertices satisfies this property. ■

4. RESULTS ON SWITCHING HOMOMORPHISM

The graph classes considered in Theorems 3.1, 3.4, 3.5, and 3.6 admit a 2-edge-colored homomorphism to an antitwinned graph. Using Lemma 2.2, we obtain good upper bounds on χ_{sw} for these classes.

Naserasr et al. [8] proved the following.

Theorem 4.1 ([8]). *Let G be a graph that admits an acyclic k-coloring. Then $\chi_{\mathrm{sw}}(G) \leqslant \lceil k/2 \rceil \cdot 2^{k-1}$.*

By Theorem 3.1, SZ_k is $\mathcal{A}_k$-universal. By Proposition 2.4 and Lemma 2.2, we obtain the following improvement of Theorem 4.1.

Theorem 4.2. *Let G_k be the graph in Theorem 3.2. For every $k \geqslant 2$, $\chi_{\mathrm{sw}}(G_k) = \chi_{\mathrm{sw}}(\mathcal{A}_k) = k \cdot 2^{k-2}$.*

We also obtain the following results.

TABLE III. Bounds on χ_{sw}

Class	Lower bound	Upper bound	Target
$\mathcal{O}_g$, for every $g \geqslant 4$	4	4	$SP_5 \setminus \{0\}$
$\mathcal{P}_3$	10	40	Half of $\mathcal{A}_5$
$\mathcal{P}_4$	6	25	SP_{25}
$\mathcal{P}_5$	4	10	$SP_9^\bullet$
$\mathcal{P}_6$	4	6	$SP_5^\bullet$

Theorem 4.3.

(1) *Every outerplanar graph of girth 4 admits a switching homomorphism to $SP_5 \setminus \{0\}$.*
(2) *Let (H) be such that $SZ_5 = AT(H)$. Every planar graph admits a switching homomorphism to (H).*
(3) *Every planar graph of girth 4 admits a switching homomorphism to SP_{25}.*
(4) *Every planar graph of girth 5 admits a switching homomorphism to $SP_9^\bullet$.*
(5) *Every planar graph of girth 6 admits a switching homomorphism to $SP_5^\bullet$.*

Proof. Every statement is of the form "every graph in $\mathcal{C}$ admits a switching homomorphism to (J)." By Lemma 2.2, it is equivalent to the statement "every graph in $\mathcal{C}$ admits a 2-edge-colored homomorphism to $AT(J)$."

- Item (1) follows from Theorem 3.5.
- Item (2) follows from Corollary 3.3 (i.e., planar graphs map to SZ_5) and Proposition 2.4 (i.e., SZ_5 is antitwinned).
- Item (3) follows from Theorem 3.6.
- Item (4) follows from Theorem 3.4(3.4).
- Item (5) follows from Theorem 3.4(3.4). ■

Concerning lower bounds, Naserasr et al. [8] constructed a planar graph G such that $\chi_{\mathrm{sw}}(G) = 10$. This result also follows from $\chi_2(\mathcal{P}_3) \geqslant 20$ in [7] and Lemma 2.2. Moreover, we obtain the following from Theorem 3.10 and Lemma 2.2.

Corollary 4.4. *If every 2-edge-colored planar graph admits a switching homomorphism to a graph of order 10, then that graph is switching equivalent to $SP_9^\bullet$.*

By Theorem 3.11, there exists a bipartite planar graph (G) such that $\chi_{\mathrm{sw}}(G) \geqslant 6$. Finally, for higher girths, note that $\chi_{\mathrm{sw}}(C_{2k}) = 4$ for even cycles with exactly one negative edge. The results discussed in this section are summarized in Table III.

5. CONCLUSION

One of our aims was to introduce and study some relevant target graphs for 2-edge-colored homomorphism and switching homomorphism. We have considered the graph $AT(G)$, 2-edge-colored Zielonka graph SZ_k, 2-edge-colored Paley graph SP_q, and 2-edge-colored Tromp–Paley graph $Tr(SP_q)$. Theorems 3.8 and 3.10 suggest that these target graphs are indeed interesting. Theorem 3.10 leads to the following question.

Open Problem 5.1. *Is $Tr(SP_9)$ $\mathcal{P}_3$-universal ?*

This would imply that $\chi_2(\mathcal{P}_3) = 20$ and $\chi_{sw}(\mathcal{P}_3) = 10$. We have checked by computer that every 4-connected planar triangulation with at most 15 vertices admits a homomorphism to $Tr(SP_9)$. The restriction to 4-connected triangulations (i.e., triangulations without separating triangles) is justified by Lemma 2.7. For the 2^{25} nonequivalent signatures of each of the 6,244 4-connected planar triangulations with 15 vertices, our computer check took 150 CPU-days. Checking 4-connected triangulations with more vertices would require too much computing power.

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