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An oriented coloring of planar graphs with girth at least five

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ABSTRACT

An oriented k-coloring of an oriented graph G is a homomorphism from G to an oriented graph H of order k. We prove that every oriented graph with a maximum average degree less than $\frac{10}{3}$ and girth at least 5 has an oriented chromatic number at most 16. This implies that every oriented planar graph with girth at least 5 has an oriented chromatic number at most 16, that improves the previous known bound of 19 due to Borodin et al. [O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, É. Sopena, On the maximum average degree and the oriented chromatic number of a graph, Discrete Math. 206 (1999) 77–89].

1. Introduction

Oriented graphs are directed graphs without loops or opposite arcs. For an oriented graph G, we denote by $V(G)$ the vertices and by $A(G)$ its set of arcs. For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v or simply uv whenever its orientation is not relevant (therefore, $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$). The number of vertices of G is the *order* of G.

An *oriented k-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of k colors such that (1) $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv}$ is an arc in G, and (2) $\varphi(u) \neq \varphi(x)$ whenever $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in G with $\varphi(v) = \varphi(w)$. In other words, an oriented k-coloring of G is a partition of the vertices of G into k stable sets $S_1, S_2, \ldots, S_k$ such that all the arcs between any pair of stable sets S_i and S_j have the same direction (either from S_i to S_j, or from S_j to S_i). The *oriented chromatic number* of an oriented graph, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-coloring.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs: $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An oriented k-coloring of G can be equivalently defined as a homomorphism from G to H, where H is an oriented graph of order k. The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The oriented chromatic number of G can then be defined as the smallest order of an oriented graph H such that $G \to H$. Links between colorings and homomorphisms are presented in more details in the recent monograph [6] by Hell and Nešetřil.

The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given girth) [13,15], planar graphs (with given girth) [1–4,10,14], graphs with bounded maximum average degree [3,4], graphs with bounded degree [7], graphs with bounded treewidth [11,15,16], and graph subdivisions [18].

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A very challenging question in this area is to determine the oriented chromatic number of planar graphs. Raspaud and Sopena [14] proved in 1994 that their oriented chromatic number is at most 80. Recently, Marshall [8] proved that there exist planar graphs with an oriented chromatic number at least 17. The gap between the lower and the upper bound is very large, but it seems very hard to reduce.

In this paper, we focus on the oriented chromatic number of graphs with bounded maximum average degree. The *average degree* of a graph G, denoted by $\mathrm{ad}(G)$, is defined as twice the number of edges over the number of vertices ($\mathrm{ad}(G) = \frac{2|E(G)|}{|V(G)|}$). The *maximum average degree* of G, denoted by $\mathrm{mad}(G)$, is then defined as the maximum of the average degrees taken over all subgraphs of G:

$$\mathrm{mad}(G) = \max_{H \subseteq G}\{\mathrm{ad}(H)\}.$$

The *girth* of a graph G is the length of a shortest cycle of G.

Borodin et al. [3,4] gave bounds of the oriented chromatic number of graphs with bounded maximum average degree:

Theorem 1 (*[3,4]*). *Let G be a graph.*

(1) *If* $\mathrm{mad}(G) < \frac{12}{5}$ *and G has girth at least* 5*, then* $\chi_o(G) \le 5$ [3].
(2) *If* $\mathrm{mad}(G) < \frac{11}{4}$ *and G has girth at least* 5*, then* $\chi_o(G) \le 7$ [4].
(3) *If* $\mathrm{mad}(G) < 3$*, then* $\chi_o(G) \le 11$ [4].
(4) *If* $\mathrm{mad}(G) < \frac{10}{3}$*, then* $\chi_o(G) \le 19$ [4].

We focus here on the class of graphs with maximum average degree less than $\frac{10}{3}$ and girth at least 5. The main result of this paper is given by the following theorem:

Theorem 2. *Let G be a graph with* $\mathrm{mad}(G) < \frac{10}{3}$ *and girth at least* 5*. Then* $\chi_o(G) \le 16$.

Actually, we prove a stronger result: we show that every oriented graph G with $\mathrm{mad}(G) < \frac{10}{3}$ and girth at least 5 admits a homomorphism to T_{16}, where T_{16} is the Tromp graph of order 16 whose construction is described in Section 2.

When considering planar graphs, the maximum average degree and the girth are linked by the following well-known relation:

Claim 3 (*[4]*). *Let G be a planar graph with girth g. Then,* $\mathrm{mad}(G) < 2 + \frac{4}{g-2}$.

In particular, by means of Theorem 1(4), we get as a corollary that every planar graph with girth at least 5 has an oriented chromatic number at most 19. Theorem 2 improves this bound and gives that every planar graph with girth at least 5 has an oriented chromatic number at most 16.

The best current knowledge for the upper bounds of the oriented chromatic number of planar graphs is then the following:

Theorem 4 (*[1–4,11,14]*). *Let G be a planar graph.*

(1) *If G has girth at least* 12*, then* $\chi_o(G) \le 5$ [3].
(2) *If G has girth at least* 11*, then* $\chi_o(G) \le 6$ [11].
(3) *If G has girth at least* 7*, then* $\chi_o(G) \le 7$ [1].
(4) *If G has girth at least* 6*, then* $\chi_o(G) \le 11$ [4].
(5) *If G has girth at least* 5*, then* $\chi_o(G) \le 16$.
(6) *If G has girth at least* 4*, then* $\chi_o(G) \le 47$ [2].
(7) *If G has no restriction of girth, then* $\chi_o(G) \le 80$ [14].

Note that among the bounds of the previous theorem, only the one for girth 12 is tight.

In the remainder, we use the following notations. For a vertex v of a graph G, we denote by $d_G^-(v)$ its *indegree*, by $d_G^+(v)$ its *outdegree*, and by $d_G(v)$ its degree (subscripts are omitted when the considered graph is clearly identified from the context). We denote by $N_G^+(v)$ the set of outgoing neighbors of v, by $N_G^-(v)$ the set of incoming neighbors of v and by $N_G(v) = N_G^+(v) \cup N_G^-(v)$ the set of neighbors of v. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. $^{\ge}$*k-vertex*, $^{\le}$*k-vertex*). If a vertex u is adjacent to a k-vertex (resp. $^{\ge}k$-vertex, $^{\le}k$-vertex) v, then v is a *k-neighbor* (resp. $^{\ge}$*k-neighbor*, $^{\le}$*k-neighbor*) of u. A path of length k (i.e. formed by k edges) is called a *k-path*. If two graphs G and H are isomorphic, we denote it by $G \cong H$.

The paper is organized as follows. The next section is devoted to the target graph T_{16} and some of its properties. We prove Theorem 2 in Section 3. We finally give some concluding remarks in the last section.

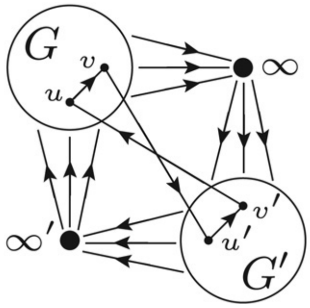


Fig. 1. The Tromp graph $Tr(G)$.

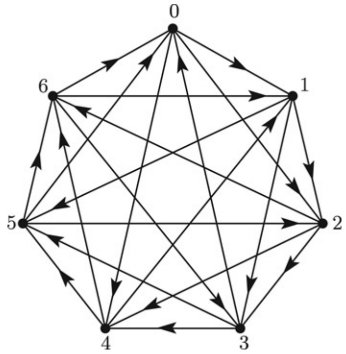


Fig. 2. The graph QR_7.

2. The Tromp graph T_{16}

In this section, we describe the construction of the target graph T_{16} used to prove Theorem 2 and give some useful properties.

Tromp's construction was proposed by Tromp [17]. Let G be an oriented graph and G' be an isomorphic copy of G. The Tromp graph $Tr(G)$ has $2|V(G)|+2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$.

Fig. 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between u and u'. Such pairs of vertices will be called *twin vertices*, and we denote by $t(u)$ the twin vertex of u. Remark that $t(t(u)) = u$. This notion can be extended to sets in a standard way: for a given $W \subseteq V(G)$, $W = \{v_1, v_2, \ldots, v_k\}$, then $t(W) = \{t(v_1), t(v_2), \ldots, t(v_k)\}$.

By construction, the graph $Tr(G)$ satisfies the following property:

$$\forall u \in Tr(G) : N^+(u) = N^-(t(u)) \quad \text{and} \quad N^-(u) = N^+(t(u)).$$

In the remainder, we focus on the specific graph family obtained via the Tromp's construction applied to Paley tournaments. For a prime power $p \equiv 3 (\text{mod } 4)$, the *Paley tournament* QR_p is defined as the oriented graph whose vertices are the integers modulo p and such that $\overrightarrow{uv}$ is an arc if and only if $v-u$ is a non-zero quadratic residue of p. For instance, the Paley tournament QR_7 has vertex set $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r (\text{mod } 7)$ for $r \in \{1, 2, 4\}$; see Fig. 2. Note that the bounds of Theorems 1(2), 1(3), and 1(4) have been obtained by proving that all the graphs of the considered classes admit a homomorphism to the Paley tournaments QR_7, QR_{11}, and QR_{19}, respectively.

Let $T_{16} = Tr(QR_7)$ be the Tromp graph on sixteen vertices obtained from QR_7. In the remainder of this paper, the vertex set of T_{16} is $\{0, 1, \ldots, 6, \infty, 0', 1', \ldots, 6', \infty'\}$ where $\{0, 1, \ldots, 6\}$ is the vertex set of the first copy of QR_7 and $\{0', 1', \ldots, 6'\}$ is the vertex set of the second copy of QR_7; thus, for every $u \in \{0, 1, \ldots, 6, \infty\}$, we have $t(u) = u'$. In addition, for every $u \in V(T_{16})$, we have by construction $|N^+_{T16}(u)| = |N^-_{T_{16}}(u)| = 7$. The graph T_{16} has remarkable symmetry and some useful properties given below.

Proposition 5 ([8]). *For any QR_p, the graph $Tr(QR_p)$ is such that:*

$$\forall u \in V(Tr(QR_p)) : N^+(u) \cong QR_p \quad \textit{and} \quad N^-(u) \cong QR_p.$$

Proposition 6 ([8]). *For any QR_p, if $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ span triangles t_1 and t_2 respectively in $Tr(QR_p)$ and the map ψ taking a_i to b_i ($1 \le i \le 3$) is an isomorphism $t_1 \to t_2$, then ψ can be extended to an automorphism of $Tr(QR_p)$.*

It is then clear that $Tr(QR_p)$ is vertex-transitive and arc-transitive.

Proposition 7. *Let G be an oriented graph such that $G \to T_{16}$. Then, for any vertex v of G, the graph G' obtained from G by reversing the orientation of every arc incident to v admits a homomorphism to T_{16}.*

Proof. Let φ be a T_{16}-coloring of G. For every $w \in V(T_{16})$, we have $N^+_{T_{16}}(w) = N^-_{T_{16}}(t(w))$ and $N^-_{T_{16}}(w) = N^+_{T_{16}}(t(w))$. Therefore, the mapping $\varphi' : V(G') \to V(T_{16})$ defined by $\varphi'(u) = \varphi(u)$ for all $u \in V(G') \setminus \{v\}$ and $\varphi'(v) = t(\varphi(v))$ is clearly a T_{16}-coloring of G'. $\square$

An *orientation n-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0, 1\}^n$ of n elements. Let $S = (v_1, v_2, \dots, v_n)$ be a sequence of n (not necessarily distinct) vertices of T_{16}; a vertex u is said to be an *α-successor of S* if for any i, $1 \le i \le n$, we have $\overrightarrow{uv_i} \in A(T_{16})$ whenever $\alpha_i = 1$ and $\overrightarrow{v_iu} \in A(T_{16})$ otherwise. For instance, the vertex $3'$ of T_{16} is a $(1, 1, 0, 1, 0, 0)$-successor of $(1, 2, 6', 1, \infty, 2')$ since the arcs $\overrightarrow{3'1}$, $\overrightarrow{3'2}$, $\overrightarrow{6'3'}$, $\overrightarrow{\infty 3'}$, and $\overrightarrow{2'3'}$ belong to $A(T_{16})$.

If, for a sequence $S = (v_1, v_2, \dots, v_n)$ of n vertices of T_{16} and an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, there exist $i \ne j$ such that $v_i = v_j$ and $\alpha_i \ne \alpha_j$, then there does not exist any α-successor of S; indeed, T_{16} does not contain opposite arcs. In addition, if there exist $i \ne j$ such that $v_i = t(v_j)$ and $\alpha_i = \alpha_j$, then there does not exist any α-successor of S; indeed, for any pair of vertices x and y of T_{16} with $x = t(y)$, we have $N^+_{T_{16}}(x) \cap N^+_{T_{16}}(y) = \emptyset$ and $N^-_{T_{16}}(x) \cap N^-_{T_{16}}(y) = \emptyset$. A sequence $S = (v_1, v_2, \dots, v_n)$ of n vertices of T_{16} is said to be *compatible* with an orientation n-vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if for any $i \ne j$, we have $\alpha_i \ne \alpha_j$ whenever $v_i = t(v_j)$, and $\alpha_i = \alpha_j$ whenever $v_i = v_j$. Note that if the n vertices of S induce an n-clique subgraph of T_{16} (i.e. $v_1, v_2, \dots, v_n$ are pairwise distinct and induce a complete graph), then S is compatible with any orientation n-vector since a vertex u and its twin $t(u)$ cannot belong together to the same clique.

In the remainder, we say that T_{16} has *Property $P_{n,k}$* if, for every sequence S of n distinct vertices of T_{16} and any orientation n-vector α which is compatible with S, there exist k α-successors of S. This set of k α-successors is denoted by $\mathrm{Succ}_\alpha(S)$.

Proposition 8. *The graph T_{16} has Properties $P_{1,7}$, $P_{2,3}$, and $P_{3,1}$.*

Proof. (1) Property $P_{1,7}$ is trivial since every vertex of T_{16} has seven successors and seven predecessors.

(2) To prove that T_{16} has Property $P_{2,3}$, we have to show that, for every sequence $S = (u, v)$ and any compatible orientation 2-vector α, there exist at least three α-successors of S. We have two cases to consider: the case $uv \in A(T_{16})$ and the case $u = t(v)$. Since T_{16} is arc-transitive, we will consider w.l.o.g. $S = (0, 1)$ and $S = (\infty, \infty')$.

A case study shows that the three α-successors of $S = (0, 1)$ are 2, $6'$, and ∞ (resp. $2'$, 6, and ∞'; $3'$, 4, and $5'$; 3, $4'$, and 5) if $\alpha = (0, 0)$ (resp. $(1, 1)$; $(0, 1)$; $(1, 0)$).

Consider now the case $S = (\infty, \infty')$. By definition, the only two compatible orientation 2-vectors with S are $(0, 1)$ and $(1, 0)$. It is then clear by construction of T_{16} that we have seven α-successors of S in each case.

(3) Property $P_{3,1}$ was proved by Marshall [8]. $\square$

Proposition 9. *Let u, v_1, and v_2 be three distinct vertices of T_{16}, and $S_i = (u, v_i)$ for every $1 \le i \le 2$. Let α be an orientation 2-vector compatible with S_1 and S_2. Then $\mathrm{Succ}_\alpha(S_1) \ne \mathrm{Succ}_\alpha(S_2)$.*

Proof. Suppose to the contrary that there exist such S_1 and S_2 with $\mathrm{Succ}_\alpha(S_1) = \mathrm{Succ}_\alpha(S_2)$.

By Proposition 7, we may assume w.l.o.g. that $\alpha_2 = 0$. If $v_1 = t(v_2)$, we clearly have $\mathrm{Succ}_\alpha(S_1) \ne \mathrm{Succ}_\alpha(S_2)$ since $N^+_{T_{16}}(v_1) \cap N^+_{T_{16}}(v_2) = \emptyset$. Thus, we may assume w.l.o.g. that $v_1v_2 \in A(T_{16})$, and since T_{16} is arc-transitive, we assume w.l.o.g. that $v_1 = 0$ and $v_2 = 1$.

Therefore, the vertices of $\mathrm{Succ}_\alpha(S_1) = \mathrm{Succ}_\alpha(S_2)$ must be the common successors of 0 and 1. We have $N^+_{T_{16}}(0) \cap N^+_{T_{16}}(1) = \{2, 6', \infty\}$. If $\alpha_1 = 0$, then a case study allows us to check that T_{16} has no vertex u distinct from 0 and 1 having 2, $6'$, and ∞ as successors. Therefore, we should have $\alpha_1 = 1$ and then we can check that u should necessarily be either $0'$ or $1'$. However, in each case, we will have $|\mathrm{Succ}_\alpha(S_i)| = 7$ and $|\mathrm{Succ}_\alpha(S_{3-i})| = 3$ for some $i \in [1, 2]$. $\square$

3. Proof of Theorem 2

In this section, we prove Theorem 2, that is that every graph G with $\mathrm{mad}(G) < \frac{10}{3}$ and girth at least 5 admits a homomorphism to T_{16}.

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $^{\geq}3$-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- G_1 is a proper subgraph of G_2;
- $n_3(G_1) < n_3(G_2)$.

Note that this partial order is well-defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \le n_3(G_2)$. So $\preceq$ is a partial linear extension of the subgraph poset.

Let H be a hypothetical minimal counterexample to Theorem 2 according to $\prec$. We first prove that H does not contain a set of thirteen configurations. Then, using a discharging procedure, we show that every graph with girth 5 which contains none of these thirteen configurations has an average degree at least $\frac{10}{3}$; this implies that H has $\mathrm{mad}(H) \ge \frac{10}{3}$, a contradiction.

3.1. Structural properties of H

A *weak 5-vertex* is a 5-vertex adjacent to three 2-vertices. A *weak* 4-*vertex* is a 4-vertex adjacent to one 2-vertex.

Lemma 10. *The graph H does not contain the following configurations:*

(C1) *a* $^{\leq}1$-*vertex;*
(C2) *a k-vertex adjacent to $(k-2)$ 2-vertices for $3 \leq k \leq 4$;*
(C3) *a k-vertex adjacent to $(k-1)$ 2-vertices for $2 \leq k \leq 7$;*
(C4) *a k-vertex adjacent to k 2-vertices for $1 \leq k \leq 15$;*
(C5) *a 3-vertex;*
(C6) *a k-vertex adjacent to $(k-2)$ 2-vertices and one weak 5-vertex for $5 \leq k \leq 6$;*
(C7) *a 4-vertex adjacent to three weak 5-vertices;*
(C8) *a weak 5-vertex adjacent to two weak 4-vertices;*
(C9) *a 5-vertex adjacent to two 2-vertices and two weak 5-vertices;*
(C10) *a 5-vertex adjacent to one 2-vertex and four weak 5-vertices;*
(C11) *a 6-vertex adjacent to three 2-vertices and three weak 5-vertices;*
(C12) *a 7-vertex adjacent to five 2-vertices and two weak 5-vertices;*
(C13) *an 8-vertex adjacent to seven 2-vertices and one weak 5-vertex.*

The drawing conventions for a *configuration* C contained in a graph G are the following. If u and v are two vertices of C, then they are adjacent in G if and only if they are adjacent in C. Moreover, the neighbors of a *white* vertex in G are exactly its neighbors in C, whereas a *black* vertex may have neighbors outside of C. Two or more black vertices in C may coincide in a single vertex in G, provided they do not share a common white neighbor. Finally, an edge will represent an arc with any of its two possible orientations.

Let G be an oriented graph, v be a k-vertex with $N(v) = \{v_1, v_2, \ldots, v_k\}$ and α be an orientation k-vector such that $\alpha_i = 0$ whenever $\overrightarrow{v_i v} \in A(G)$ and $\alpha_i = 1$ otherwise. Let φ be a T_{16}-coloring of $G \setminus \{v\}$ and $S = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_k))$. Recall that a necessary condition to have α-successors of S is that α must be compatible with S, that is for any pair of vertices v_i and v_j, $\varphi(v_i) \neq \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $\varphi(v_i) \neq t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$. Hence, every vertex v_j forbids one color for each vertex v_i, $i \in [1, k]$, $i \neq j$. We define $f_{v_i}^{\varphi}(v_j)$ to be the forbidden color for v_i by $\varphi(v_j)$ (i.e. $f_{v_i}^{\varphi}(v_j) = \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $f_{v_i}^{\varphi}(v_j) = t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$). Therefore, α is compatible with S if and only if we have $\varphi(v_i) \neq f_{v_i}^{\varphi}(v_j)$ for every pair i, j, $1 \leq i < j \leq k$. Note that if $\varphi(v_i) \neq f_{v_i}^{\varphi}(v_j)$, then we necessarily have $\varphi(v_j) \neq f_{v_j}^{\varphi}(v_i)$.

For each configuration, we suppose that H contains it and we consider a reduction H' with a girth at least 5 such that $H' \prec H$ and $\text{mad}(H') < \frac{10}{3}$; therefore, by minimality of H, H' admits a T_{16}-coloring φ. We will then show that we can choose φ so that it can be extended to H by Proposition 8, contradicting the fact that H is counterexample.

In what follows, if H contains a configuration, then H^* will denote the graph obtained from H be removing all the white vertices of this configuration.

Proof of Configurations (C1)–(C4). Trivial. □

For Configurations (C1)–(C4), the reductions H' have been obtained from H by removing some vertices and/or arcs; therefore, we clearly had $\text{mad}(H') \leq \text{mad}(H)$. To prove that Configuration (C5) is forbidden in H, we considered a reduction H' obtained from H by removing one 3-vertex and by adding new vertices and arcs. The following lemma shows that this reduction H' has nevertheless a maximum average degree less that $\frac{10}{3}$.

Let G be a graph containing a 3-vertex v adjacent to three vertices u_1, u_2, and u_3; see Fig. 4(a). We denote by $R(G)$ the graph obtained from $G \setminus \{v\}$ by adding 2-paths joining respectively u_1 and u_2, u_2 and u_3, and u_3 and u_1; see Fig. 4(b).

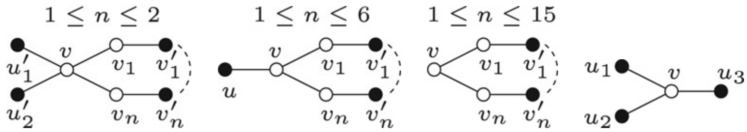
Lemma 11. *If* $\text{mad}(G) < \frac{10}{3}$, *then* $\text{mad}(R(G)) < \frac{10}{3}$.

The proof of this lemma is left to the reader.

Proof of Configuration (C5). Suppose that H contains the configuration depicted in Fig. 3(d). Since Configurations (C1) and (C2) are forbidden, u_1, u_2, and u_3 are $^{\geq}3$-vertices. Let H' be the graph obtained from H^* by adding, for every $1 \leq i < j \leq 3$, a 2-path joining u_i to u_j in such a way that its orientation is the same orientation of the path $[u_i, v, u_j]$ in H. We have $H' \prec H$ since $n_3(H') = n_3(H) - 1$, $\text{mad}(H') < \frac{10}{3}$ by Lemma 11, and H' has clearly girth at least 5. Any T_{16}-coloring φ of H' induces a coloring of H^* such that $\varphi(u_i) \neq f_{u_i}^{\varphi}(u_j)$ for any i, j, $1 \leq i < j \leq 3$. □

Configurations (C6)–(C13) all contain a weak 5-vertex. To shorten the proofs, we will often use the following lemma, later called Main Lemma.

Lemma 12 (*Main Lemma*). *Let G be an oriented graph containing a weak 5-vertex u (see Fig. 5) and let φ be a T_{16}-coloring of G^*. Then, for a fixed coloring of u'_1, u'_2, u'_3, and v_1, at most two colors are forbidden for v_2.*



(a) C2. (b) C3. (c) C4. (d) C5.

Fig. 3. Configurations C2–C5.

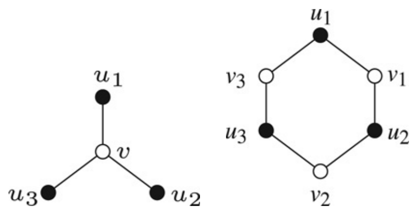

(a) The graph G. (b) The graph $R(G)$.

Fig. 4. Configurations of Lemma 11.

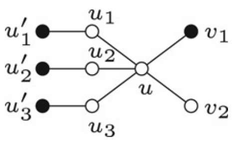


Fig. 5. Configuration of Lemma 12.

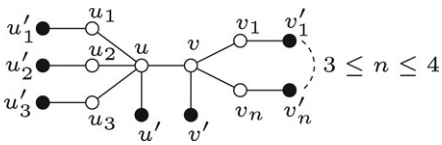


Fig. 6. Configuration (C6): a k-vertex adjacent to $(k-2)$ 2-vertices and one weak 5-vertex for $5 \leq k \leq 6$.

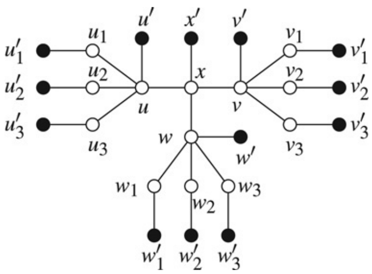


Fig. 7. Configuration (C7): a 4-vertex adjacent to three weak 5-vertices.

Proof. The color $\varphi(v_1)$ together with each of the fifteen colors for v_2 distinct from $f_1 = f^{\varphi}_{v_2}(v_1)$ give three possible colors for u by Property $P_{2,3}$. Proposition 9 ensures that at most one of these fifteen colors, say f_2, gives the three colors $f^{\varphi}_u(u'_1)$, $f^{\varphi}_u(u'_2)$, and $f^{\varphi}_u(u'_3)$ for u. Thus, for any $\varphi(v_2) \notin \{f_1, f_2\}$, we have three available colors for u whose one of them is distinct from $f^{\varphi}_u(u'_1)$, $f^{\varphi}_u(u'_2)$, and $f^{\varphi}_u(u'_3)$. □

Proof of Configuration (C6). Suppose that H contains the configuration depicted in Fig. 6 and let φ be a T_{16}-coloring of H^*. By Main Lemma, the weak 5-vertex u forbids two colors for v, say f_1 and f_2. By Property $P_{1,7}$, we can choose φ such that $\varphi(v) \notin \{f_1, f_2, f^{\varphi}_v(v'_1), \ldots, f^{\varphi}_v(v'_n)\}$. □

Proof of Configuration (C7). Suppose that H contains the configuration depicted in Fig. 7 and let φ be a T_{16}-coloring of H^*. By Main Lemma, each of the weak 5-vertices u, v, and w forbids two colors for x, say $f_1, f_2, \ldots, f_6$. By Property $P_{1,7}$, we can choose φ such that $\varphi(x) \notin \{f_1, f_2, \ldots, f_6\}$. □

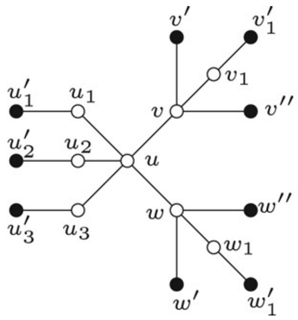


Fig. 8. Configuration (C8): a weak 5-vertex adjacent to two weak 4-vertices.

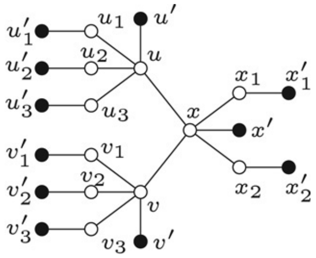


Fig. 9. Configuration (C9): a 5-vertex adjacent to two 2-vertices and two weak 5-vertices.

Proof of Configuration (C8). Suppose that H contains the configuration depicted in Fig. 8 and let $H' = H \setminus \{u, u_1, u_2, u_3, v_1, w_1\}$. Let φ be a T_{16}-coloring of H'. We clearly have $\varphi(v') \neq f^{\varphi}_{v'}(v'')$ (resp. $\varphi(w') \neq f^{\varphi}_{w'}(w'')$) since v (resp. w) is colored in H'. Property $P_{2,3}$ ensures that we have two available colors for v (resp. w), say c_1 and c_2 (resp. d_1 and d_2), distinct from $f^{\varphi}_{v}(v'_1)$ (resp. $f^{\varphi}_{w}(w'_1)$).

Therefore, we have to show that there exists $i, j \in [1, 2]$ such that assigning c_i to v and d_i to w allows us to extend φ to a T_{16}-coloring of H. Let $W = \{f^{\varphi}_{u}(u'_1), f^{\varphi}_{u}(u'_2), f^{\varphi}_{u}(u'_3)\}$, and $V_1 = \{c_1, c_2\}$, $V_2 = \{d_1, d_2\}$. Let $\alpha = (\alpha_1, \alpha_2)$ be an orientation 2-vector such that $\alpha_1 = 0$ (resp. $\alpha_2 = 0$) whenever $\overrightarrow{vu} \in A(G)$ (resp. $\overrightarrow{wu} \in A(G)$), and $\alpha_1 = 1$ (resp. $\alpha_2 = 1$) otherwise.

Note that we must have $\varphi(u) \notin W$. Suppose first that $V_1 = V_2$ (more precisely, $c_i = d_i$ for every $i \in [1, 2]$). If $\alpha_1 = \alpha_2$, then we set $\varphi(v) = \varphi(w)$, and we get $|\operatorname{Succ}_{\alpha}(\varphi(v), \varphi(w))| = 7$. Thus, φ can be extended to H. If $\alpha_1 \neq \alpha_2$, then let $S_1 = (c_1, d_2)$ and $S_2 = (c_2, d_1)$. The sequences S_1 and S_2 are compatible with α, and by Property $P_{2,3}$ we have $|\operatorname{Succ}_{\alpha}(S_i)| \geq 3$ for every $i \in [1, 2]$. Moreover, a case study shows that $\operatorname{Succ}_{\alpha}(S_1) = t(\operatorname{Succ}_{\alpha}(S_2))$. Therefore, there exists $i \in [1, 2]$ such that $\operatorname{Succ}_{\alpha}(S_i) \neq W$, and so φ can be extended to H.

Suppose now that $V_1 \neq V_2$. If there exists $i \in [1, 2]$ such that the arcs $c_i d_1$ and $c_i d_2$ exist in T_{16}, say $i = 1$, then $c_1 \neq d_1 \neq d_2 \neq c_1$ and therefore the sequences $S_1 = (c_1, d_1)$ and $S_2 = (c_1, d_2)$ are compatible with α and Proposition 9 ensures that there exist $i \in [1, 2]$ such that $\operatorname{Succ}_{\alpha}(S_i) \neq W$. If there exists $i \in [1, 2]$ such that the arcs $c_i d_1$ and $c_i d_2$ do not exist in T_{16}, say $i = 1$, then it means that $c_1 = d_1$ and $c_1 = t(d_2)$. This leads us to the previous case, that is that the two arcs $c_2 d_1$ and $c_2 d_2$ exist in T_{16} and $c_2 \neq d_1 \neq d_2 \neq c_2$. The last case to consider is the one where $c_1 d_1$ and $c_2 d_2$ exist in T_{16}, and $c_1 d_2$ and $c_2 d_1$ do not exist in T_{16}. We can check that we then have either (1) $c_1 = d_2$ and $c_2 = t(d_1)$, or (2) $c_1 = t(d_2)$ and $c_2 = t(d_1)$. If $\alpha_1 \neq \alpha_2$, then for both Cases (1) and (2), the sequence $S = (c_2, d_1)$ is compatible with α and we clearly have $\operatorname{Succ}_{\alpha}(S) \neq W$ since $|\operatorname{Succ}_{\alpha}(S)| = 7$. Finally, if $\alpha_1 = \alpha_2$, then for Case (1), the sequence $S = (c_1, d_2)$ is compatible with α and we clearly have $\operatorname{Succ}_{\alpha}(S) \neq W$ since $|\operatorname{Succ}_{\alpha}(S)| = 7$; for Case (2), the sequences $S_1 = (c_1, d_1)$ and $S_2 = (c_2, d_2)$ are compatible with α, and since $N^{+}_{T_{16}}(c_1) \cap N^{+}_{T_{16}}(d_2) = \emptyset$, we clearly have $\operatorname{Succ}_{\alpha}(S_1) \neq \operatorname{Succ}_{\alpha}(S_2)$ and thus there exists $i \in [1, 2]$ such that $\operatorname{Succ}_{\alpha}(S_i) \neq W$. □

Proof of Configuration (C9). Suppose that H contains the configuration depicted in Fig. 9 and let φ be a T_{16}-coloring of H^*. The weak 5-vertices u and v forbid four colors for x, say f_1, f_2, f_3, f_4, by Lemma 12. By Property $P_{1,7}$, we can choose φ such that $\varphi(x) \notin \{f_1, f_2, f_3, f_4, f^{\varphi}_{x}(x'_1), f^{\varphi}_{x}(x'_2)\}$. □

Proof of Configuration (C10). Suppose that H contains the configuration depicted in Fig. 10 and let φ be a T_{16}-coloring of H^*. By Main Lemma, the weak 5-vertices u, v, w, and x forbid eight colors for y, say $f_1, \ldots, f_8$. We clearly can choose φ such that $\varphi(y) \notin \{f_1, \ldots, f_8, f^{\varphi}_{y}(y'_1)\}$. □

Proof of Configuration (C11). Suppose that H contains the configuration depicted in Fig. 11 and let φ be a T_{16}-coloring of H^*. The weak 5-vertices v, w, and x forbid six colors for u, say $f_1, \ldots, f_6$, by Lemma 12. We clearly can choose φ such that $\varphi(u) \notin \{f_1, \ldots, f_6, f^{\varphi}_{u}(u'_1), f^{\varphi}_{u}(u'_2), f^{\varphi}_{u}(u'_3)\}$. □

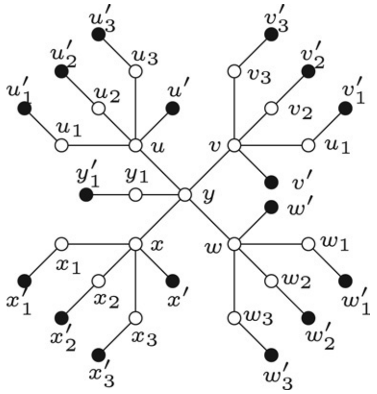


Fig. 10. Configuration (C10): a 5-vertex adjacent to one 2-vertex and four weak 5-vertices.

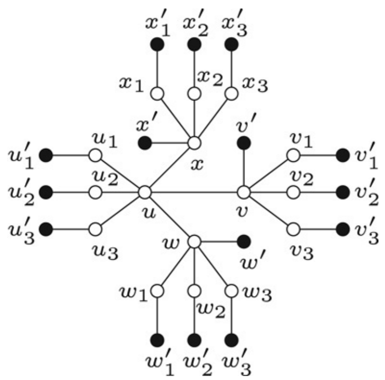


Fig. 11. Configuration (C11): a 6-vertex adjacent to three 2-vertices and three weak 5-vertices.

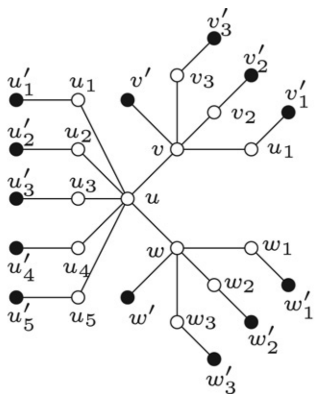


Fig. 12. Configuration (C12): a 7-vertex adjacent to five 2-vertices and two weak 5-vertices.

Proof of Configuration (C12). Suppose that H contains the configuration depicted in Fig. 12 and let φ be a T_{16}-coloring of H^*. The weak 5-vertices v and w forbid four colors for u, say $f_1, \dots, f_4$, by Lemma 12. We clearly can choose φ such that $\varphi(u) \notin \{f_1, \dots, f_4, f_u^{\varphi}(u'_1), f_u^{\varphi}(u'_2), \dots, f_u^{\varphi}(u'_5)\}$. $\square$

Proof of Configuration (C13). Suppose that H contains the configuration depicted in Fig. 13 and let φ be a T_{16}-coloring of H^*. The weak 5-vertex u forbids two colors for v, say f_1, f_2, by Lemma 12. We clearly can choose φ such that $\varphi(v) \notin \{f_1, f_2, f_v^{\varphi}(v'_1), f_v^{\varphi}(v'_2), \dots, f_v^{\varphi}(v'_7)\}$. $\square$

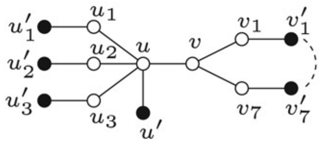


Fig. 13. Configuration (C13): an 8-vertex adjacent to seven 2-vertices and one weak 5-vertex.

3.2. Discharging procedure

To complete the proof of Theorem 2, we use a discharging procedure. We define the weight function ω by $\omega(v) = 3d(v) - 10$ for every $v \in V(H)$. Since $\text{mad}(H) < \frac{10}{3}$, we have:

$$\sum_{v \in V(H)} \omega(v) = \sum_{v \in V(H)} (3d(v) - 10) < 0.$$

In what follows, we will define discharging rules (R1), (R2) and (R3) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is fixed by the discharging rules. Nevertheless, we can show that $\omega^*(v) \geq 0$ for every $v \in V(H)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{v \in V(H)} \omega^*(v) = \sum_{v \in V(H)} \omega(v) < 0.$$

Therefore, no such counterexample H exists.

The discharging rules are defined as follows:

(R1) Each weak 4-vertex gives 2 to its 2-neighbor.
(R2) Each non weak 4-vertex gives 1 to their weak 5-neighbors.
(R3) Each $^{\geq}$5-vertex gives 2 to their 2-neighbors and 1 to their weak 5-neighbors.

Let v be a k-vertex of H. Note that $k > 1$ by (C1) and $k \neq 3$ by (C5).

- If $k = 2$, then $\omega(v) = -4$. Since every 2-vertex of H has two $^{\geq}$4-neighbors by (C2) and (C3), v receives 2 from each neighbor by (R1) and (R3). Hence $\omega^*(v) = 0$.
- If $k = 4$, then $\omega(v) = 2$. By (C2), a 4-vertex has at most one 2-neighbor. If v has one 2-neighbor (i.e. v is weak), then it gives 2 by (R1). If v has no 2-neighbor, then it has at most two weak 5-neighbors by (C7). Therefore, v gives at most 1×2 by (R2). Hence $\omega^*(v) \geq 2 - \max\{2; 1 \times 2\} = 0$.
- If $k = 5$, then $\omega(v) = 5$. By (C3), a 5-vertex has at most three 2-neighbors. If v has three 2-neighbors (i.e. v is weak), then it has no weak 5-neighbors by (C6); it thus gives 2×3 by (R3). Moreover, by (C8), v has at most one weak 4-neighbor; therefore, v has at least either one non weak 4-neighbor or one $^{\geq}$5-neighbor; thus, v receives at least 1 by (R2) or (R3). If v has two 2-neighbors, then it has at most one weak 5-neighbor by (C9), and then gives at most $2 \times 2 + 1$ by (R3). If v has one 2-neighbor, then it has at most three weak 5-neighbors by (C10), and then gives at most $2 + 1 \times 3$ by (R3). Finally, if v has no 2-neighbor, it gives at most 1×5 by (R3). Hence, $\omega^*(v) \geq 5 - \max\{2 \times 3 - 1; 2 \times 2 + 1; 2 + 1 \times 3; 1 \times 5\} = 0$.
- If $k = 6$, then $\omega(v) = 8$. By (C3), a 6-vertex has at most four 2-neighbors. If v has four 2-neighbors, then it has no weak 5-neighbor by (C6), and then gives 2×4 by (R3). If v has three 2-neighbors, then it has at most two weak 5-neighbors by (C11), and then gives at most $2 \times 3 + 1 \times 2$ by (R3). Finally, if v has l 2-neighbors, $0 \leq l \leq 2$, then v has at most $(6-l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (6-l)$ by (R3). Hence, $\omega^*(v) \geq 8 - \max\{2 \times 4; 2 \times 3 + 1 \times 2; 2 \times l + 1 \times (6-l)\} = 0$ for any $0 \leq l \leq 2$.
- If $k = 7$, then $\omega(v) = 11$. By (C3), a 7-vertex has at most five 2-neighbors. If v has five 2-neighbors, then it has at most one weak 5-neighbor by (C12) and then gives at most $2 \times 5 + 1$ by (R3). Finally, if v has l 2-neighbors, $0 \leq l \leq 4$, then it has at most $(7 - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (7 - l)$ by (R3). Hence, $\omega^*(v) \geq 11 - \max\{2 \times 5 + 1; 2 \times l + 1 \times (7 - l)\} = 0$ for any $0 \leq l \leq 4$.
- If $k = 8$, then $\omega(v) = 14$. By (C4), an 8-vertex has at most seven 2-neighbors. If v has seven 2-neighbors, then it has no weak 5-neighbor by (C13) and then gives 2×7 by (R3). Finally, if v has l 2-neighbors, $0 \leq l \leq 6$, then it has at most $(8-l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (8-l)$ by (R3). Hence, $\omega^*(v) \geq 14 - \max\{2 \times 7; 2 \times l + 1 \times (8-l)\} = 0$ for any $0 \leq l \leq 6$.
- If $k = 9$, then $\omega(v) = 17$. By (C4), a 9-vertex has at most eight 2-neighbors. If v has l 2-neighbors, $0 \leq l \leq 8$, then it has at most $(9-l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (9-l)$ by (R3). Hence, $\omega^*(v) \geq 17 - 2 \times l + 1 \times (9-l) \geq 0$ for any $0 \leq l \leq 8$.
- If $k \geq 10$, then $\omega(v) = 3k - 10$. If v has l 2-neighbors, $0 \leq l \leq k$, then v has at most $(k - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (k - l)$ by (R3). Hence, $\omega^*(v) \geq 3k - 10 - 2 \times l + 1 \times (k - l) \geq 0$ for any $0 \leq l \leq k$.

Thus, for every $v \in V(H)$, we have $\omega^*(v) \geq 0$ once the discharging is finished, that completes the proof.

4. Concluding remarks

4.1. Graphs with maximum average degree of less that $\frac{10}{3}$

In this paper, we proved that every oriented graph with maximum average degree less than $\frac{10}{3}$ and girth at least 5 has oriented chromatic number at most 16. We recently proved in a companion paper that the restriction of girth can be dropped:

Theorem 13 ([12]). *Let G be an oriented graph with maximum average degree less than $\frac{10}{3}$. Then, $\chi_o(G) \leq 16$.*

The proof of this theorem is based on the same techniques than the one used in this paper. The discharging procedure is the same. The difference lies in the forbidden configurations. Recall that the case where two black vertices coincide in a configuration (provided they do not share a white neighbor) is taken into account in the proofs. However, the cases where black vertices coincide with white vertices (creating cycles of lengths 3 and 4) was not taken into account here since we considered graphs with girth at least 5. So, to drop the restriction of girth, and thus get Theorem 13, we considered configurations with cycles of lengths 3 and 4.

4.2. Strong oriented coloring

In 1999, Nešetřil and Raspaud [9] introduced the notion of *strong oriented coloring*, which is a stronger version of the notion of oriented coloring studied in this paper.

Let M be an additive abelian group. An *M-strong-oriented coloring* of an oriented graph G is a mapping φ from $V(G)$ to M such that $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv}$ is an arc in G and $\varphi(v) - \varphi(u) \neq -(\varphi(t) - \varphi(z))$ whenever $\overrightarrow{uv}$ and $\overrightarrow{zt}$ are two arcs in G. The *strong oriented chromatic number* of an oriented graph is the minimal order of a group M such that G has an M-strong-oriented coloring. It is clear that any strong oriented coloring of an oriented graph G is an oriented coloring of G and therefore the oriented chromatic number of G is less than its strong oriented chromatic number.

Nešetřil and Raspaud showed that a strong oriented coloring of an oriented graph G can be equivalently defined as a homomorphism φ from G to H, where H is an oriented graph with k vertices labeled by the k elements of an abelian additive group M, such that for any pair of arcs $\overrightarrow{uv}$ and $\overrightarrow{zt}$ of $A(H)$, $v - u \neq -(t - z)$. For every prime power $p \equiv 3 (\mathrm{mod}\ 4)$, the Paley graph QR_p (defined in Section 2,) is clearly an oriented graph with p vertices labeled by the p elements of the field $\frac{\mathbb{Z}}{p\mathbb{Z}}$ and such that for any pair of arcs $\overrightarrow{uv}$ and $\overrightarrow{zt}$ of $A(QR_p)$, $v - u \neq -(t - z)$.

Borodin et al. [4] proved that the oriented chromatic number of the graphs with maximum average degree less that $\frac{10}{3}$ is at most 19 by showing that these graphs admit a homomorphism to the Paley graph QR_{19}. Therefore, their result applies for the strong oriented chromatic number: the graphs with maximum average degree of less that $\frac{10}{3}$ have a *strong* oriented chromatic number at most 19. So, a natural question to ask is:

Question 14. *Does there exist an abelian additive group M on 16 elements such that we can label the vertices of T_{16} with the elements of M in such a way that $v - u \neq -(t - z)$ whenever $\overrightarrow{uv}$ and $\overrightarrow{zt}$ are two arcs of T_{16}?*

If it is true, that would imply that 16 colors are enough for a *strong* oriented coloring of an oriented graph with maximum average degree less than $\frac{10}{3}$, and therefore, for a *strong* oriented coloring of a planar graph with girth at least 5.

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