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The acircuitic directed star arboricity of subcubic graphs is at most four

Alexandre Pinlou, Éric Sopena

LaBRI, Université Bordeaux I, 351, cours de la Libération, 33405 Talence Cedex, France

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Abstract

A directed star forest is a forest all of whose components are stars with arcs emanating from the center to the leaves. The acircuitic directed star arboricity of an oriented graph G (that is a digraph with no opposite arcs) is the minimum number of arc-disjoint directed star forests whose union covers all arcs of G and such that the union of any two such forests is acircuitic. We show that every subcubic graph has acircuitic directed star arboricity at most four.

Keywords: Acircuitic directed star arboricity; Directed star forest

1. Introduction

We consider finite simple oriented graphs, that is digraphs with no opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs.

In [1], Algor and Alon introduced the notion of the *directed star arboricity* of a digraph G, defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$. (A directed star forest is a forest all of whose components are directed stars, that is stars with arcs emanating from the center.) In the same vein, we study here the new notion of the *acircuitic* directed star arboricity of an oriented graph G, defined as the minimum number of arc-disjoint directed star forests needed to cover $A(G)$ in such a way that the union of any two such forests contains no circuit (that is directed cycle). In [5], Guiduli proved that every oriented graph with indegree and outdegree both less than D has directed star arboricity at most $D + 20 \log D + 84$ colors.

In this paper, we prove the following.

Theorem 1. *Every graph with maximum degree at most* 3 *has acircuitic directed star arboricity at most* 4.

The notion of acircuitic directed star arboricity arises from the study of arc-coloring of oriented graphs. In [4], Courcelle introduced the notion of vertex-coloring of oriented graphs as follows: a *k-vertex-coloring* of an oriented graph G is a mapping f from $V(G)$ to a set of k colors such that (i) $f(u) \neq f(v)$ whenever $\overrightarrow{uv}$ is an arc in G, and (ii) $f(u) \neq f(x)$ whenever $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in G with $f(v) = f(w)$. Vertex-coloring of oriented graphs have been studied by several authors in the last past years (see e.g. [2,6,8] for an overview).

E-mail addresses: Alexandre.Pinlou@labri.fr (A. Pinlou), Eric.Sopena@labri.fr (É. Sopena).

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Recall that an *acyclic* coloring of an undirected graph U is a proper coloring of U such that every cycle in U uses at least three colors. Raspaud and Sopena proved in [7] that every orientation of an undirected graph that admits an acyclic k-coloring admits an oriented $(k \cdot 2^{k-1})$-coloring.

One can define arc-colorings of oriented graphs in a natural way by saying that, as in the undirected case, an arc-coloring of an oriented graph G is a vertex-coloring of the line digraph of G. (Recall that the line digraph $L(G)$ of G is given by $V(L(G)) = A(G)$ and $(\overrightarrow{uv}, \overrightarrow{vw}) \in A(L(G))$ whenever $\overrightarrow{uv} \in A(G)$ and $\overrightarrow{vw} \in A(G)$.) It is not difficult to see that every oriented graph having a k-vertex-coloring admits a k-arc-coloring (from a k-vertex-coloring f, we obtain a k-arc-coloring g by setting $g(\overrightarrow{uv}) = f(u)$).

By adapting the proof of the above-mentioned result of Raspaud and Sopena, it is not difficult to prove that every oriented graph with acircuitic directed star arboricity at most k admits a $(k \cdot 2^{k-1})$-arc-coloring.

This paper is organized as follows: we introduce the main definitions and notation in the next section and prove our main result in Section 3.

2. Definitions and notation

In the rest of the paper, oriented graphs will be simply called graphs. For a vertex v, we denote by $d^-(v)$ the indegree of v, by $d^+(v)$ its outdegree and by $d(v)$ its degree, that is $d(v) = d^+(v) + d^-(v)$. A *source vertex* is a vertex v with $d^-(v) = 0$. The *maximum degree* and *minimum degree* of a graph G are, respectively, denoted by $\Delta(G)$ and $\delta(G)$. A graph G is said to be *cubic* if $\Delta(G) = \delta(G) = 3$ and *subcubic* if $\Delta(G) \leqslant 3$.

We denote by $\overrightarrow{uv}$ the arc from u to v or simply uv whenever its orientation is not relevant (therefore $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$). If $a = \overrightarrow{uv}$ is an arc, then u is the *tail* and v is the *head* of a.

For a graph G and a vertex v of $V(G)$, we denote by $G \backslash v$ the graph obtained from G by removing v together with the set of its incident arcs; similarly, for an arc a of $A(G)$, $G \backslash a$ denotes the graph obtained from G by removing a. These two notions are extended to sets in a standard way: for a set of vertices V', $G \backslash V'$ denotes the graph obtained from G by successively removing all vertices of V' and their incident arcs, and for a set of arcs A', $G \backslash A'$ denotes the graph obtained from G by removing all arcs of A'.

The notions of arboricity discussed in the previous section may be defined in terms of arc-colorings or partitions of the set of arcs. More precisely, a *k-directed-star-coloring* (or simply *k-dst-coloring*) of a graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$. Equivalently, a k-dst-coloring of G is a k-coloring f of $A(G)$ such that (i) $\overrightarrow{uv}, \overrightarrow{vw} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overrightarrow{vw})$ and (ii) $\overrightarrow{uv}, \overrightarrow{tv} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overrightarrow{tv})$. The *directed star arboricity* of G, denoted by *dst*(G), is then the smallest k for which G admits a k-dst-coloring.

A graph G is *acircuitic* if it does not contain any circuit. A *k-acircuitic-directed-star-coloring* (or simply *k-adst-coloring*) of a graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$ such that for all $i, j \in [1, k]$, $F_i \cup F_j$ is acircuitic. Equivalently, a k-adst-coloring of G is a k-dst-coloring of G such that no circuit in G is bichromatic. The *acircuitic directed star arboricity* of G, denoted by $\text{adst}(G)$, is the smallest k for which G admits a k-adst-coloring.

Note that from the above definitions we get that every edge-coloring of an undirected graph H is a dst-coloring of any orientation of H. Similarly, every acyclic edge-coloring of H is an adst-coloring of any orientation of H.

The following notation will be extensively used in the rest of the paper. Consider a graph G and let $A' = \{a_1, a_2, \dots, a_n\}$ be a subset of $A(G)$. We denote by $C_G(a_1, a_2, \dots, a_n)$, or simply $C_G(A')$, the set of circuits of G that contain all the arcs $a_1, a_2, \dots, a_n$.

Drawing conventions. In all the figures, we shall use the following convention: a vertex whose neighbors are totally specified will be black, whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

3. Proof of Theorem 1

Suppose that Theorem 1 is false and consider a minimal counter-example G. We prove a series of lemmas. In each of them, we reduce G to a smaller graph G' (that is $|A(G)| > |A(G')|$) which admits a 4-adst-coloring f' which is also a *partial* adst-coloring of G (that is an adst-coloring only defined on some subset A' of $A(G)$). We extend such a partial adst-coloring f' to an adst-coloring f of G. In this case, it should be understood that we set $f(a) = f'(a)$ for every arc $a \in A(G')$. We then explain how to set $f(a)$ for every uncolored $a \in A(G)$. The existence of f proves that G does not contain some specific configurations. This set of configurations will finally lead to a contradiction.

Consider a circuit C and let $u, v \in V(C)$. We denote by $P_C(u, v)$ the directed path from u to v in C.

The following observation will be extensively used in the sequel:

Observation 2. *Let C be a circuit, f an adst-coloring of C, and C' the circuit obtained from C by replacing $P_C(u, v)$ by a directed path $P_{C'}(u, v)$. If f' is a dst-coloring of C' such that $f'(a) = f(a)$ for every $a \notin P_{C'}(u, v)$ and $\{f(a);\ a \in P_C(u, v)\} \subseteq \{f'(a');\ a' \in P_{C'}(u, v)\}$ then f' is an adst-coloring of C'.*

This directly follows from the fact that $|f'(C')| \geqslant |f(C)| \geqslant 3$.

We first show that a minimal counter-example to Theorem 1 is necessarily a cubic graph.

Lemma 3. *If G is a minimal counter-example to Theorem* 1, *then $\delta(G) \geqslant 3$.*

Proof. Let $v \in V(G)$ with $d(v) \leqslant 2$. We consider two cases:

Case 1: $d_G(v) = 1$. Consider the dangling arc uv in G and let f' be any 4-adst-coloring of the graph $G' = G \backslash \{v\}$. We extend f' to a 4-adst-coloring f of G by setting $f(uv) = a$ for some color a distinct from the colors of the at most two arcs incident to uv.

Case 2: $d_G(v) = 2$. Consider the two arcs uv and wv in G and let f' be any 4-adst-coloring of the graph G' obtained from G by contracting uv in a single vertex x. We extend f' to a 4-adst-coloring f of G by setting $f(wv) = f'(wx)$ and $f(uv) = a$ for any a distinct from the colors of the three arcs incident to uv. (By Observation 2, no circuit in G can be bichromatic).

In both cases we thus obtain a 4-adst-coloring f of G, a contradiction. □

Lemma 4. *If G is a minimal counter-example to Theorem* 1, *then G does not contain any source vertex.*

Proof. Let $v \in V(G)$ be a source vertex. By Lemma 3, we know that $d^+(v) = 3$. Let u_1, u_2 and u_3 be the three neighbors of v and f' be any 4-adst-coloring of the graph $G' = G \backslash v$. Each of the arcs $\overrightarrow{vu_1}$, $\overrightarrow{vu_2}$ and $\overrightarrow{vu_3}$ has at least two available colors. Since they can get the same color, we can extend f' to a 4-adst-coloring f of G, a contradiction. □

We now prove that a minimal counter-example to Theorem 1 contains no triangle.

Lemma 5. *If G is a minimal counter-example to Theorem* 1, *then G is triangle-free.*

Proof. If G contains three pairwise adjacent triangles, then G is an orientation of the complete graph K_4. By Lemma 4, we only have to consider the two orientations of K_4 depicted on Figs. 1(a) and (b) that both admit a 4-adst-coloring.

If G contains two adjacent triangles, then G contains the configuration of Fig. 1(c). Consider the graph $G' = G \backslash \{w, x\}$ and let f' be a 4-adst-coloring of G' such that $f'(uv) \neq f'(yz)$ (this can be done since we have two possible choices for coloring each of uv and yz). Suppose without loss of generality that $f'(uv) = 1$ and $f'(yz) = 2$. In this case, we can produce an acyclic 4-edge-coloring as depicted in Fig. 1(c). Indeed, this coloring is a proper edge-coloring and no path linking u and z is bichromatic. Hence, for all possible orientations of the arcs of the configuration, this coloring gives a 4-adst-coloring f of G.

Suppose finally that G contains the configuration of Fig. 1(d), and let f' be any 4-adst-coloring of the graph G' obtained from G by contracting the triangle $v_1v_2v_3$ in a single vertex v. Therefore, every circuit $C \in C_G(\overrightarrow{u_iv_i}, \overrightarrow{v_ju_j})$ corresponds to a circuit $C' \in C_{G'}(\overrightarrow{u_iv}, \overrightarrow{vu_j})$.

We now extend the partial adst-coloring f' to a 4-adst-coloring f of G as follows. We distinguish two cases:

Case 1: $f'(vu_1) \neq f'(vu_2) \neq f'(vu_3) \neq f'(vu_1)$. Without loss of generality, suppose that $f'(v_1u_1) = 1$, $f'(v_2u_2) = 2$ and $f'(v_3u_3) = 3$. We then set $f(v_3v_1) = 2$, $f(v_1v_2) = 3$ and $f(v_2v_3) = 1$.

Case 2: $\exists i, j \in \{1, 2, 3\}, i \neq j$, $f'(v_iu_i) = f'(v_ju_j) = a$. In this case we necessarily have $\overrightarrow{v_iu_i}, \overrightarrow{v_ju_j} \in A(G)$. Let $k \in \{1, 2, 3\}$, $k \neq i, j$. We then set $f(v_iv_k)$, $f(v_jv_k)$ and $f(v_iv_j)$ as follows:

1. $f(v_iv_k) = b$ for any $b \notin \{a, f'(u_kv_k)\}$,
2. $f(v_jv_k) = c$ for any $c \notin \{a, b, f'(u_kv_k)\}$,
3. $f(v_iv_j) = d$ for any $d \notin \{a, b, c\}$.

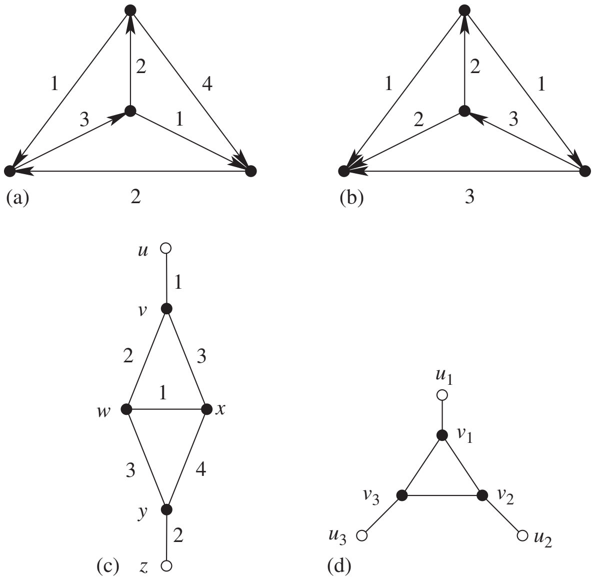

Fig. 1. Configurations of Lemma 5.

This can be done since we have four available colors.

In both cases, thanks to Observation 2, we obtain a 4-adst-coloring f of G, a contradiction. □

Let G be a graph and C a circuit in G. An arc having exactly one of its endpoints in C is said to be *incident* to C. Moreover, two such incident arcs are *neighboring* if their endpoints in C are linked by an arc of C.

The four next lemmas will allow us to prove that a minimal counter-example G to Theorem 1 is necessarily acircuitic.

Lemma 6. *If G is a minimal counter-example to Theorem* 1, *then G does not contain a circuit all of whose vertices have indegree one and outdegree two.*

Proof. Suppose that there exists a circuit $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$ in G such that $d^+(v_i) = 1$ and $d^-(v_i) = 2$ for $i \in [0, k-1]$ and let f' be any 4-adst-coloring of the graph $G' = G \backslash C$. Let $\{\overrightarrow{v_iu_i} | i \in [0, k-1]\}$ be the set of arcs incident to C.

We extend the partial coloring f' to a 4-adst-coloring f of G as follows. Due to the orientation of G, C is the only circuit of G that does not belong to G'. Therefore, we only need to color the arcs of C in such a way that C is not bichromatic. We distinguish two cases depending on the colors of the arcs incident to C.

1. All arcs incident to C are colored with the same color. In this case, we color the arcs of C using the three other remaining colors.
2. Two neighboring arcs incident to C have distinct colors. Suppose without loss of generality that $f'(\overrightarrow{v_0u_0}) = c_0$ and $f'(\overrightarrow{v_1u_1}) \neq c_0$. In this case, we set
   (a) $f(\overrightarrow{v_0v_1}) = c_0$,
   (b) $\forall i \in [1, k-2]$, $f(\overrightarrow{v_iv_{i+1}}) = c_i$ for any $c_i \notin \{c_{i-1}, f(\overrightarrow{v_{i+1}u_{i+1}})\}$,
   (c) $f(\overrightarrow{v_{k-1}v_0}) = c_{k-1}$ for any $c_{k-1} \notin \{c_0, c_1, c_{k-2}\}$.

The circuit C is clearly not bichromatic since $c_0 \neq c_1 \neq c_{k-1} \neq c_0$.

In both cases, we obtain a 4-adst-coloring f of G, a contradiction. □

Lemma 7. *If G is a minimal counter-example to Theorem* 1, *then G does not contain a circuit all of whose vertices have indegree two and outdegree one.*

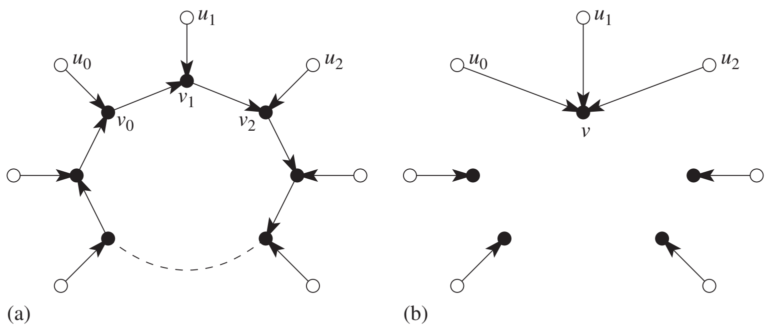

Fig. 2. The configuration of Case 1 of Lemma 7 and its reduction: (a) the graph G; (b) the graph G'.

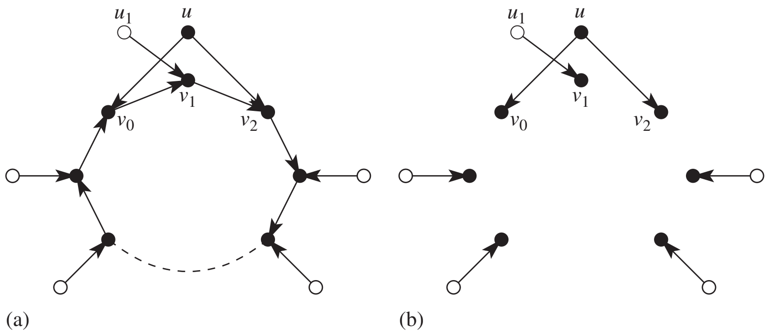

Fig. 3. The configuration of Case 2 of Lemma 7 and its reduction: (a) the graph G; (b) the graph G'.

Proof. Suppose that there exists a circuit $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$ in G (see Figs. 2(a) or 3(a)) such that $d^+(v_i) = 2$ and $d^-(v_i) = 1$ for $i \in [0, k-1]$. Let $\{\overrightarrow{u_iv_i} | i \in [0, k-1]\}$ be the set of arcs incident to C. By Lemma 5, the tails of two neighboring arcs incident to C are necessarily distinct.

We consider two cases depending on whether the vertices u_0 and u_2 are distinct or not. We first show that in both cases there exists a reduction G' of G (see Figs. 2(b) and 3(b))) which admits a 4-adst-coloring f' such that $f'(\overrightarrow{u_0v_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{u_2v_2}) \neq f'(\overrightarrow{u_0v_0})$.

Case 1: $u_0 \neq u_2$ (see Fig. 2(a)). Let f' be any 4-adst-coloring of the graph G' obtained from $G \backslash C$ by identifying v_0, v_1 and v_2 in a single vertex v (see Fig. 2(b)). We clearly have $f'(\overrightarrow{u_0v}) \neq f'(\overrightarrow{u_1v}) \neq f'(\overrightarrow{u_2v}) \neq f'(\overrightarrow{u_0v})$.

Case 2: $u_0 = u_2 = u$ (see Fig. 3(a)). Note that by Lemma 4 we have $u_1 \neq u$. Let f' be any 4-adst-coloring of the graph $G' = G \backslash C$ (see Fig. 3(b)). Since we have at least three available colors for the arcs $\overrightarrow{uv_0}$ and $\overrightarrow{uv_2}$, we can choose f' in such a way that $f'(\overrightarrow{uv_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{uv_2}) \neq f'(\overrightarrow{uv_0})$.

Assume now that $f'(\overrightarrow{u_0v_0}) = c_1$, $f'(\overrightarrow{u_1v_1}) \neq c_1$ and $f'(\overrightarrow{u_2v_2}) \neq c_1$. As in the previous lemma, C is the only circuit of G that does not belong to G'. Therefore, we only need to color the arcs of C in such a way that C is not bichromatic. We then set f as follows:

1. $f(\overrightarrow{v_1v_2}) = c_1$,
2. $\forall\, i \in [2, k-1]$, $j = i+1 \pmod{k}$, $f(\overrightarrow{v_iv_j}) = c_i$ for any $c_i \notin \{c_{i-1}, f(\overrightarrow{u_iv_i}), f(\overrightarrow{u_jv_j})\}$,
3. $f(\overrightarrow{v_0v_1}) = c_0$ for any $c_0 \notin \{c_{k-1}, c_1, f(\overrightarrow{u_1v_1})\}$.

Note that $c_{k-1} \neq f(\overrightarrow{u_0v_0}) = c_1$. Therefore, $c_{k-1} \neq c_0 \neq c_1 \neq c_{k-1}$ and C is not bichromatic. We thus obtain a 4-adst-coloring f of G, a contradiction. □

From the two previous lemmas, we get that if C is a circuit in a minimal counter-example to Theorem 1, there exist two neighboring arcs incident with C having opposite directions (with respect to C). The next two lemmas will show that this situation is also not possible.

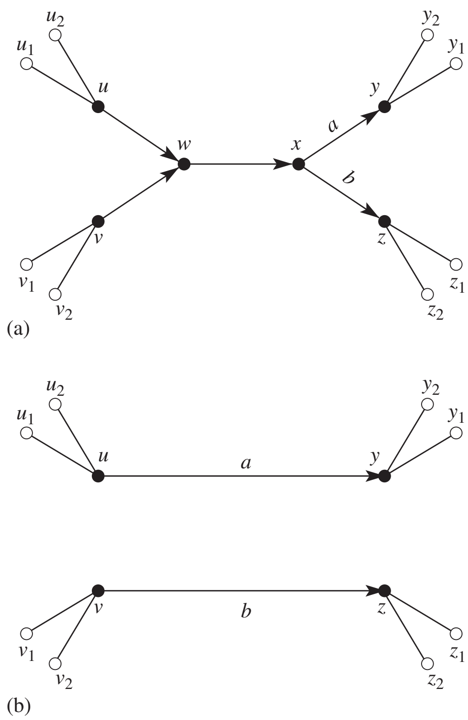

Fig. 4. The configuration of Lemma 8 and its reduction: (a) the graph G; (b) the reduction G'.

Lemma 8. *If G is a minimal counter-example to Theorem* 1, *then G does not contain the configuration depicted in Fig.* 4(a) *where* u_1u, u_2u, y_1y, y_2y, v_1v, v_2v, z_1z *and* z_2z *are pairwise distinct.*

Proof. Suppose that the graph G contains the configuration of Fig. 4(a), and let f' be any 4-adst-coloring of the graph G' obtained from $G\backslash\{w, x\}$ by adding the arcs $\overrightarrow{uy}$ and $\overrightarrow{vz}$ (see Fig. 4(b)). Suppose that $f'(\overrightarrow{uy}) = a$ and $f'(\overrightarrow{vz}) = b$.

We extend the partial 4-adst-coloring f' to a 4-adst-coloring f of G as follows.

Let $S_1 = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xy}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xz})$ and $S_2 = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$. We first set $f(\overrightarrow{xy}) = a$ and $f(\overrightarrow{xz}) = b$. Clearly, all circuits in G not belonging to $S_1 \cup S_2$ also belong to G', and thus are already not bichromatic. Moreover, by Observation 2, the circuits in S_1 will not be bichromatic. Therefore, we only have to pay attention to the circuits in S_2.

We consider two cases depending on the colors a and b:

Case 1: $a \neq b$. We set $f(\overrightarrow{uw}) = a$, $f(\overrightarrow{vw}) = b$ and $f(\overrightarrow{wx}) = c$ for any $c \notin \{a, b\}$. Since $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{a, c, b\}| = 3$ and $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{b, c, a\}| = 3$, no circuit in S_2 is bichromatic.

Case 2: $a = b$. We consider three subcases.

1. $\{\overrightarrow{uu_1}, \overrightarrow{uu_2}, \overrightarrow{vv_1}, \overrightarrow{vv_2}\} \cap A(G) \neq \emptyset$. We assume without loss of generality that $\overrightarrow{vv_1} \in A(G)$. In this case, we first set $f(\overrightarrow{uw}) = c$ for any $c \notin \{a, f(uu_1), f(uu_2)\}$ and $f(\overrightarrow{vw}) = d$ for any $d \notin \{a, c, f(vv_2)\}$. Now, we can color the arc $\overrightarrow{wx}$ with the fourth color $e \notin \{a, c, d\}$. We recall that $f(\overrightarrow{xy}) = f(\overrightarrow{xz})$. Therefore, $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{c, e, a\}| = 3$, $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, e, a\}| = 3$, and so no circuit in S_2 is bichromatic.
2. $\overrightarrow{u_1u}, \overrightarrow{u_2u}, \overrightarrow{v_1v}, \overrightarrow{v_2v} \in A(G)$ and $\{f(\overrightarrow{u_1u}), f(\overrightarrow{u_2u})\} \neq \{f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\}$. Note that since $a \notin \{f(\overrightarrow{u_1u}), f(\overrightarrow{u_2u}), f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\}$, we necessarily have $\{f(\overrightarrow{u_1u}), f(\overrightarrow{u_2u})\} \cap \{f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\} \neq \emptyset$. Therefore, we can assume without loss of generality that $f(\overrightarrow{u_1u}) = f(\overrightarrow{v_1v}) = c$, $f(\overrightarrow{u_2u}) = d$ and $f(\overrightarrow{v_2v}) = e$, with a, c, d, e being pairwise distinct. In this case, we set $f(\overrightarrow{uw}) = e$ and $f(\overrightarrow{vw}) = d$. Now, we can color the arc $\overrightarrow{wx}$ with the color c. Therefore, $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{e, c, a\}| = 3$, $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, c, a\}| = 3$, and so no circuit in$S_2$ is bichromatic.
3. $\overrightarrow{u_1u}, \overrightarrow{u_2u}, \overrightarrow{v_1v}, \overrightarrow{v_2v} \in A(G)$ and $\{f(\overrightarrow{u_1u}), f(\overrightarrow{u_2u})\} = \{f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\}$.

We assume without loss of generality that $f(\overrightarrow{u_1u}) = f(\overrightarrow{v_1v}) = c$ and $f(\overrightarrow{u_2u}) = f(\overrightarrow{v_2v}) = d$, $a \neq c \neq d \neq a$. We then set $f(\overrightarrow{vw}) = a$ and $f(\overrightarrow{uw}) = e$ with $e \notin \{a, c, d\}$.

Since $\overrightarrow{uw}$ and $\overrightarrow{xz}$ are colored with distinct colors, no circuit in $C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz})$ is bichromatic.

We still have to set the color of the arc $\overrightarrow{wx}$. We consider three subcases.

(a) $\{\overrightarrow{y_1y}, \overrightarrow{y_2y}\} \cap A(G) \neq \emptyset$. We assume without loss of generality that $\overrightarrow{y_1y} \in A(G)$. So, if $f(yy_2) = c$ (resp. $f(yy_2) = d$), we set $f(\overrightarrow{wx}) = d$ (resp. $f(\overrightarrow{wx}) = c$), otherwise ($f(yy_2) = e$), we use either c or d. Therefore, $|\{f(\overrightarrow{wx}), f(\overrightarrow{xy}), f(\overrightarrow{yy_2})\}| = 3$, and thus no circuit in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$ is bichromatic.

(b) $\overrightarrow{yy_1}, \overrightarrow{yy_2} \in A(G)$ and $\{f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\} \neq \{f(\overrightarrow{yy_1}), f(\overrightarrow{yy_2})\}$. We assume without loss of generality that $f(\overrightarrow{yy_2}) = e$. Now, if $f(\overrightarrow{yy_1}) = c$ (resp. $f(\overrightarrow{yy_1}) = d$) we set $f(\overrightarrow{wx}) = d$ (resp. $f(\overrightarrow{wx}) = c$). This implies that for any $i \in \{1, 2\}$, $|\{f(\overrightarrow{wx}), f(\overrightarrow{xy}), f(\overrightarrow{yy_i})\}| = 3$, and thus no circuit in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$ is bichromatic.

(c) $\overrightarrow{yy_1}, \overrightarrow{yy_2} \in A(G)$ and $\{f(\overrightarrow{v_1v}), f(\overrightarrow{v_2v})\} = \{f(\overrightarrow{yy_1}), f(\overrightarrow{yy_2})\}$. We can suppose without loss of generality that $f(\overrightarrow{yy_1}) = c$ and $f(\overrightarrow{yy_2}) = d$. We then set $f(\overrightarrow{wx}) = c$. If there is no arc emanating from y_1 and colored with a, no circuit in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$ is bichromatic. If there exists an arc emanating from y_1 and colored with a, then there exists at least one available color distinct from c that can be used to recolor the arc $\overrightarrow{yy_1}$ in such a way that we forbid bichromatic circuits in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$.

In all cases, we obtain a 4-adst-coloring f of G, a contradiction. □

In the configuration of the previous lemma, the arcs uu'_i and vv'_j on one hand, yy'_i and zz'_j on the other hand, are necessarily distinct since, by Lemma 5, a minimal counter-example to Theorem 1 contains no triangle. The next lemma deals with the case where two arcs uu'_i (or vv'_i) and yy'_j (or zz'_j) are the same. Without loss of generality, we will suppose that the arcs v'_1v and z'_1z are the same.

Lemma 9. *If G is a minimal counter-example to Theorem* 1, *then G does not contain the configuration depicted on Fig.* 5(a).

Proof. Suppose that the graph G contains the configuration of Fig. 5(a) (in this configuration, two arcs linking a black and a white vertex may be the same provided it does not produce a triangle). We consider two cases depending on the orientation of the arc vz.

Case 1: $\overrightarrow{vz} \in A(G)$. Consider the graph G'_1 (see Fig. 5(b)) obtained from $G \backslash \{w, x\}$ by adding the arcs $\overrightarrow{uv}$ and $\overrightarrow{zy}$ (see Fig. 5(b)) and let f'_1 be any 4-adst-coloring of G'_1. Assume that $f'_1(\overrightarrow{uv}) = a$, $f'_1(\overrightarrow{vz}) = b$ and $f'_1(\overrightarrow{zy}) = c$ (see Fig. 5(b)). We extend the partial 4-adst-coloring f' to a 4-adst-coloring f of G as follows. We first set $f(\overrightarrow{uw}) = a$, $f(\overrightarrow{wx}) = f(\overrightarrow{vz}) = b$ and $f(\overrightarrow{xy}) = c$ (see Fig. 5(a)). By Observation 2, no circuit in G is thus bichromatic. We then color the arcs $\overrightarrow{vw}$ and $\overrightarrow{xz}$ so that $f(\overrightarrow{vw}) \notin \{a, b, f(v'v)\}$ and $f(\overrightarrow{xz}) \notin \{b, f(zz')\}$.

Case 2: $\overrightarrow{zv} \in A(G)$. Consider the graph G'_2 obtained from $G \backslash \{w, x\}$ by adding the arcs $\overrightarrow{uz}$ and $\overrightarrow{vy}$ and let f'_2 be any 4-adst-coloring of G'_2. Assume that $f'_2(\overrightarrow{uz}) = a$, $f'_2(\overrightarrow{zv}) = b$ and $f'_2(\overrightarrow{vy}) = c$ (see Fig. 5(c)). We extend the partial 4-adst-coloring f' to a 4-adst-coloring f of G as follows. As in the previous case, we set $f(\overrightarrow{uw} = a$, $f(\overrightarrow{wx}) = f(\overrightarrow{vz}) = b$ and $f(\overrightarrow{xy}) = c$ (see Fig. 5(a)). By Observation 2, we only have to pay attention to the circuits in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xz})$. We then color the arcs $\overrightarrow{vw}$ and $\overrightarrow{xz}$ in such a way that $f(\overrightarrow{vw}) \notin \{a, b, f(v'v)\}$ and $f(\overrightarrow{xz}) = a$ (this can be done since $f'(zz') \neq a$). Since $a \neq b \neq f(\overrightarrow{vw}) \neq a$, no circuit in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xz})$is bichromatic.
In both cases we obtain a 4-adst-coloring f of G, a contradiction. □

Using the previous lemmas, we can now prove our main result.

Proof of Theorem 1. By Lemmas 6–9, a minimal counter-example G to Theorem 1 does not contain any circuit. Therefore, any 4-dst-coloring of G is a 4-adst-coloring of G. Moreover, it follows from the definitions that any k-edge-coloring of the underlying undirected graph of G is a k-dst-coloring of G. Therefore, by Vizing's theorem [10], the graph G admits a 4-edge-coloring and thus a 4-adst-coloring, acontradiction. □

The bound given in Theorem 1 is optimal. To see that, consider the orientation $\overrightarrow{K_4}$ of the complete graph K_4 given in Fig. 6. If we want to color this graph with three colors, the only way to color the arcs $\overrightarrow{uw}$, $\overrightarrow{xu}$, $\overrightarrow{wx}$, $\overrightarrow{vw}$

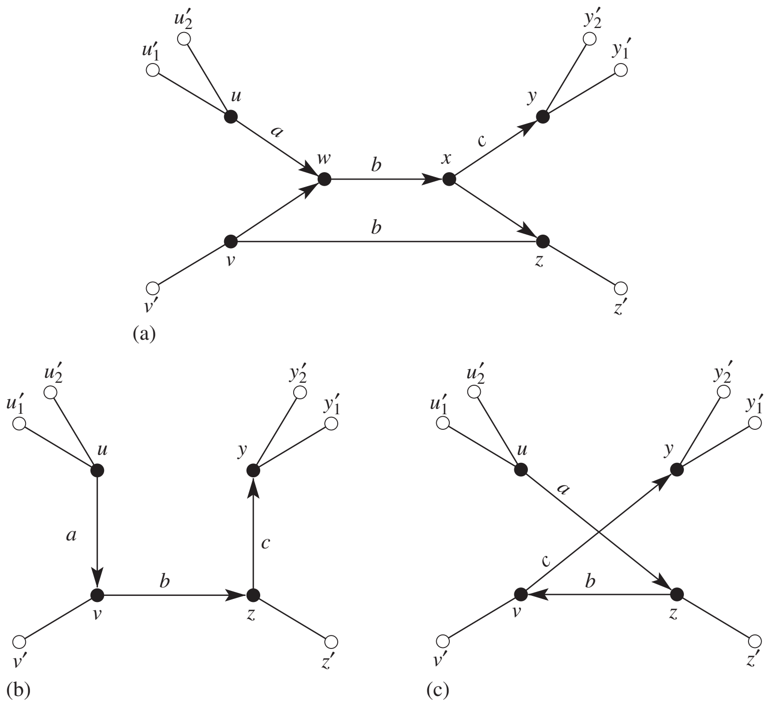

Fig. 5. The configuration of Lemma 9 and its reductions: (a) the graph G; (b) the reduction G_1'; (c) the reduction G_2'.

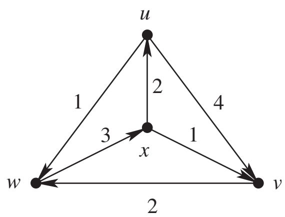

Fig. 6. The orientation $\overrightarrow{K_4}$ such that $\mathrm{adst}(\overrightarrow{K_4}) = 4$.

and $\overrightarrow{xv}$ is clearly the one depicted in Fig. 6. But in this case, we need one more color for the arc $\overrightarrow{uv}$ and thus, $\mathrm{adst}(\overrightarrow{K_4}) = 4$.

4. Discussion

In [3] Burnstein proved that every graph with maximum degree 4 admits an acyclic 5-vertex-coloring. Since the line graph of a subcubic graph has maximum degree at most 4, we get that every subcubic graph admits an acyclic 5-edge-coloring and thus a 5-adst-coloring. Our result shows that this bound can be decreased to 4 when considering oriented graphs and acircuitic arc-colorings.

We also provided an oriented cubic graph with acircuitic directed star arboricity 4. However, we do not know any other example of a cubic oriented graph that does not admit a 3-adst-coloring.

From our result, we get that every oriented graph with maximum degree 3 admits a $4 \cdot 2^{4-1} = 32$-arc-coloring. However, every such graph admits an 11-vertex-coloring [9] and thus an 11-arc-coloring.

Using similar techniques, we are able to prove that every K_4-minor free oriented graph G has acircuitic directed star arboricity at most $\min\{\Delta(G), \Delta^-(G) + 2\}$, where $\Delta^-(G)$ stands for the maximum indegree of G. This class of graphs contains in particular outerplanar graphs. It would thus be interesting to determine the acircuitic directed star arboricity of planar graphs.

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