Locally identifying colouring planar graphs of small maximum degree and girth 5 with four colours is NP-hard

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Abstract

We show that deciding whether a planar graph of maximum degree at most 5 and girth 5 admits a locally identifying 4-colouring is an NP-complete problem.

1 The reduction

A *proper vertex colouring* is a colouring of the vertices of a graph such that adjacent vertices are assigned distinct colours. For a proper colouring c of the vertices of the graph G and for any $S \subseteq V(G)$, we will note $c(S)$ the set of colours of the vertices of S. $N[v]$ is the closed neighbourhood of v. A *locally identifying colouring* of a graph (lid-colouring for short) is a proper vertex colouring, such that for two adjacent vertices u and v, if $N[u] \neq N[v] \Rightarrow c(N[u]) \neq c(N[v])$. It was introduced in [2], see also [3, 5] for further work on the topic.

The k-LID-COLOURING problem is defined as follows:

INSTANCE: A graph G.

QUESTION: Does G have a lid-colouring with k colours?

In [2], it is proved that 3-LID-COLOURING is NP-complete, even for bipartite graphs of maximum degree 3 and arbitrarily high girth (although every bipartite graph is 4-lid-colourable). However, 3-LID-COLOURING is polynomial-time solvable on planar graphs [2].[1]

The 3-COLOURING problem is defined as follows:

INSTANCE: A graph G.

QUESTION: Does G have a proper colouring with three colours?

Theorem 1 *4-LID-COLOURING is NP-complete for planar graphs of maximum degree* 5 *and girth* 5.

Proof

The problem is clearly in NP. We will prove the NP-hardness by reducing from 3-COLOURING for planar graphs with maximum degree 4, which is NP-complete [4].

First, let us remark that a graph containing a triangle is never 4-lid-colourable.

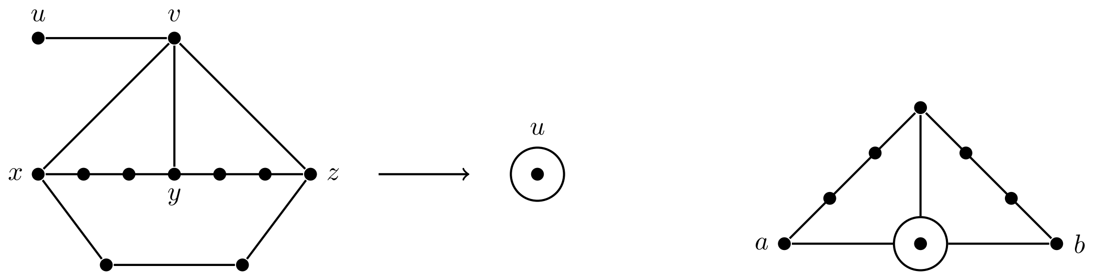
Consider three vertices x, y and z connected among themselves by a path of length 3. We will represent a vertex u adjacent to a vertex v such that $\{x, y, z\} \subset N(v)$, by a special vertex, as shown in Figure 1(a). One can easily check that this graph is 4-lid-colourable and for a valid 4-lid-colouring of this graph, vertices x, y, z and v have to be assigned distinct colours:

Claim 1 *For any* 4*-colouring* c *of the graph of Figure 1(a),* $|c(N[v] \setminus \{u\})| = 4$.

We get the following claim as an immediate consequence of Claim 1.

Claim 2 *For any* 4*-colouring* c *of the graph of Figure 1(b),* $c(a) = c(b)$.

[1] This is claimed only for maximum degree 3 in [2] using a reduction to PLANAR NAE-3-SAT, which is polynomial-time solvable. But in fact one can observe that this holds for arbitrary planar graphs by using the fact that PLANAR NAE-k-SAT is polynomial-time solvable for any value of k [1].



(a) Vertex u adjacent to a vertex v with $|c(N[v] \setminus \{u\})| = 4$

(b) The colours of a and b are the same

Figure 1: Two useful subgraphs

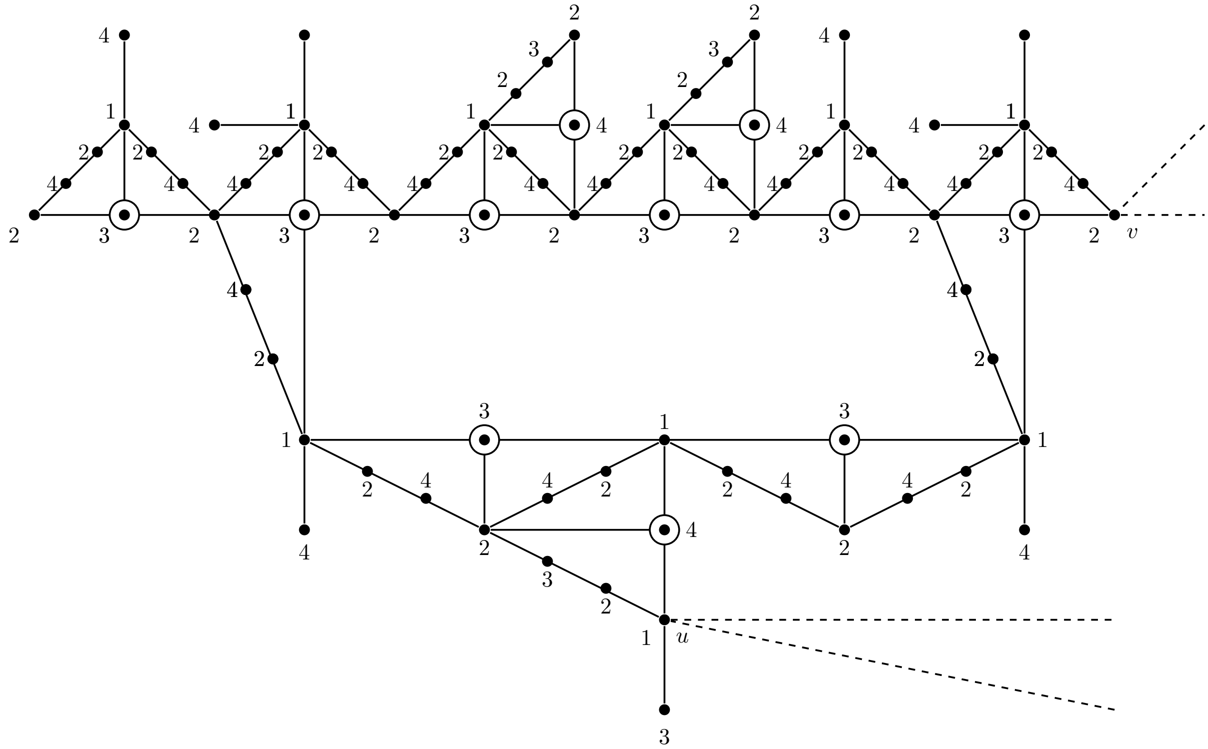


Figure 2: Colouring a part of the vertex gadget

Claim 3 *The graph of Figure 2 is* 4*-lid-colourable.*

A 4-lid-colouring is given in the Figure. Note that in the figure the vertex u coloured 1 which has adjacent dotted edges, is such that $|c(N[u]) = 4|$, hence adding adjacent vertices to u does not imply any changes on the given colouring of the graph which remains valid. Replacing the dotted part adjacent to v (coloured 2 in the figure), will not change the given colouring of the graph neither.

Given a planar graph G with maximum degree 4, we construct a planar graph G' as follows. We replace every vertex v of G by a copy G_v of the vertex gadget depicted in Figure 3. For every edge uv, we add an edge between x_i of G_u and x_j of G_v and we identify one vertex y_i^k of G_u with one vertex y_j^l of G_v, such that the obtained graph G' has a planar representation. Note that G' has maximum degree 5 and the smallest cycle is of length 5.

It is left as an exercise to the reader to see that the vertex gadget of Figure 3 is 4-lid-colourable using Claim 3. By Claim 2, one can observe that in any 4-lid-colouring of G', vertices y_i^k with $i \in \{1, ..., 4\}$ and $k \in \{1, 2\}$, receive the same colour (colour 2 in the figure) and we call it *forbidden colour* of G_u. Using the same argument, vertices x_i' receive the same colour (colour 1 in the figure) and we say that it is the *colour of* G_u. For every edge uv in G, in G' the colours of G_u and G_v are always different and the forbidden colour is the same. Hence, the forbidden colour is the same for all the graphs G_v with $v \in V(G)$ (G is connected).

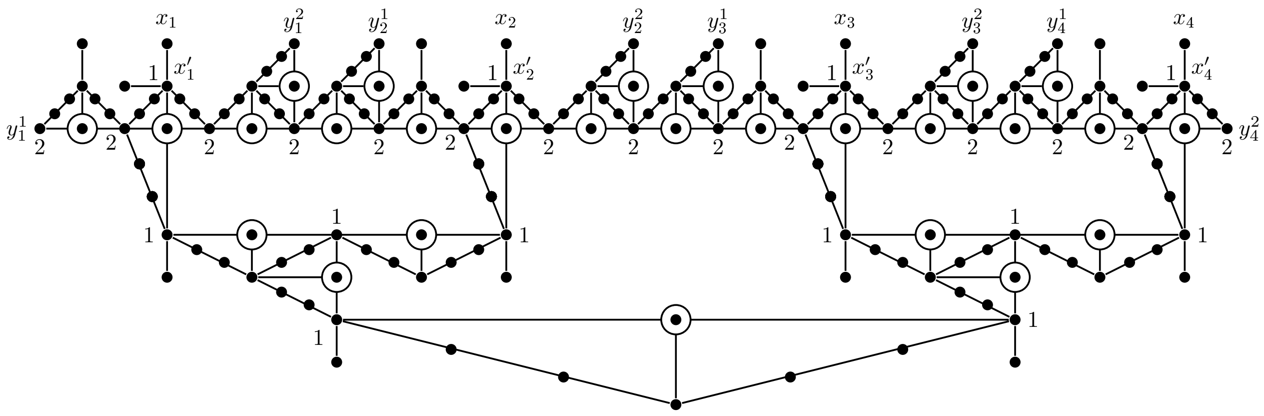


Figure 3: Vertex gadget

Now we are ready to prove the reduction. Suppose G is 3-colourable and consider a proper vertex colouring c of G with three colours. For a vertex $v \in V(G)$, we colour G_v of G' with $c(v)$ and extend c to a 4-lid-colouring c' of G'. Note that vertex x'_i has a neighbour of degree 1 and its colour can be chosen such that $|c(N[x'_i])| = 4$. Conversely, suppose G' is 4-lid-colourable and consider one such colouring. A proper vertex 3-colouring of G can be obtained by assigning v of G, the colour of G_v of G'. We recall that the forbidden colour of G_v is never used in the colouring of G. Therefore, G' is 4-lid-colourable if and only if G is 3-colourable. $\square$

References

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