Aspects of GEOMETRIC LOGIC

Talk given at Workshop
Logic, Categories, Semantics
Bordeaux 12 Nov 2010

Steve Vickers
School of Computer Science
University of Birmingham

- Logic, Categorical semantics

USPs:

- Geometric types
- Ontology – observational
- Geometricity = continuity
- Fibrewise topology (bundles)

Geometric logic

- theories – must use sequents
- examples: topological, algebraic, ...
- inference rules

Geometric logic

First order, many sorted, positive, infinitary

Signature Σ: Sorts, functions, predicates

Formulae ϕ: use $\top, \wedge, \bot, \bigvee, =, \exists$

disjunctions can be infinite

Formulae in context $(\vec{x}.\ \phi)$

finite list of sorted variables; All free variables are in $\vec{x}$

Sequent $\phi \vdash_{\vec{x}} \psi$ $(\vec{x}.\phi)$, $(\vec{x}.\psi)$ both formulae in context

$(\forall \vec{x})(\phi \rightarrow \psi)$

Theory $\mathbb{T}$ over Σ: set of sequents

axioms

Example I: Reals

Topology: P_{qr} – open interval (q,r)

Signature: no sorts

propositions (nullary predicates) P_{qr} $(q,r \in \mathbb{Q})$

Axioms:

$$P_{qr} \wedge P_{q'r'} \dashv\vdash \bigvee \{P_{st} \mid \max(q,q') < s < t < \min(r,r')\}$$

$$\top \vdash \bigvee \{P_{q-\varepsilon,q+\varepsilon} \mid q \in \mathbb{Q}\} \quad (0<\varepsilon\in\mathbb{Q})$$

read $\bigvee$ as $\bigcup$
$\wedge$: $\cap$
$\vdash$: $\subseteq$

Example II: Commutative rings

Algebra

Signature: sort – R

functions – $0, 1 : 1 \to R$

$- : R \to R$

$+, \cdot : R^2 \to R$

Axioms:

$\top \vdash_{xyz} x + (y+z) = (x+y)+z \qquad \top \vdash_{xyz} x\cdot(y\cdot z) = (x\cdot y)\cdot z$

etc.

$\top \vdash_{xy} x+y = y+x \qquad \top \vdash_{xy} x\cdot y = y\cdot x$

$\top \vdash_{x} x+0 = x \qquad \top \vdash_{x} x\cdot 1 = x$

$\top \vdash_{x} x+(-x) = 0$

$\top \vdash_{xyz} x\cdot(y+z) = x\cdot y + x\cdot z$

Example III: Commutative local rings

Neither topology nor pure algebra

Signature: Same as commutative rings

Axioms: Same as commutative rings

$+ \qquad (\exists z)\ (x+y)\cdot z = 1 \vdash_{xy} (\exists z)\, x\cdot z = 1 \vee (\exists z)\, y\cdot z = 1$

$0 = 1 \vdash \bot$

Invertibles form complement of a proper ideal

Follow account in Elephant

Inference rules I: Propositional

Sequent based: because no $\to$. No other surprises

$$\frac{}{\phi \vdash_{\vec{x}} \phi} \qquad \frac{\phi \vdash_{\vec{x}} \psi \quad \psi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \chi} \qquad \frac{}{\phi \vdash_{\vec{x}} \top}$$

$$\frac{}{\phi\wedge\psi \vdash_{\vec{x}} \phi} \qquad \frac{}{\phi\wedge\psi \vdash_{\vec{x}} \psi} \qquad \frac{\chi \vdash_{\vec{x}} \phi \quad \chi \vdash_{\vec{x}} \psi}{\chi \vdash_{\vec{x}} \phi\wedge\psi}$$

$$\frac{}{\phi \vdash_{\vec{x}} \bigvee S}\ (\phi \in S) \qquad \frac{\phi \vdash_{\vec{x}} \psi\ (\text{all } \phi \in S)}{\bigvee S \vdash_{\vec{x}} \psi}$$

$$\frac{}{\phi \wedge \bigvee S \vdash_{\vec{x}} \bigvee\{\phi\wedge\psi \mid \psi \in S\}}$$

Need this in the absence of $\to$

Inference rules II: Predicate

Always assuming formulae in correct contexts

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$$

($\vec{s}$ a vector of terms in context $\vec{y}$ with sorts matching $\vec{x}$)

Justifies $\dfrac{\phi \vdash_{\vec{x}} \psi}{\phi \vdash_{\vec{x}y} \psi}$

$$\frac{}{\top \vdash_{x} x = x}$$

$$\frac{}{\vec{x} = \vec{y} \wedge \phi \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]}$$

$$\frac{\phi \vdash_{\vec{x}y} \psi}{(\exists y)\phi \vdash_{\vec{x}} \psi}$$

$$\frac{}{\phi \wedge (\exists y)\psi \vdash_{\vec{x}} (\exists y)(\phi\wedge\psi)}$$

Need this in absence of $\to$

Again no surprises except ...

$(\forall x)\phi$

Suppose have $\top \vdash_x \phi$ as axiom

Deduce:

$$\dfrac{\top \vdash_x \phi \qquad \dfrac{\overline{(\exists x)\phi \vdash (\exists x)\phi}}{\phi \vdash_x (\exists x)\phi}}{\top \vdash_x (\exists x)\phi}$$

CANNOT conclude $\top \vdash (\exists x)\phi$ — even though no free variables

Rules work correctly for empty carriers

Categorical semantics

- models in Grothendieck toposes
- axiom of unique choice
- geometric types
- geometricity of constructions

Semantics - categorical

Again, see Elephant

Syntax	Interpretation
sort	object — carriers
sort tuple	product
in context: term	morphism
in context: formula	subobject
$\wedge$	pullback
$=$	equalizer
$\exists$	image
$\vee$	image of coproduct
sequent	truth value (order relation between subobjects)

Categorical structure required

to interpret geometric logic & its rules validly

= geometric category

In practice want more: Grothendieck topos

In particular, often want unique choice

- non-logical principle
- every total, single-valued relation defines a morphism
- categorically: monic epis are isos — balanced

Can use less: arithmetic universe — Joyal

- if disjunctions countable & internalizable as $\exists$

$$\bigvee_{n\in\mathbb{N}} \phi_n(x) \longmapsto (\exists n:N)\,\phi(n,x)$$

USPs:
→ Geometric types
· Ontology – observational
· Geometricity = continuity
· Fibrewise topology (bundles)

Geometric types
- characterized uniquely up to iso by geometric structure & axioms
- geometric v. non-geometric constructions in Grothendieck toposes

---

Geometric types — characterized uniquely upto iso by geometric structure & axioms

e.g. Sorts A, L, — $L = \mathrm{List}(A)$

Functions $\mathrm{nil}: 1 \to L$, $\mathrm{cons}: A \times L \to L$

Axioms $\mathrm{cons}(a,l) = \mathrm{nil} \vdash_{a,l} \bot$

$\mathrm{cons}(a,l) = \mathrm{cons}(a',l') \vdash_{aa'll'} a = a' \wedge l = l'$

$\top \vdash_l \bigvee_{n \in \mathbb{N}} (\exists a_0, \ldots, a_{n-1} : A)\ \underbrace{l = \mathrm{cons}(a_0, \mathrm{cons}(a_1, \ldots \mathrm{cons}(a_{n-1}, \mathrm{nil}) \ldots))}_{\text{recursively defined family of formulae, indexed by } n}$ — abbreviate $[a_0, \ldots, a_{n-1}]$

Interpretation of L has to be parametrized list object of that of A. Impossible with finitary logic

---

Proof sketch Given functions $f: B \to Y$, $g: A \times Y \to Y$

want unique $r = \mathrm{rec}(f,g): L \times B \to Y$

$$\begin{tikzcd} B \arrow[r, "\langle \mathrm{nil}!, \mathrm{id} \rangle"] \arrow[dr, "f"'] & L \times B \arrow[d, "r"] & A \times L \times B \arrow[l, "\mathrm{cons} \times B"'] \arrow[d, "A \times r"] \\ & Y & A \times Y \arrow[l, "g"] \end{tikzcd}$$

Logically define graph of r, $\gamma \subseteq L \times B \times Y$
- $\gamma(l, b, y) \equiv \bigvee_n (\exists a_0, \ldots, a_{n-1})(l = [a_0, \ldots, a_{n-1}] \wedge y = g(a_0, g(a_1, \ldots, g(a_{n-1}, f(b)) \ldots)))$
- Prove γ total & single valued
- Appeal to unique choice to get r
- Prove uniqueness of r.

Geometric constructions – e.g.

finite limits, set-indexed colimits } cf. Giraud's theorem characterizing Grothendieck toposes

free algebras – eg. $\mathbb{N}$, list objects

finite power set = free semilattice — also get finitely bounded $\forall$

$\mathbb{Z}, \mathbb{Q}$

Non-geometric constructions

exponentials (function types)

Ω, powersets $\mathcal{P}X$

$\mathbb{R}, \mathbb{C}$ (as sets) — various kinds

"set" = object of topos

Geometric types: two views

Syntactic sugar

Nice but not strictly necessary.

Can do it all with infinite disjunctions

Improve foundations

Avoid dependence on external infinities (at least for countable V)

Arithmetic universes

Useful either way

Example: the reals

Sorts: none

none declared – but $\mathbb{Q}$ constructed out of nothing

Predicates: $L, R \subseteq \mathbb{Q}$

Axioms: $\top \vdash (\exists q : \mathbb{Q})\, L(q)$ $\quad \top \vdash (\exists r : \mathbb{Q})\, R(r)$

$L(q) \dashv\vdash_{q:\mathbb{Q}} (\exists q' : \mathbb{Q})(q < q' \wedge L(q'))$ $\quad R(r) \dashv\vdash_{r:\mathbb{Q}} (\exists r' : \mathbb{Q})(r > r' \wedge R(r'))$

$L(q) \wedge R(q) \vdash_{q:\mathbb{Q}} \bot$ $\quad q < r \vdash_{q,r:\mathbb{Q}} L(q) \vee R(r)$

- Directly describes Dedekind sections
- Equivalent to propositional version

USPs:
- Geometric types
- Ontology – observational
- Geometricity = continuity
- Fibrewise topology (bundles)

Ontology

- observational
- sets: existence + equality of elements

Ontology – matching logic to what you're talking about

Geometric logic ↔ observational ontology

Formula — finitely observable property

$\wedge$ — observe all conjuncts

$\bigvee$ — observe one disjunct

$\neg, \rightarrow$ – no observational account

Topology via Logic

Sequent — background assumption
– scientific hypothesis

Popperian refutation

refutability

Suppose -

- theory T includes some axioms $\phi \vdash \perp$
- experimental observations E expressed as axioms $T \vdash \psi$
- in $T \cup E$ can infer $T \vdash \perp$

Then theory T is refuted by experimental results E.

Predicate ontology

(Observations "serendipitous"
Serendipity = the faculty of making happy chance finds)

"Observable set"

- How can you know that you have apprehended an element?
- How can you know that two elements are equal?

e.g. G a finitely presented group

- To apprehend element: write word in generators
- To affirm equality: find proof from relations

NOTE – If word problem undecidable, then inequality is not observable in same sense.

Ontology of $\exists$

$(\exists y)\ \phi(x,y) \qquad x:X, y:Y$

To apprehend element:
apprehend x, y & affirm $\phi(x,y)$

To find equality $\langle x,y\rangle = \langle x',y'\rangle$, affirm $x = x'$

Ontology of $\psi(x) \vdash_x (\exists y)\phi(x,y)$

If you have apprehended x & affirmed ψ then

- You already have y — too strong
- You know how to find y — constructivist
- There is a y somewhere — cf. scientific hypothesis

UNIQUE CHOICE $\Rightarrow$ functions interpreted the same way.

Ontology of list objects

Given observable set A:

To apprehend element of List(A) –
get a natural number n, and for each $0 \leq i < n$ apprehend an element a_i of A

To affirm $\langle n, (a_i)_0^{n-1}\rangle = \langle n', (a'_i)_0^{n'-1}\rangle$,
find $n = n'$ and affirm $a_i = a'_i$ for each $0 \leq i < n$

Toposes as spaces

USPs:
- Geometric types
- Ontology – observational
- → Geometricity = continuity
- Fibrewise topology (bundles)

general

classifying categories

functors as model transformers

toposes

classifying toposes

geometric morphisms

geometricity = continuity

Classifying categories: general technique

Logic $\mathcal{L}$ interpreted using categorical structure $\mathbb{C}$.

Given $\mathcal{L}$ theory $\mathbb{T}$:
- For $\mathbb{C}$-cat $\mathcal{C}$: category $\mathrm{Mod}_{\mathbb{T}}(\mathcal{C})$ of models in $\mathcal{C}$
- For $\mathbb{C}$-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, have functor $\mathrm{Mod}_{\mathbb{T}}(F): \mathrm{Mod}_{\mathbb{T}}(\mathcal{C}) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{D})$
- Classifying category $\mathcal{C}_{\mathbb{T}}$, with generic $\mathbb{T}$-model M_g
  $\mathbb{C}\text{-cat}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{C})$, $F \mapsto \mathrm{Mod}_{\mathbb{T}}(F)(M_g)$ is equivalence

  Idea: $\mathcal{C}_{\mathbb{T}}$ freely generated as $\mathbb{C}$-cat by M_g
- Trivial theory $\mathbb{T}_0$ classified by initial $\mathbb{C}$-cat
- $\mathbb{C}$-functor $\mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$ $\approx$ model of $\mathbb{T}_1$ in $\mathcal{C}_{\mathbb{T}_2}$

Classifying categories as spaces

- Work in opposite of cat of $\mathbb{C}$-cats
- Initial $\mathbb{C}$-cat becomes final. Write it as 1
- Point $1 \to \mathcal{C}_{\mathbb{T}}$ $\approx$ model of $\mathbb{T}$ in $\mathcal{C}_{\mathrm{init}}$
- Generalized point $\mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ $\approx$ - - - - - - - - $\mathcal{C}$

  Points of $\mathcal{C}_{\mathbb{T}}$ = models of $\mathbb{T}$ … $\mathcal{C}_{\mathbb{T}}$ = "space of $\mathbb{T}$-models"
- $f: \mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$

match
1. transforms points $M \mapsto f \circ M$ $(M: \mathcal{C} \to \mathcal{C}_{\mathbb{T}_1})$
2. transforms models. f = model of $\mathbb{T}_2$ in $\mathcal{C}_{\mathbb{T}_1}$ —

$f(M_g)$ constructed by $\mathbb{C}$ constructions out of M_g

M as functor preserves $\mathbb{C}$-constructions

$f(M)$ made by same construction

Classifying toposes

Grothendieck toposes

Geometric logic

Preserves colimits, finite limits

Classifying topos $\mathcal{S}[\mathbb{T}]$

Set

Classifying categories: general technique

Logic $\mathcal{L}$ interpreted using categorical structure $\mathbb{C}$.

Given $\mathcal{L}$ theory $\mathbb{T}$:
- For $\mathbb{C}$-cat $\mathcal{C}$: category $\mathrm{Mod}_{\mathbb{T}}(\mathcal{C})$ of models in $\mathcal{C}$
- For $\mathbb{C}$-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, have functor $\mathrm{Mod}_{\mathbb{T}}(F): \mathrm{Mod}_{\mathbb{T}}(\mathcal{C}) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{D})$
- Classifying category $\mathcal{C}_{\mathbb{T}}$, with generic $\mathbb{T}$-model M_g
  $\mathbb{C}\text{-cat}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{C})$, $F \mapsto \mathrm{Mod}_{\mathbb{T}}(F)(M_g)$ is equivalence

  Idea: $\mathcal{C}_{\mathbb{T}}$ freely generated as $\mathbb{C}$-cat by M_g
- Trivial theory $\mathbb{T}_0$ classified by initial $\mathbb{C}$-cat
- $\mathbb{C}$-functor $\mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$ $\approx$ model of $\mathbb{T}_1$ in $\mathcal{C}_{\mathbb{T}_2}$

Geometric morphisms — $f: \mathcal{E} \to \mathcal{F}$ is

$f^*: \mathcal{F} \to \mathcal{E}$ preserving colimits, finite limits $\simeq$ $\mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{F}$, f^* preserving finite limits

Topos = "space of models" for some geometric theory

Write $[\mathbb{T}]$ instead of $\mathcal{S}[\mathbb{T}]$ in dual category

Classifying categories as spaces

- Work in opposite of cat of $\mathbb{C}$-cats
- Initial $\mathbb{C}$-cat becomes final. Write it as 1
- Point $1 \to C_{\mathbb{T}}$ $\simeq$ model of $\mathbb{T}$ in C_{init}
- Generalized point $C \to C_{\mathbb{T}}$ $\simeq$ - - - - - - - - C

  Points of $C_{\mathbb{T}}$ = models of $\mathbb{T}$ — $C_{\mathbb{T}}$ = "space of $\mathbb{T}$-models"
- $f: C_{\mathbb{T}_1} \to C_{\mathbb{T}_2}$

match
① transforms points $M \mapsto f \circ M$ $(M: C \to C_{\mathbb{T}_1})$
② transforms models. f = model of $\mathbb{T}_2$ in $C_{\mathbb{T}_1}$ —
$f(M_g)$ constructed by $\mathbb{C}$ constructions out of M_g
M as functor preserves $\mathbb{C}$-constructions
$f(M)$ made by same construction

geometric morphism = model transformer

Geometric morphism = model transformer

To define $f: [\mathbb{T}_1] \to [\mathbb{T}_2]$

Let x be a model of $\mathbb{T}_1$ Then $f(x)$ = ⋮ is a model of $\mathbb{T}_2$

geometric construction

geometric logic is incomplete

NB No problem if $\mathbb{T}_1$ has insufficient models in Set
Geometric construction works uniformly for all models in all toposes – including generic model

Geometricity = continuity

For propositional geometric logic:
same trick works giving frames & frame homomorphisms

complete lattice, $\wedge$ distributes over $\bigvee$

preserve $\wedge \bigvee$

– localic analogue of continuous maps

Propositional theory $\mathbb{T}$ $\mapsto$ frame $\Omega[\mathbb{T}]$ — treat $\mathbb{T}$ as frame generators & relations

Theorem $\mathcal{S}[\mathbb{T}]$ = topos of sheaves over $\Omega[\mathbb{T}]$

Geometric morphism $[\mathbb{T}_1] \longrightarrow [\mathbb{T}_2]$ — $\mathcal{S}[\mathbb{T}_1] \underset{\to}{\overset{\leftarrow}{\perp}} \mathcal{S}[\mathbb{T}_2]$

$\simeq$ locale map $[\mathbb{T}_1] \longrightarrow [\mathbb{T}_2]$ — $\Omega[\mathbb{T}_1] \leftarrow \Omega[\mathbb{T}_2]$

Geometricity = continuity

More generally: take "(continuous) maps" to be geometric constructions / morphisms

e.g. sheaves $\mathbb{T}_{obj}$: one sort, nothing else
model = object

Geometric morphism $[\mathbb{T}] \longrightarrow [\mathbb{T}_{obj}]$
= object of $\mathcal{S}[\mathbb{T}]$ - - - - → sheaf
= geometric construction
model M of $\mathbb{T}$ $\mapsto$ set — stalk at M

sheaf = "continuous set valued map —
map : space of models of $\mathbb{T}$ → space of sets"

For simplicity:
work with locales
(all theories propositional)

USPs:
- Geometric types
- Ontology – observational
- Geometricity = continuity
- → Fibrewise topology (bundles)

Bundles

– maps $\begin{array}{c}[\mathbb{T}_1]\\ \downarrow \\ [\mathbb{T}_2]\end{array}$ thought of as space (fibre) parametrized by base point

≈ internal locale in $\mathcal{S}[\mathbb{T}_2]$

Fourman / Scott / Joyal / Tierney

– get fibrewise topology of bundles

Locale constructions

If construction (on frames) is topos-valid:
also gives construction on bundles

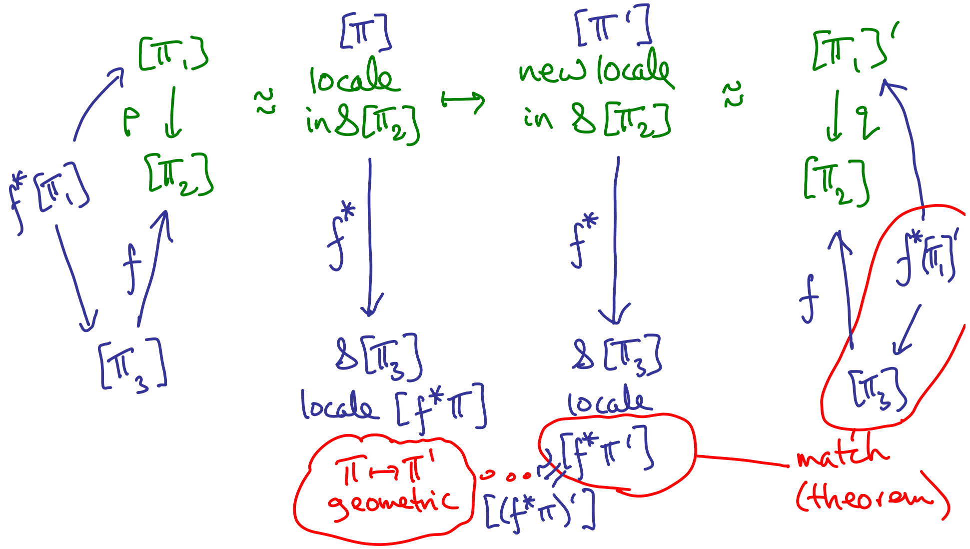
$$\begin{array}{c}[\mathbb{T}_1]\\ p\downarrow \\ [\mathbb{T}_2]\end{array} \approx \begin{array}{c}\text{locale}\\ \text{in } \mathcal{S}[\mathbb{T}_2]\end{array} \mapsto \begin{array}{c}\text{new locale}\\ \text{in } \mathcal{S}[\mathbb{T}_2]\end{array} \approx \begin{array}{c}[\mathbb{T}_1]'\\ \downarrow q \\ [\mathbb{T}_2]\end{array}$$

Geometric locale constructions

Internal frame has internal presentation
$\Omega[\mathbb{T}]$ ≀ (frame), $\mathbb{T}$ ≀ (presentation)

If construction can be done
geometrically on presentations
then bundle construction preserved by pullbacks
∴ works fibrewise .... fibres are pullbacks along global points

Examples
Powerlocales (localic hyperspaces)
Valuation locales .... get geometric characterization of, e.g. compactness



Geometricity in general

- Locale construction is geometric if (on bundles) preserved by pullback
- Generalizes geometricity for set constructions (bundle = local homeomorphism)
- Logically: define bundle $p: [\mathbb{T}_1] \to [\mathbb{T}_2]$ by

> Let x be a model of $\mathbb{T}_2$
> Then $p^{-1}(x) =$
>
> is a locale

geometric construction

Selected bibliography

Geom. logic, cat. semantics

Johnstone: Sketches of an elephant vol 2

Joyal & Tierney: An extension of the Galois theory of Grothendieck

Bundles

Vickers: Topology via logic

Ontology

Issues of logic, algebra & topology in ontology (chapter in Theory & Applications of Ontology vol 2)

Locales & toposes as spaces (chapter in Handbook of Spatial Logics)

geom. types

Topical categories of domains

geometricity = continuity, bundles

The double powerlocale and exponentiation