Types as Weak ∞-Groupoids

Thomas Streicher (TU Darmstadt)

November 2010

Identity Types (1)

are the most intriguing concept of intensional Martin-Löf type theory (ITT). They are given by the rules

$$\frac{\Gamma \vdash A}{\Gamma, x, y{:}A \vdash \mathsf{Id}_A(x,y)} \ (\mathsf{Id}\text{-}F)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash r_A(x) : \mathsf{Id}_A(x,x)} \ (\mathsf{Id}\text{-}I)$$

$$\frac{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash C(x,y,z) \qquad \Gamma, x{:}A \vdash d : C(x,x,r_A(x))}{\Gamma, x, y{:}A, z : \mathsf{Id}_A(x,y) \vdash J((x)d)(z) : C(x,y,z)} \ (\mathsf{Id}\text{-}E)$$

together with the conversion rule

$$J((x)d)(r_A(t)) = d[t/x]$$

and motivated by the intention that *all* concepts appear as inductively defined (families of) types.

Identity Types (2)

Using J one can define operations

$cmp_A \in (\Pi x, y, z{:}A)\,\mathsf{Id}_A(x,y) \to \mathsf{Id}_A(y,z) \to \mathsf{Id}_A(x,z)$

$inv_A \in (\Pi x, y{:}A)\,\mathsf{Id}_A(x,y) \to \mathsf{Id}_A(y,x)$

validating (where we write id_x for $r_A(x)$)

(a) $(\Pi x, y, z, u{:}A)$
$(\Pi f{:}\mathsf{Id}_A(x,y))(\Pi g{:}\mathsf{Id}_A(y,z))(\Pi f{:}\mathsf{Id}_A(z,u))$
$\mathsf{Id}_{\mathsf{Id}_A(x,u)}(cmp(f, cmp(g,h)), cmp(cmp(f,g),h))$

(b) $(\Pi x, y{:}A)\ \mathsf{Id}(cmp(id_x, f), f) \wedge \mathsf{Id}(cmp(g, id_y), g)$

(c) $(\Pi x, y{:}A)(\Phi f{:}\mathsf{Id}_A(x,y))$
$\mathsf{Id}(cmp(f, inv(f)), id_x) \wedge \mathsf{Id}(cmp(inv(f), f), id_y)$

rendering type A as an **internal groupoid where the groupoid equations hold only in the sense of propositional equality**, i.e. for instance (a) means that there is a term $assoc_A(f,g,h)$ of type
$\mathsf{Id}_{\mathsf{Id}_A(x,u)}(cmp(f, cmp(g,h)), cmp(cmp(f,g),h))$
which may be thought of as a **2-cell** in the sense of *higher dimensional categories*.

The Groupoid Model

In early 1990ies I observed that one can prove

$$(\Pi A{:}\mathsf{Set})(\Pi x, y{:}A)(\Pi{:}f, g{:}\mathsf{Id}_{\mathsf{Id}_A(x,y)}(f, g))$$

i.e. *proof irrelevance for equality proofs* (PIE). using the following natural extension of MLTT

$$\frac{\Gamma, x{:}A, z : \mathsf{Id}_A(x, x) \vdash C(x, z) \qquad \Gamma, x{:}A \vdash d : C(x, r_A(x))}{\Gamma, x{:}A, z : \mathsf{Id}_A(x, x) \vdash K((x)d)(z) : C(x, z)} \ (\mathsf{Id}\text{-}E')$$

In 1994 [HS95] M. Hofmann and I constructed a groupoid model for ITT where K does not exists and (a)-(c) hold in the sense of judgemental equality. The **key idea** was to interpret **types as groupoids** and **families of types as fibrations of groupoids** and

$$\mathsf{Id}_A(x, y) \quad \text{as} \quad A(x, y)$$

which may contain more than one element if the groupoid is not posetal. Thus

PIE fails in the groupoid model!

Towards Weak ω-Groupoids

Already in [HS95] we observed that

> the **bureaucracy of identity types** forces one to **check all coherence conditions** when reasoning up to isomorphism

i.e. when treating ‘isomorphic’ as ‘equal’ (as categorists like to do) which sometimes is a source of mistakes when done naively!

Already in [HS95] it was observed that ∞-groupoids might be more appropriate since in ITT the types $\mathrm{Id}_A(x, y)$ are groupoids themselves and not just sets.

We also observed that strict ω-groupoids are not sufficient either because in ITT the conditions (a), (b) and (c) do **not hold in the sense of judgemental equality** but **only in the sense of propositional equality**, i.e. that **weak ω-groupoids are more appropriate**.

Towards Weak ω-Groupoids

Many definitions of hdc's and hdg's are fairly complex. Therefore, in a talk in Uppsala (November 2006) I suggested to consider the easiest notion of weak higher dimensional groupoid, namely **Kan complexes** in the category (topos) $\mathcal{SS} = \widehat{\mathbf{\Delta}}$ of *simplicial sets*. Accordingly, families of types will be modeled as **Kan fibrations**.

The latter form part of the classical Quillen model structure on $\mathcal{SS}$. Awodey & Warren promoted the idea of interpreting Id-types in Quillen model structures. But there is a problem with BC which can be overcome when having a universe available.

Independently, V. Voevodsky (October 2006) suggested to interpret type theory in simplicial sets (see `www.math.ias.edu/~vladimir`). In particular, he came up with a construction of universes and suggested his **Equivalence Axiom** roughly saying that types are equal iff there is a weak equivalence between them.

A Recap of $\mathcal{SS}$

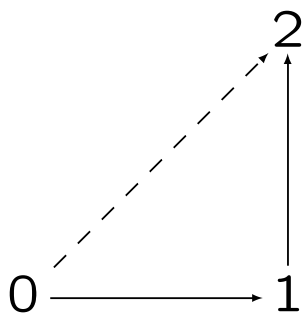
Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and order preserving maps between them. We write $[n]$ for $\{0, 1, \ldots, n\}$. The maps of $\mathbf{\Delta}$ are generated by the morphisms

$$d^i_n : [n{-}1] \to [n] \qquad s^i_n : [n] \to [n{-}1]$$

where the first one is monic and omits i and the second one is epic and "repeats" i.
We write $\mathcal{SS}$ for $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ and Δ^n for Yoneda of $[n]$.
For $0 \leq i \leq n$ let $\partial_i \Delta^n$ be the subobject of Δ^n consisting all maps $u : [m] \to [n]$ with $i \notin \mathrm{im}[u]$. Let $\partial\Delta^n = \bigcup_{i=0}^{n} \partial_i \Delta^n$ called the *boundary* of Δ^n.
For $0 \leq k \leq n$ let $\Lambda^n_k = \bigcup_{i \neq k} \partial_i \Delta^n$, i.e. the union of all $(n{-}1)$-faces of Δ^n containing the node k.
Λ^n_k is an inner horn iff $0 < k < n$.

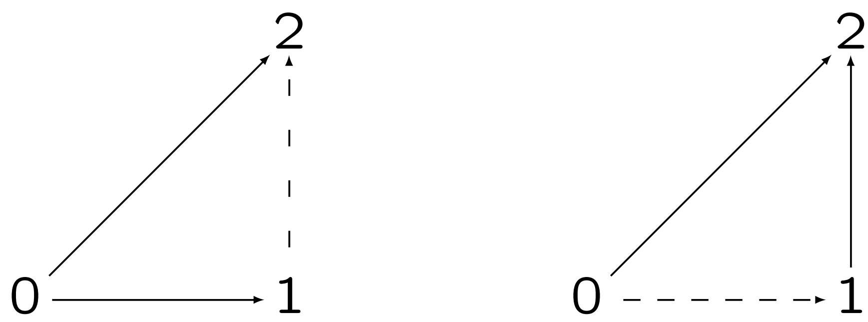
Pictures of Horns

The horn Λ^2_1 can be depicted as



where the omitted faces are indicated by broken lines.

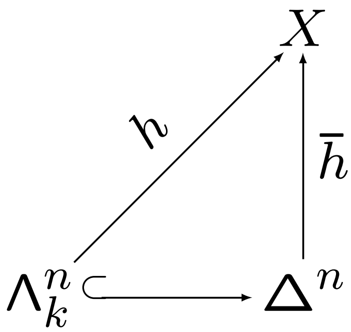
Λ^2_1 is an inner horn as opposed to the horns Λ^2_0 and Λ^2_2 depicted as



respectively.

Kan Fibrations

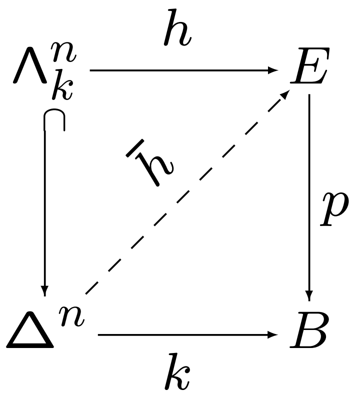
A **horn** in a simplicial set X is a morphism $h : \Lambda^k[n] \to X$. A **Kan complex** is a simplicial set X such that every horn $h : \Lambda^n_k \to X$ in X can be extended to some $\bar{h} : \Delta^n \to X$ making



commute (this extension need not be unique!).

Remark Requiring this only for inner horns gves rise to Joyal's notion of **quasi-category**.

A **Kan fibration** is a morphism $p : E \to B$ in $\mathcal{SS}$ such that every commuting square



has some (generally non-unique) filler $\bar{h}$.

Quillen structure on $\mathcal{SS}$

There is an obvious functor from Δ to $\mathbf{Sp}$ whose left Kan extension we denote as

$$|\cdot| : \mathcal{SS} \to \mathbf{Sp}$$

and call **geometric realization**. We call a map w in $\mathcal{SS}$ a **weak equivalence** iff $|w|$ is a homotopy equivalence in $\mathbf{Sp}$.

The **classical Quillen model structure** on $\mathcal{SS}$ is given by $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ where

$\mathcal{C}$ = class of monomorphisms

$\mathcal{W}$ = class of weak equivalences

$\mathcal{F}$ = class of Kan fibrations.

Closure Properties of $\mathcal{F}$

Since $\mathcal{SS}$ is a topos it is in particular locally cartesian closed. As $\mathcal{F}$ is defined by a weak orthogonality condition it is obvious that $\mathcal{F}$ is closed under Σ and Π, i.e.

(Σ) $\mathcal{F}$ is closed under composition

(Π) $\Pi_f(g) \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$.

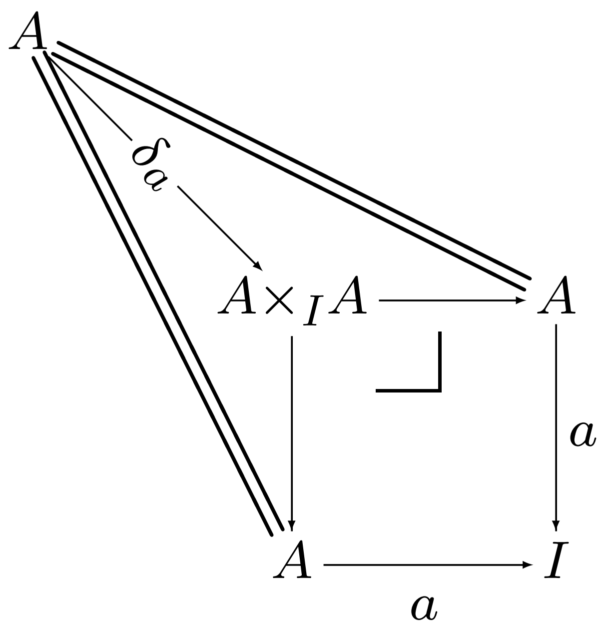
Thus $(\mathcal{SS}, \mathcal{F})$ gives a model of type theory without Id-types.

Let $\Delta \dashv \Gamma : \mathcal{SS} \to \mathrm{Set}$. Then all discrete simplicial sets $\Delta(S)$ are Kan complexes and all $\Delta(f)$ are Kan fibrations.

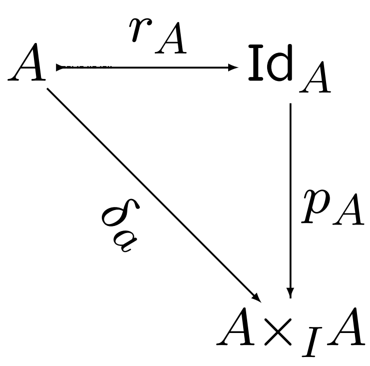
Thus $(\mathcal{SS}, \mathcal{F})$ contains Set as a submodel.

Interpreting Id-**Types (1)**

Awodey and Warren have suggested to interpret Id-types in Quillen model structures as follows. For a fibration $a : A \to I$ the map δ_a



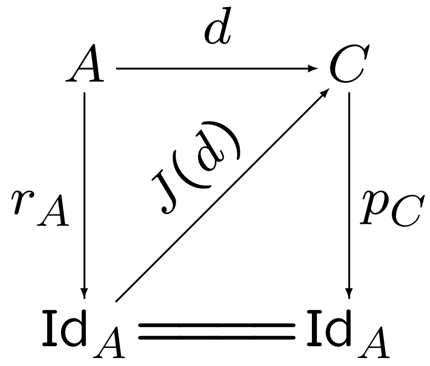
is a fibration, too. We may consider



with $p_A \in \mathcal{F}$ and $r_A \in \mathcal{C} \cap \mathcal{W}$.

Interpreting Id-Types (2)

Given a fibration $p_C : C \to \mathrm{Id}_A$ and $d : A \to C$ with $p_C \circ f = r_A$ then we have



for some $J(d)$.

But the **problem** is that $J(d)$ is not unique and thus one does not know how to make a choice which is stable under pullbacks along substitutions $u : J \to I$.

This problem, however, can be overcome when instantiating I by the *generic* context

$$A : \mathrm{Set}, C : (x, y{:}A)\mathrm{Set}^{\mathrm{Id}_A(x,y)}, d : (x{:}A)C(x, x, r_A(x))$$

where Set is some appropriate universe since then one has to split just once and for all !

Lifting Universes

If $\mathcal{U}$ is a (Grothendieck) universe in $\mathbf{Set}$ and $\mathcal{C}$ is a small category then this gives rise to a type-theoretic universe $p_U : \widetilde{U} \to U$ in $\mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$.
The object U is defined as

$$U(I) = \mathcal{U}^{(\mathcal{C}/I)^{\mathsf{op}}} \qquad U(\alpha) = \mathcal{U}^{\Sigma_\alpha^{\mathsf{op}}}$$

where for $\alpha : J \to I$ the functor $\Sigma_\alpha : \mathcal{C}/J \to \mathcal{C}/I$ is postcomposition with α.
The presheaf $\widetilde{U}$ is defined as

$$\widetilde{U}(I) = \{\langle A, a\rangle \mid A \in U(I) \text{ and } a \in A(id_I)\}$$

and

$$\widetilde{U}(\alpha)(\langle A, a\rangle) = \langle U(\alpha)(A), A(\alpha \xrightarrow{\alpha} id_I)(a)\rangle$$

for $\alpha : J \to I$ in $\mathcal{C}$.
The map $p_U : \widetilde{U} \to U$ sends $\langle A, a\rangle$ to A.
One easily checks that p_U is **generic** for maps with fibres small in the sense of $\mathcal{U}$, i.e. these maps are up to iso precisely those which can be obtained as pullback of p_U along some map in $\widehat{\mathcal{C}}$.

Lifting Universes to $\mathcal{SS}$

Now in case $\mathcal{C} = \Delta$ we adapt this idea in such a way that p_U is generic for Kan fibrations with fibres small in the sense of $\mathcal{U}$. For this purpose we **redefine** U as

$$U(n) = \{A \in \mathcal{U}^{(\Delta/[n])^{\mathrm{op}}} \mid P_A \text{ is a Kan fibration}\}$$

where $P_A : \mathsf{Elts}(A) \to \Delta[n]$ is obtained from A by the Grothendieck construction. For maps α in Δ we can define $U(\alpha)$ as above since Kan fibrations are stable under pullbacks. We define $\widetilde{U}$ and p_U using the same formulas as above but understood as restricted to U in its present form.

Obviously, families of simplicial sets with $\mathcal{U}$-small fibres are closed under Σ, Π.

Voevodsky has shown that U is a Kan complex and p_U is a Kan fibration. Thus p_U gives rise to a universe Set appropriate for interpreting Id-types.

Prop **in** $\mathcal{SS}$

Starting from $\mathcal{U} = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$ one gets a universe $p_U = \mathsf{El} \to \mathsf{Prop}$. Apparently Prop is closed under arbitrary Π's.

Notice that Prop($[n]$) consists of all monos $m : P \rightarrowtail [n]$ which are Kan fibrations. These are known to be trivial, i.e. either minimal or maximal. Thus, in $\mathcal{SS}$ we have Prop $\cong 2 = 1 + 1$ and

$$\begin{array}{ccc} \mathsf{Prf} & \rightarrowtail & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \top} \\ \mathsf{Prop} & \xrightarrow{[\top, \bot]} & \Omega \end{array}$$

i.e. this way we obtain an interpretation of Prop which is 2-valued, boolean and proof-irrelevant.

Although the interpretation of logic is quite as in **Set** equality on Set is fairly noncanonical because it validates Voevodsky's

Equivalence Axiom

We first introduce a few abbreviations

$$\mathsf{iscontr}(X : \mathsf{Set}) = (\Sigma x : X)(\Pi y : X)\,\mathsf{Id}_X(x, y)$$

$$\mathsf{hfiber}(X, Y : \mathsf{Set})(f : X \to Y)(y : Y) = \\ = (\Sigma x : X)\,\mathsf{Id}_Y(f(x), y)$$

$$\mathsf{isweq}(X, Y : \mathsf{Set})(f : X \to Y) = \\ = (\Pi y : Y)\,\mathsf{iscontr}(\mathsf{hfiber}(X, Y, f, y))$$

$$\mathsf{Weq}(X, Y : \mathsf{Set}) = (\Sigma f : X \to Y)\,\mathsf{isweq}(X, Y, f)$$

Using the eliminator J for identity types one easily constructs a map

$$\mathsf{eqweq}(X, Y : \mathsf{Set}) : \mathsf{Id}_{\mathsf{Set}}(X, Y) \to \mathsf{Weq}(X, Y)$$

Then the Equivalence Axiom

$$\mathsf{EquAx} : (\Pi X, Y : \mathsf{Set})\,\mathsf{isweq}(\mathsf{eqweq}(X, Y))$$

postulates that all maps eqeweq(X, Y) are themselves weak equivalences.

Voevodsky has shown that the Equivalence Axiom holds in the above model in simplicial sets.

Conclusion and Problems

- Simplicial sets provide a classical model of impredicative type theory extending the naive model in $\mathbf{Set}$.

- Types are interpreted as Kan complexes, i.e. weak higher dimensional groupoids. Families of types are Kan fibrations.

- Types in the universe Set validate the Equivalence Axiom roughly saying that types in Set are propositionally equal iff there is a weak equivalence between them. $\mathrm{Id}_{\mathsf{Set}}(X, Y)$ is not a proposition but a type in a universe containing Set as an element.

- There is no obvious computational meaning of the Equivalence Axiom!