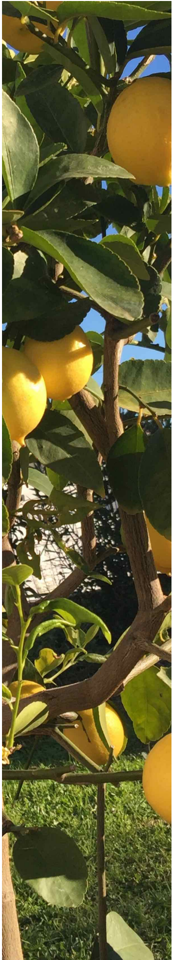


FROM

HANDSOME PROOF-NETS

TO

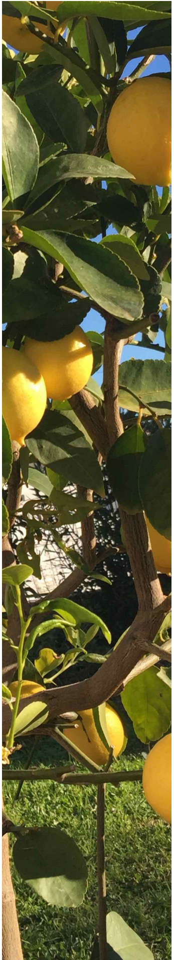
DEEP INFERENCE

$\sim 1994 \quad \longrightarrow \quad \sim 1999$

Christian Retoré (LIRMM, Univ Montpellier, CNRS)

2024, September 18

***in memoriam* Alessio Guglielmi (1964–2004)**



1. Thanks to Alessio

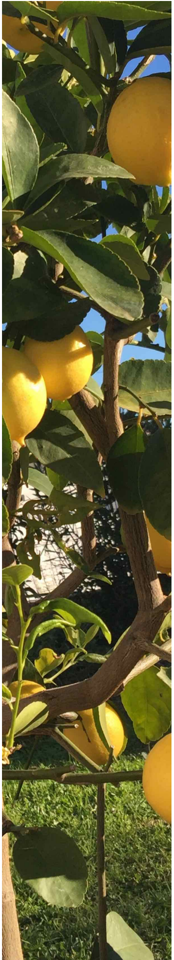
My work on proof nets and on pomset logic is known because of Alessio

who substantially extended it into a whole area of research

in directions i never though of

DEEP INFERENCE

Thanks, Alessio!



2. Thirty years earlier: 1994

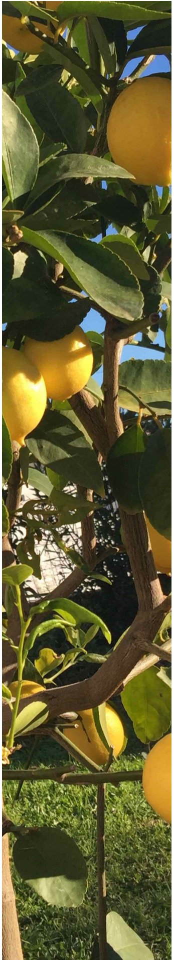
PhD (univ Paris 7 Nov 1987 – Feb 1993, adv. JY Girard):
linear logic (proof-nets, coherence semantics) introducing **pomset logic** as it's called,thanks to a suggestion by S.Abramsky (Sept 1989 - Dec 1990: research assistant, Imperial College)

Multiplicative Linear Logic + multiplicative connective $\triangleleft$
non commutative, associative, self-dual connective
$(A \triangleleft B)^{\perp} \equiv A^{\perp} \triangleleft B^{\perp} \quad (A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C) \quad A \triangleleft B \not\equiv B \triangleleft A$

Postdoc INRIA Nice during the academic year 1993-1994.

At INRIA, email and Sun machines, great for 1993!

In 1994 was contacted by email by Alessio who was interested in my "before" connective. This connective was called **seq(uential)** by Alessio and Lutz later on.



3. How did Alessio heard about ”before”?

PhD in 1993 but no publication on pomset logic before 1997

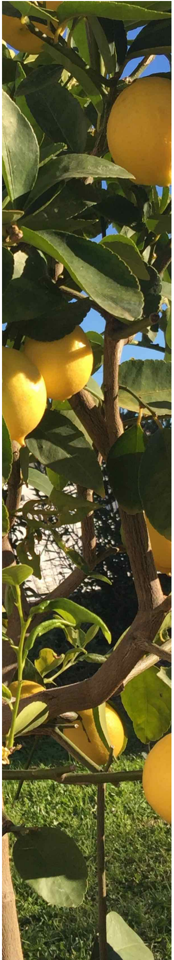
PhD Univ Paris 7, *Ch. Retoré — Réseaux et séquents ordonnés*

TLCA 1997 *Ch. Retoré — Pomset logic: a non commutative extension of classical linear logic*

Girard suggested the idea, but was not anymore interested (geometry of interaction).
Girard in a talk: unlikely but possible.

Berry, Boudol (INRIA) $\longrightarrow$ Levi, Montanari,...
Possibly: all these people were travelling all around the world in concurrency conferences.

Another possibility, via Samson Abramsky: Uday S. Reddy who wrote in 1993 *A Linear Logic Model of State, manuscript, Urbana Champaign* using pomset logic and ”before”.



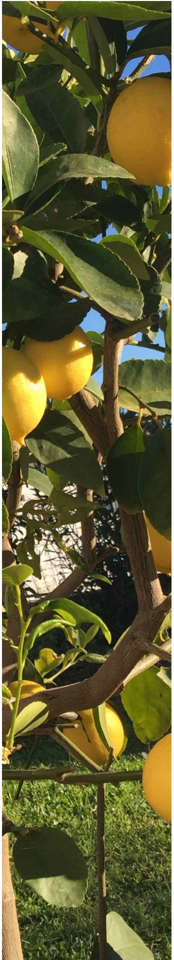
4. Alessio's view point?

Alessio told me he was interested to model concurrent computations, but also how unhappy he was with process calculi like π calculus.

He was interested in the chemical abstract machine by Berry and Boudol (1992, TCS) but he wanted an algebraic process calculus (with rewriting?)

Pietro Di Gianantonio, Alessio Guglielmi, Giorgio Levi: Chemical Logic Programming? ICLP Workshop on Blackboard-Based Logic Programming 1993

Alessio Guglielmi Concurrency and Plan Generation in a Logic Programming Language with a Sequential Operator ICLP, 1994.



5. Why was Alessio interested in ”before”?

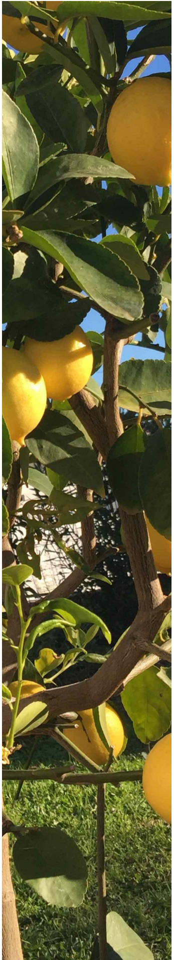
Personnally I was still struggling to understand process calculi, although i spent 1989-1990 as a research assistant / PhD student at Imperial College where i met Pietro Di Gianantonio (they were all studying CCS π-calculus . . .)

Alessio was inquiring about linear logic,
with propositions as actions:

$a \otimes b$ actions a and b happen and are fully mixed, only an $a^\perp$ may bring back b

$a \parr b$ one of a and b happen but you cannot tell which one

$a < b$ simply means $a;b$, a then b that the sequential operator.



6. The story behind ”before”: coherence spaces

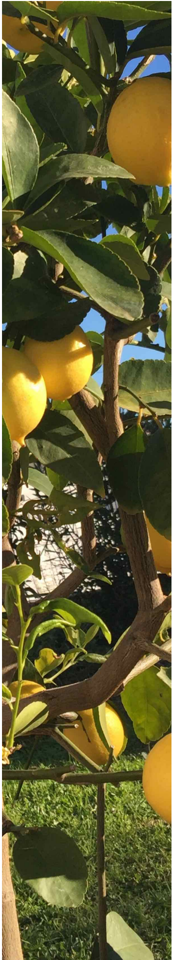
Formula X object of the category, simple graph $(|X|, \frown)$

Negation:
If $A = (|A|, \frown_A)$ then $A^\perp = (|A|, \frown_A^\perp)$ with $\alpha \frown \alpha'[A^\perp]$ iff $\alpha \not\frown \alpha'[A]$

proof $\pi : A \vdash B$: morphism $[\![\pi]\!] : A \mapsto B$ (linear map)

proof $\pi : \vdash B$: $[\![\pi]\!]$ clique in B = map $1 \mapsto B$

whenever $\pi \rightsquigarrow \pi'$: $[\![\pi]\!] = [\![\pi']\!]$.



7. Commutative Multiplicative Connectives

Multiplicative connectives $A * B$: $|A * B| = |A| \times |B|$. Unit $= \mathbb{1} = \{*\}$.

We may assume they are covariant in both their arguments (otherwise: use negation).

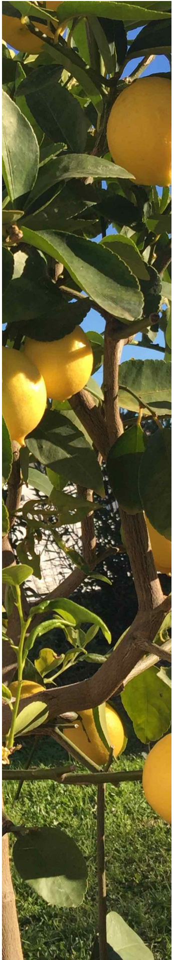
$\asymp$ coherent or equal, $\frown$ coherent but different, $\smile$ incoherent and different

Commutative multiplicative (binary) connectives, just two of them, both assocative:

$A \parr B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$A \otimes B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

They are the dual one of the other:
$(A \parr B)^\perp \equiv A \otimes B$ and $(A \otimes B)^\perp \equiv A \parr B$



8. Before (pomset logic)= Seq (deep inference)

But, there is another (non commutative) multiplicative connective:

$A \lhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

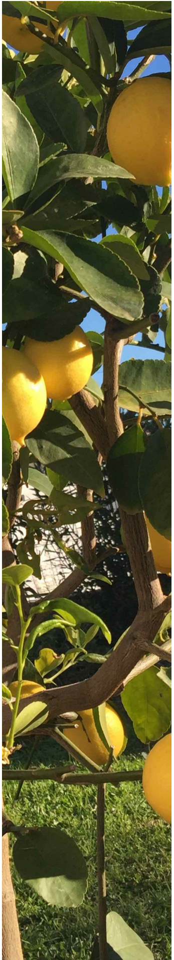
$A \rhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$(\alpha,\beta) \frown (\alpha',\beta')[A \lhd B]$ whenever $\begin{cases} \alpha \frown \alpha'[A] \\ \text{or} \\ \alpha = \alpha' \text{ and } \beta \frown \beta'[A] \end{cases}$

self dual $(A \lhd B)^{\perp} = A^{\perp} \lhd B^{\perp}$ **(no swap!)**

associative

non commutative



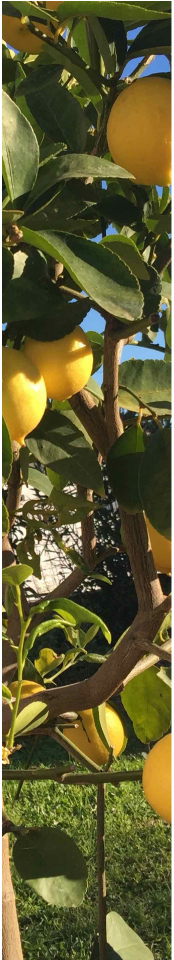
9. Partial orders

To have some ”before” contexts (in the sequent calculus) ought to be endowed with partial orders. As

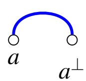
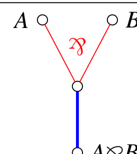
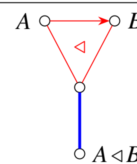
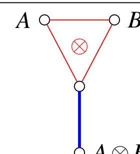
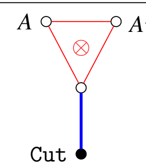
Generalisation: $\prec$ finite (partial) order over $I = \{1, \dots, n\}$, $\Pi^I A_i$:

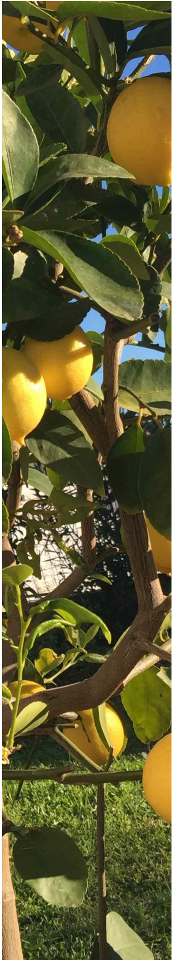
- web: $|A_1| \times \cdots \times |A_n|$
- strict coherence: $(\alpha_1, \dots, \alpha_n) \frown (\alpha'_1, \dots, \alpha'_n)$
  when there exists i s.t. $\alpha_i \frown \alpha'_i$ and $\alpha_j = \alpha'_j$ for all $j \prec i$.

The ”ordered” coherence spaces that can be defined with $\lhd$ and $\parr$ are the ones with a sepries parallel partial order, cf. infra.



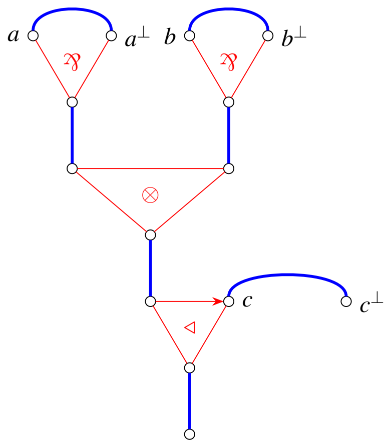
10. Pomset logic: proof net syntax (with links)

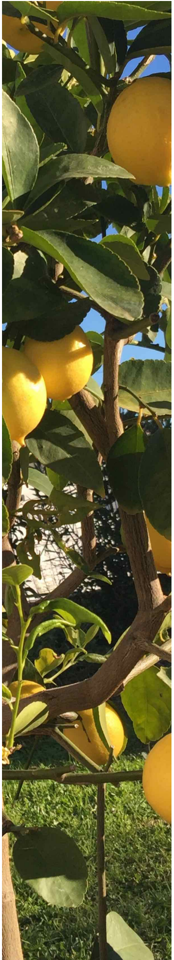
	Axiom	Par $\parr$	Before $\triangleleft$	Times $\otimes$	Cut
Premisses	None	A and B	A and B	A and B	K and $K^{\perp}$
RnB link					
Conclusion(s)	a and $a^{\perp}$	$A \parr B$	$A \triangleleft B$	$A \otimes B$	None



11. Correctness criterion

No alternate elementary circuit (directed cycle).



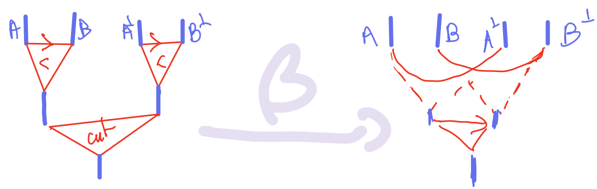


12. Cut elimination

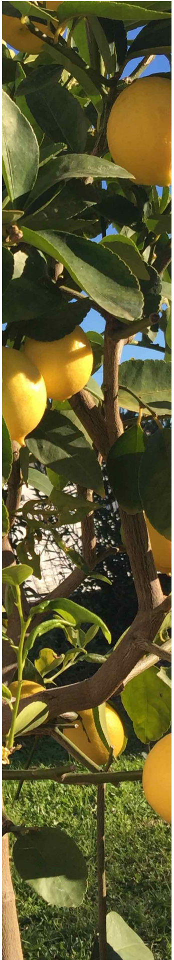
Cut elimination preserves the criterion.

a cut may be viewed as $K \otimes K^\perp$

two cuts in parallel ($⅋$ of the cuts)



two cuts one before the other ($<$ of the cuts)



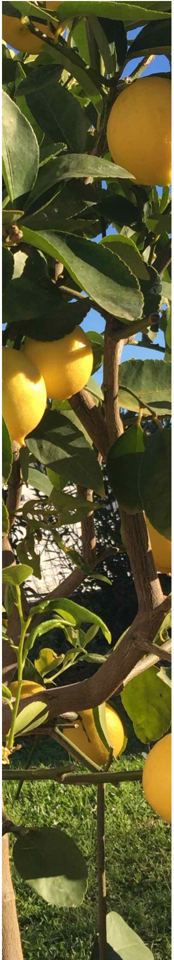
13. Semantics

A proof structure is interpreted as a set of tokens in the corresponding coherence space (experiment method).

Theorem: cut elimination preserves the semantics.

Theorem: a proof structure is correct **if and only if** its interpretation if a clique of the corresponding coherence space.

So the syntax matches the semantics.



14. Sequent calculus? still open IMHO

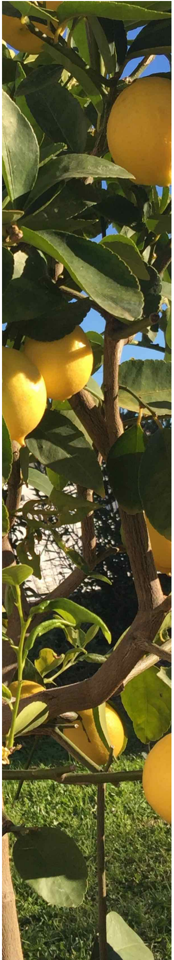
$$\{A,B\} \sim A \parr B \qquad \langle A;B\rangle \sim A \lhd B$$

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \langle \Gamma;\Delta\rangle}\,\mathit{dimix} \qquad \frac{\vdash \Gamma}{\vdash \Gamma'}\,\text{entropy}(\Gamma' \text{ sub sp order of } \Gamma)$$

$$\frac{}{\vdash \{a,a^{\perp}\}}$$

$$\frac{\vdash \{A,\Gamma\} \qquad \vdash \{B,\Delta\}}{\vdash \{\Gamma,(A\otimes B),\Delta\}}\,\otimes \text{ / cut when } A=B^{\perp}$$

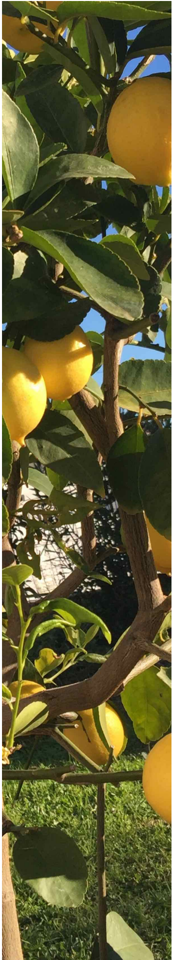
$$\frac{\vdash \Gamma[\{A,B\}]}{\vdash \Gamma[A\parr B]}\,\parr(A \overset{..}{\sim} B) \qquad \frac{\vdash \Gamma[\langle A;B\rangle]}{\vdash \Gamma[A\lhd B]}\,\lhd(A \overset{\rightarrow}{\sim} B)$$



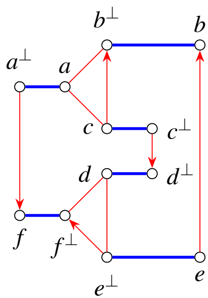
15. Sequent proof proof net example

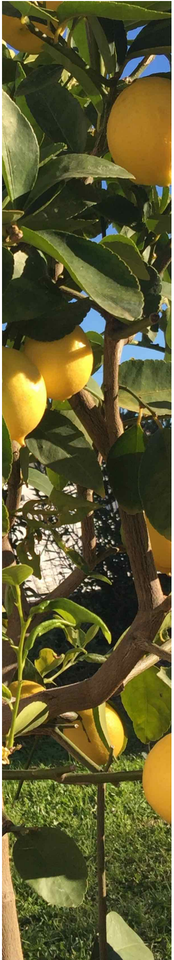
$$
\begin{prooftree}
\AxiomC{$\vdash \{a, a^\perp\}$}
\UnaryInfC{$\vdash a \parr a^\perp$}
\AxiomC{$\vdash \{b, b^\perp\}$}
\UnaryInfC{$\vdash b \parr b^\perp$}
\BinaryInfC{$\vdash (a \parr a^\perp) \otimes (b \parr b^\perp)$}
\AxiomC{$\vdash c, c^\perp$}
\RightLabel{$dimix$}
\BinaryInfC{$\vdash \langle (a \parr a^\perp) \otimes (b \parr b^\perp); \{c, c^\perp\} \rangle$}
\RightLabel{$entropy$}
\UnaryInfC{$\vdash \{\langle (a \parr a^\perp) \otimes (b \parr b^\perp); c \rangle, c^\perp\}$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdash \{a, a^\perp\}$}
\UnaryInfC{$\vdash a \parr a^\perp$}
\AxiomC{$\vdash \{b, b^\perp\}$}
\UnaryInfC{$\vdash b \parr b^\perp$}
\BinaryInfC{$\vdash (a \parr a^\perp) \otimes (b \parr b^\perp)$}
\AxiomC{$\vdash c, c^\perp$}
\RightLabel{$dimix$}
\BinaryInfC{$\vdash \langle (a \parr a^\perp) \otimes (b \parr b^\perp); \{c, c^\perp\} \rangle$}
\RightLabel{$entropy$}
\UnaryInfC{$\vdash \{\langle (a \parr a^\perp) \otimes (b \parr b^\perp); c \rangle, c^\perp\}$}
\end{prooftree}
$$

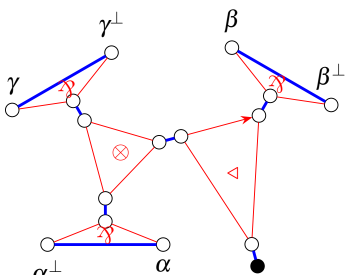


16. Not derivable in sequent calculus



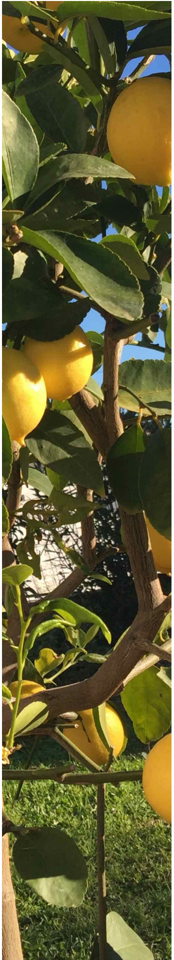


17. Folding/unfolding 1

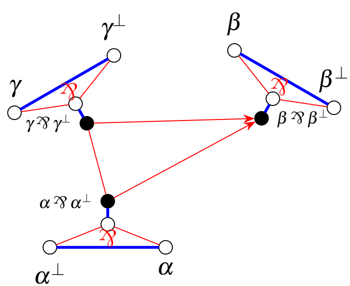


$((\alpha \parr \alpha^\perp) \otimes (\gamma \parr \gamma^\perp)) \triangleleft (\beta \parr \beta^\perp)$

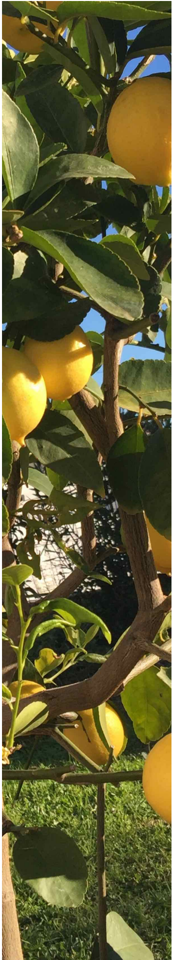
(d) π_4



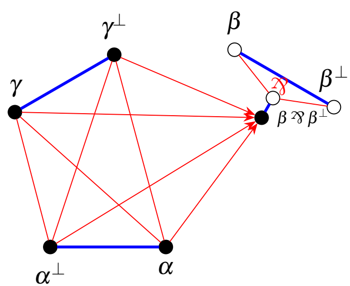
18. Folding/unfolding 2



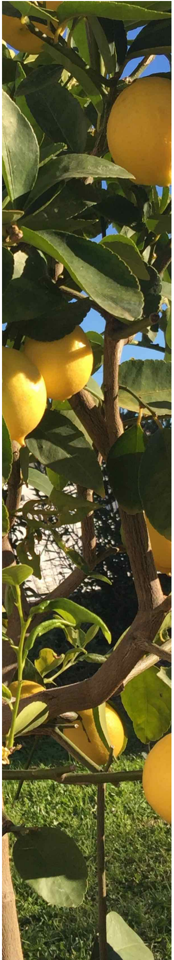
(c) π_3



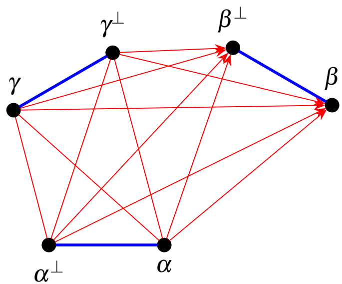
19. Folding/unfolding 3



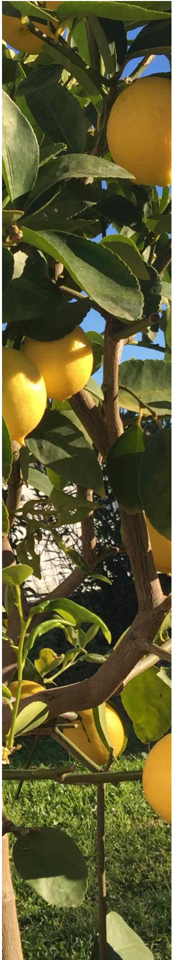
(b) π_2



20. Folding/unfolding 4



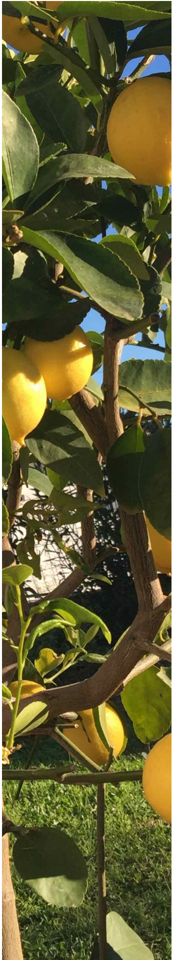
(a) π_1



21. Series parallel partial orders

An empty relation on a single vertex is a series parallel partial order. Given two SP orders (E_1, R_1) and (E_2, R_2) we can define two SP orders on $E_1 \uplus E_2$:

- directed series composition $R_1 \lhd R_2 = R_1 \uplus R_2 \uplus (E_1 \times E_2)$
- parallel composition $R_1 \parr R_2 = R_1 \uplus R_2$

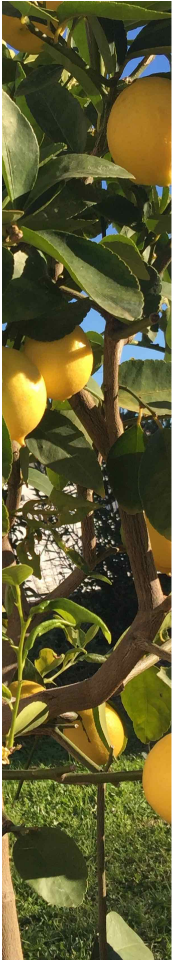


22. Series parallel partial orders: properties

The class of series parallel partial orders is characterised by the fact that it is N-free: the restrction on the order to 4 vertices never is $a < b, c < b, c < d$: if those 3 relations hold, at least one more relation holds.

There is a unique was to write an SP order as a term with $\lhd$ and $\parr$, up to the associativity of $\lhd$ and $\parr$ and to the commutativity of $\parr$.

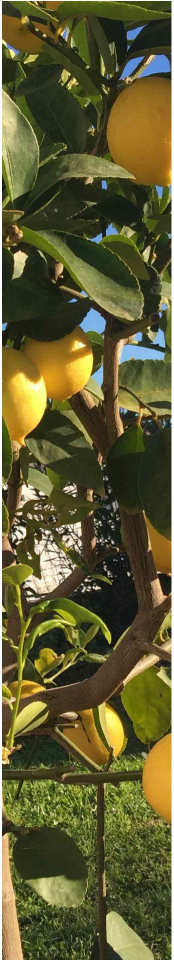
I proved this in my PhD Réseaux et séquents ordonné 1993 where i called them ”ordres contractiles” but all this was known (thanks to Maurice Pouzet for letting me know).



23. Cographs

An empty relation on a single vertex is a cograph. Given two cographs (E_1, R_1) and (E_2, R_2) we can define two cographs on $E_1 \uplus E_2$

- symmetric series composition $R_1 \otimes R_2 = R_1 \uplus R_2 \uplus (E_1 \times E_2) \uplus (E_2 \times E_1)$
- parallel composition $R_1 \parr R_2 = R_1 \uplus R_2$

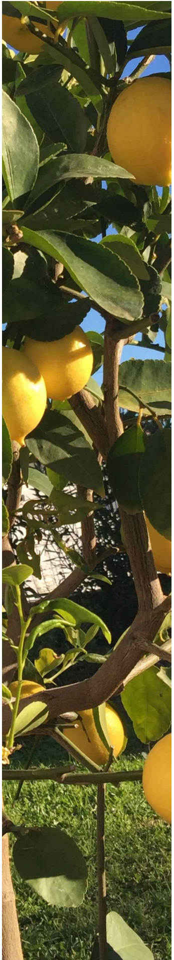


24. Cographs properties

The class of cographs is characterised by the fact that it is P_4-free: the restrction on the relation to 4 vertices never is $a-b, b-c, c-d$: if those 3 relations holds, at least one more relation holds.

There is a unique was to write a cograph as a term with $\otimes$ and $\parr$, up to the associativity and the commutativity of $\parr$ and $\otimes$.

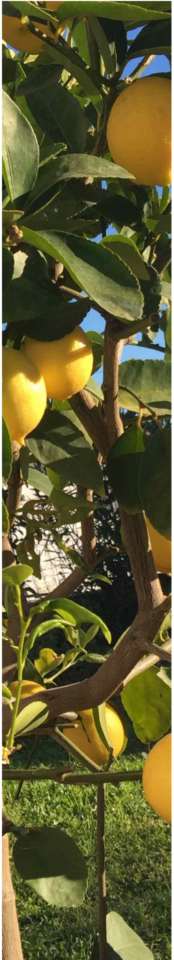
I proved this in the ENTCS 1996 paper (presented in Tokyo by François Lamarche) where i called them ”series parallel graphs” but all this was known (thanks to Maurice Pouzet for letting me know).



25. Directed cographs, a.k.a. dicographs

An empty relation on a single vertex is a directed cograph. Given two dicographs (E_1,R_1) and (E_2,R_2) we can define three dicographs on $E_1 \uplus E_2$

- symmetric series composition $R_1 \otimes R_2 = R_1 \uplus R_2 \uplus (E_1 \times E_2) \uplus (E_2 \times E_1)$
- directed series composition $R_1 \lhd R_2 = R_1 \uplus R_2 \uplus (E_1 \times E_2)$
- parallel composition $R_1 \parr R_2 = R_1 \uplus R_2$



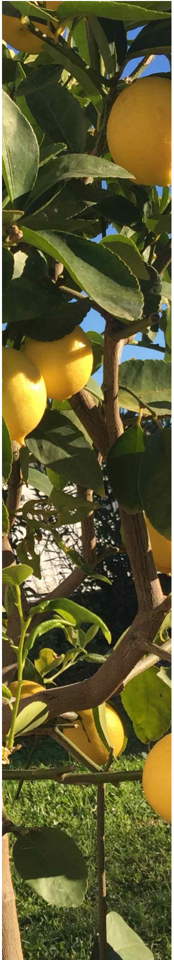
26. Dicographs: properties

The class of dicographs is characterised by the fact that

- the directed part is a series parallel partial order (N-free)
- the symmetric part is a cograph (P4 free)
- weak transitivity:
  whenever $a - b$ and $b \to c$ one has $a \to c$
  whenever $a \to b$ and $b - c$ one has $a \to c$

($a - b$ is $a \to b$ and $a \leftarrow b$)

These three conditions can be replaced with the exclusion of all the impossibilities on 4 vertices that are consequences of the three conditions.



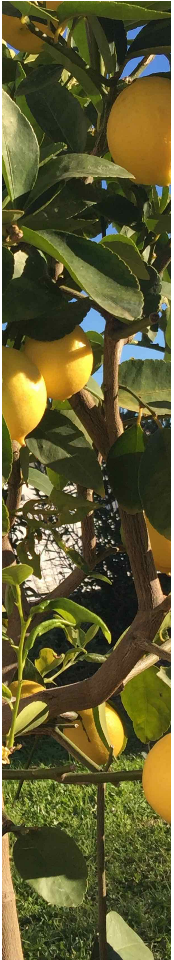
27. Dicographs: properties, continued

There is a unique was to write a dicograph as a term with $\lhd$, $\otimes$ and $\parr$, up to the associativity of $\lhd$, $\otimes$ and $\parr$ and to the commutativity of $\parr$ and $\otimes$.

Characterisation and inclusion of dicographs:

Denis Bechet, Philippe de Groote Christian Retoré A complete axiomatisation for the inclusion of series-parallel partial orders in H. Comon, editor, Rewriting Techniques and Applications, RTA'97, pages 230–240, vol. 1232 of LNCS, Springer 1997. https://doi.org/10.1007/3-540-62950-5$_7$4

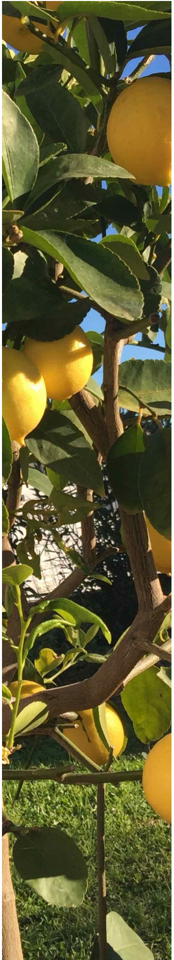
Christian Retoré Pomset logic as a calculus of directed cographs In M. Abrusci, C. Casadio and G. Sandri eds, Fourth Roma Workshop : Dynamic perspectives in Logic and Linguistics., CLUEB, 1998. [Complete version INRIA RR-3714]



28. Dicograph inclusion as rewriting

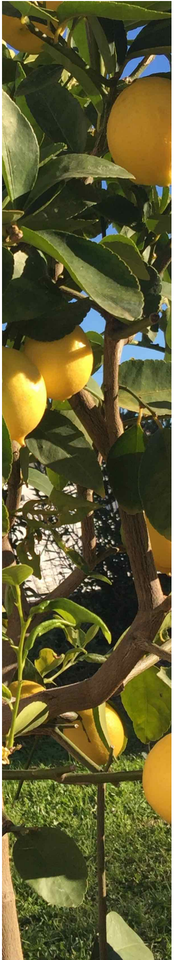
Variation on the interchange (exchange, medial,. . .) law, naturality equation.

One or two of the four formulas can be the common unit, 1. Any inclusion can be obtained by a sequence of these rules, which process up to the associativity of $\triangleleft$, $\otimes$ and $\parr$ and to the commutativity of $\parr$ and $\otimes$.



29. Dicograph inclusion as rewriting: rules

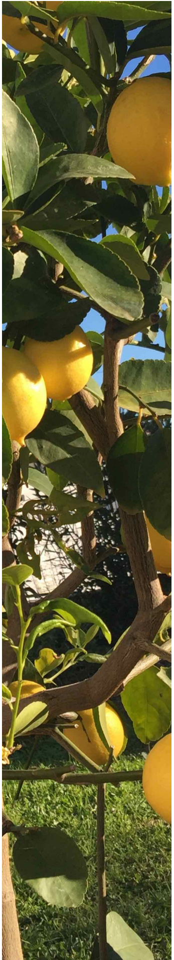
rule name	*dicograph*	$\rightsquigarrow$	*dicograph*$'$
☠ $\widehat{\otimes}\widehat{\parr}4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\parr 3$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} Y$
$\otimes\parr 2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$
$\otimes\triangleleft 4$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft 3l$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} Y$
$\otimes\triangleleft 3r$	$Y \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft 2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} Y$
$\triangleleft\parr 4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr 3l$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} Y$
$\triangleleft\parr 3r$	$Y \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr 2$	$Y \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$



30. How did I dive into this?

Personal interest in graph theory, paths, connectivity etc. (connecting proof theory with some geometry/topology)

I needed to change the vision of proofnet for pomset logic in order to obtain sequentialisation (to divide every proof net into two smaller proof nets).



31. First step: aggregates, 1991

Idea: standard constructions of graph theory describe proof nets paths and criteria much better.

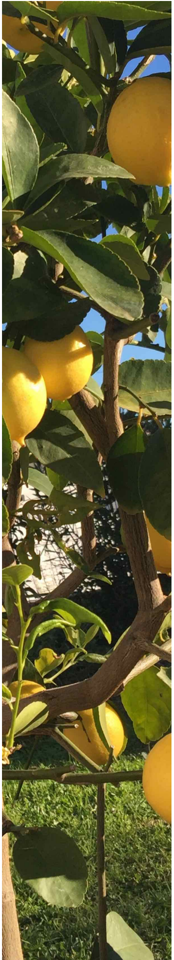
Path study: one point per axiom, one color per times and a times edge beween all the axioms one one side of the times to the axioms on the other side of the same times.

if every cycle of an aggregate has two edges of the same color, then one colour (a complete bipartite graph) is splitting the graph.

this yields sequentialisation for MLL with a splitting par (à la Danos) with a splitting times (à la Girard)

PhD Univ Paris 7, *Ch. Retoré — Réseaux et séquents ordonnés*

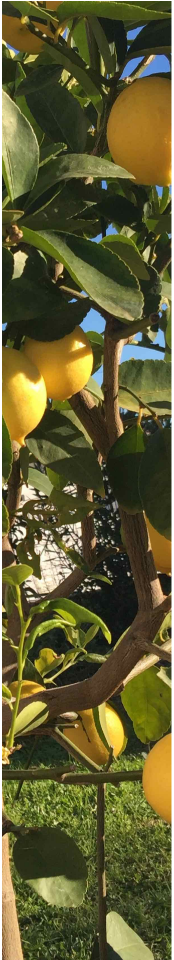
Technical report Equipe de Logique, univ Paris 7 n 47, 1993 *Ch. Retoré — Graph Theory from Linear Logic: Aggregates*



32. Towards handsome proof nets

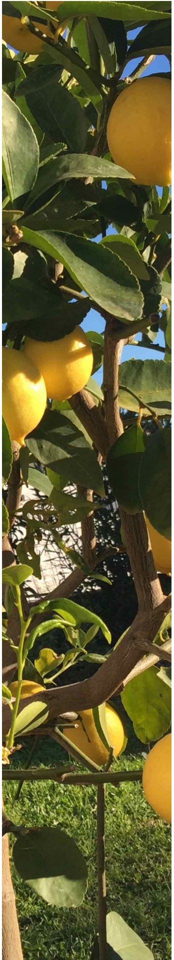
Jean-Claude Bermond, in 1994 (while i was at INRIA with Berry and Boudol, at the moment i met Alessio by email) suggested me to pinch all the edges of a times (of the same color) with one edge, to get a matching.

I took the idea but i applied differently: axioms as matching, formula trees as cographs (first) then as dicograph.



33. Handsome proof nets

- vertices atoms $a\ a^{\perp}\ b\ b^{\perp}\ \cdots$
- B (blue, bold) edges axioms, perfect matching
- R (red, regular) directed cograph (directed part: series parallel partial order; symmetric part: cograph; weak transitivity between both)



Criterion: every alternate elementary circuit contains a chord (an edge or an arc not in the circuit but between two vertices of the circuit)

an adaptation of the previous graph theoretical result yields:

Theorem: an edge bicoloured undirected (without $\triangleleft$) proofnet (in which every ea cycle contains a chord) contains a B bridge

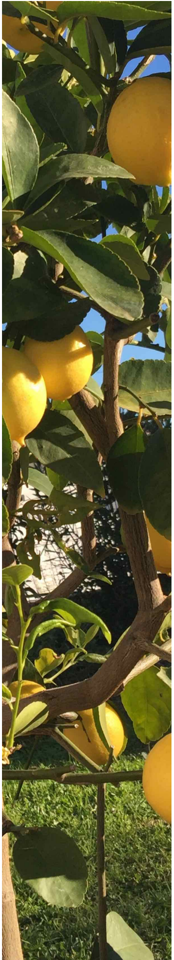
From this i obtained **sequentialisation**:

Theorem: every correct handsome proofnet does correspond to a proof in MLL sequent calculus

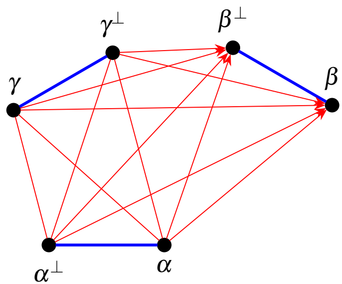
this was presented in Tokyo by François Lamarche (beware that in the title ”series-parallel graphs” refers to ”cographs”):

Ch. Retoré Perfect matchings and series-parallel graphs: multiplicatives proof nets as R&B-graphs, ENTCS, volume 5 1996, pages 167-182.

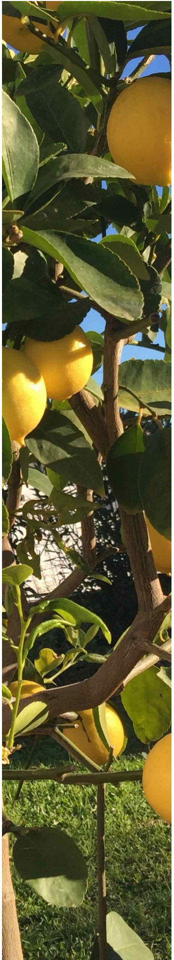
Christian Retoré Handsome proof nets: perfect matchings and cographs. Theoretical Computer Science. 294(3), 2003, pages 473–488 seee the much more complete and interesting INRIA research report RR 3652(1999)



34. A handsome proof net



(a) π_1



35. Nancy 1994-1997

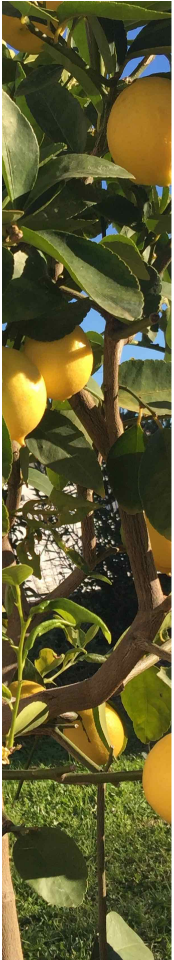
Meanwhile what about Alessio?

Since we first get in touch in 1994 while i was at INRIA Nice,

I was hired at INRIA in Nancy in september 1994.

We were starting the Calligramme Team with Philippe de Groote, joined by François Lamarche, Jean-Yves Marion (post-doc), Guy Perrier (PhD)...

Alessio wanted to join and he applied in Nancy (Marie Curie postdoc ?) and he got it, i would say in spring (?) 1997 and he settled down in Nancy with his wife Paola.



36. Changing places

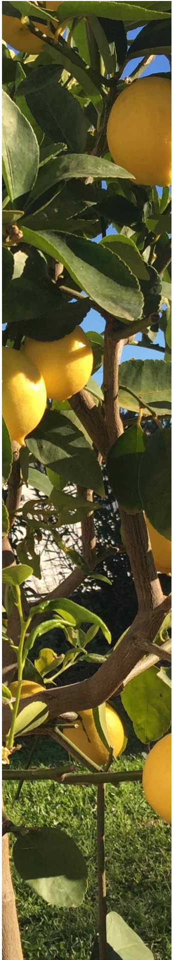
However I had small (6yo and 4yo) children and therefore normal schedules, whereas Alessio's were very off-beat, “nocturnal”, and we didn't see as much of each other at work as we'd planned. I mainly remember diners together.

I was moving towards computational linguistics and logically to Lambek calculus, its variants and extenstion (de Groote).

I moved to Rennes/Angers at the end of septembre 1997 for family reasons, it was better for my son who is deaf.

My work on proof nets as standard graphs and on pomset logic was not popular. I manage to publish on pomset logic at TLCA 1997, that's all. I kept working on those issue ”in secret” with my PhD student Syvain Pogodalla without much success.

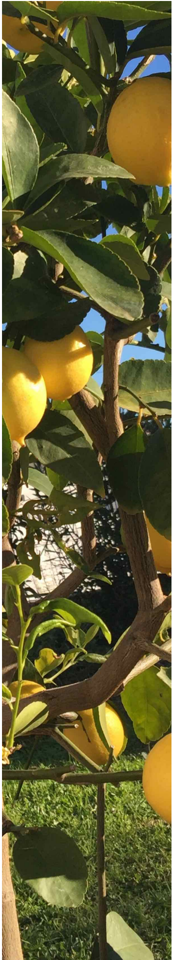
But i kept in touch with Alessio, by mail or by visits in Nancy.



37. Correct rewriting = deep inference

Christian Retoré Pomset logic as a calculus of directed cographs In M. Abrusci, C. Casadio and G. Sandri eds, Fourth Roma Workshop : Dynamic perspectives in Logic and Linguistics., CLUEB, 1998. [Complete version INRIA RR-3714]

rule name	*dicograph*	$\rightsquigarrow$	*dicograph'*
☠ $\widehat{\otimes}\widehat{\parr}4$	$(X \;\widehat{\parr}\; Y) \;\widehat{\otimes}\; (U \;\widehat{\parr}\; V)$	$\rightsquigarrow$	$(X \;\widehat{\otimes}\; U) \;\widehat{\parr}\; (Y \;\widehat{\otimes}\; V)$
$\otimes\parr 3$	$(X \;\widehat{\parr}\; Y) \;\widehat{\otimes}\; U$	$\rightsquigarrow$	$(X \;\widehat{\otimes}\; U) \;\widehat{\parr}\; Y$
$\otimes\parr 2$	$Y \;\widehat{\otimes}\; U$	$\rightsquigarrow$	$U \;\widehat{\parr}\; Y$
$\otimes\lhd 4$	$(X \;\widehat{\lhd}\; Y) \;\widehat{\otimes}\; (U \;\widehat{\lhd}\; V)$	$\rightsquigarrow$	$(X \;\widehat{\otimes}\; U) \;\widehat{\lhd}\; (Y \;\widehat{\otimes}\; V)$
$\otimes\lhd 3l$	$(X \;\widehat{\lhd}\; Y) \;\widehat{\otimes}\; U$	$\rightsquigarrow$	$(X \;\widehat{\otimes}\; U) \;\widehat{\lhd}\; Y$
$\otimes\lhd 3r$	$Y \;\widehat{\otimes}\; (U \;\widehat{\lhd}\; V)$	$\rightsquigarrow$	$U \;\widehat{\lhd}\; (Y \;\widehat{\otimes}\; V)$
$\otimes\lhd 2$	$Y \;\widehat{\otimes}\; U$	$\rightsquigarrow$	$U \;\widehat{\lhd}\; Y$
$\lhd\parr 4$	$(X \;\widehat{\parr}\; Y) \;\widehat{\lhd}\; (U \;\widehat{\parr}\; V)$	$\rightsquigarrow$	$(X \;\widehat{\lhd}\; U) \;\widehat{\parr}\; (Y \;\widehat{\lhd}\; V)$
$\lhd\parr 3l$	$(X \;\widehat{\parr}\; Y) \;\widehat{\lhd}\; U$	$\rightsquigarrow$	$(X \;\widehat{\lhd}\; U) \;\widehat{\parr}\; Y$
$\lhd\parr 3r$	$Y \;\widehat{\lhd}\; (U \;\widehat{\parr}\; V)$	$\rightsquigarrow$	$U \;\widehat{\parr}\; (Y \;\widehat{\lhd}\; V)$
$\lhd\parr 2$	$Y \;\widehat{\lhd}\; U$	$\rightsquigarrow$	$U \;\widehat{\parr}\; Y$

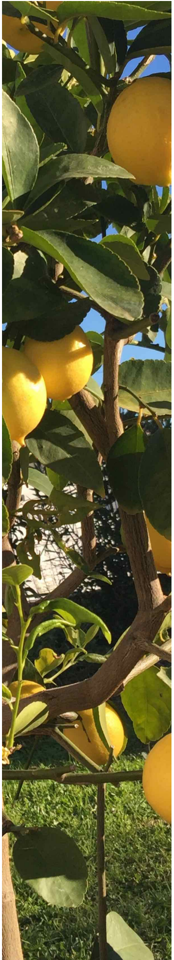


38. Rewriting and Cut-elimination

I introduced the rewriting to prove cut elimination for handsome proof nets but .... as Alessio noticed i never seriously considered rewriting especially rewriting as a deductive systems yielding an inductive definition of proofs.

I just noticed this for MLL and i remeber explaining this to ALessio in Nancy befor leaving.

To me it was just a possible tool for sequentialisation of pomset logic, and i had the feeling that handsome proof nets where better.



39. Rewriting in MLL

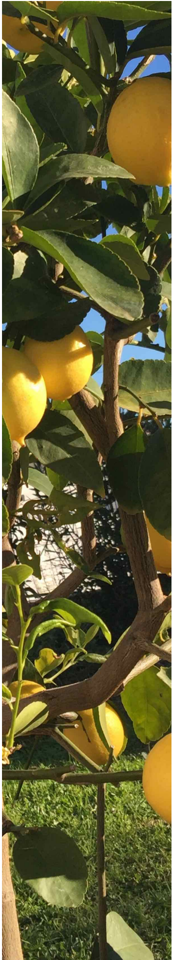
The rewriting rule up to commutativity and associativity of $\parr$ and $\otimes$

$$A \otimes (B \parr C)) \rightsquigarrow (A \otimes B) \parr C$$

derives all theorem of MLL from axioms $\bigotimes_{1 \leqslant i \leqslant n} a \parr a^{\perp}$

To have the theorems of MLL+mix add

$$(A \otimes B) \rightsquigarrow (A \parr B)$$



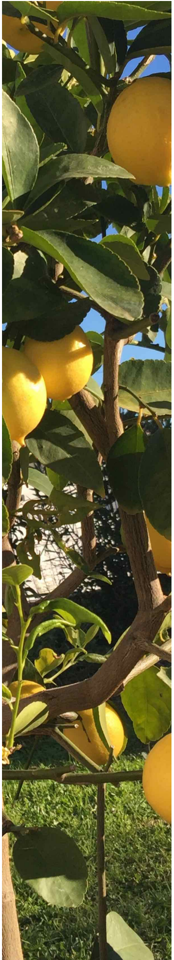
40. Rewriting for Pomset Logic

I think Alessio already started thinking about rewriting inside formulas as a deductive sytem with the MLL(+mix) case discussed above.

I remember that at the defense of Paul Ruet on October 27, 1997 I told Girard about those rewriting, and here is what he told me:

for MLL(+mix) he knew the result but globally he told me one should not study that, let alone that he thought that research on pomset logic (that was his suggestion) should be stopped. Reasons:

- no sequent calculus,
- $\nvdash A^{\perp} \lhd A$.



41. Calculus of order and interaction 1999

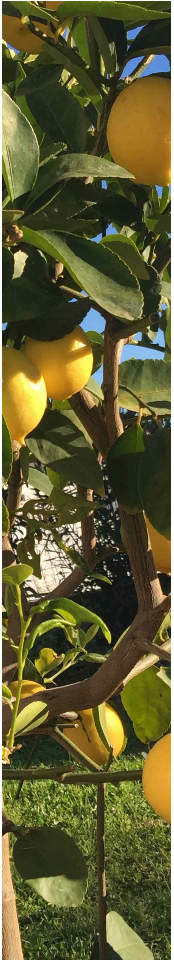
After i did not follow closely the work of Alessio, mainly because his view was much more algebraic than mine, with terms and equations, while i had a more geometric or topological viewpoint.

I remember a critic by Lambek on my habilitation telling me that my research goes against history, agains the trend in mathematics to replace geometric and topological methods with algebraic ones: algebraic geometry, algebraic topology, etc.

The advances of Alessio were difficult for me to follow because it was purely an algebraic calculus of terms.

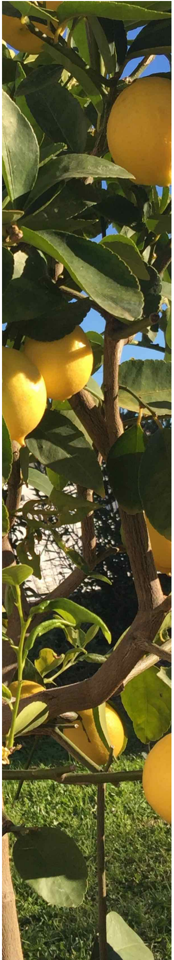
I think at some point Alessio sent me his 1999 paper ”A calculus of order and interaction” paper.

I admit i skimmed it and later on i had the duty to read seriously this or a subsequent paper of his — since i was a reviewer. And actually I really liked it.



42. BV and SBV

$$
\begin{aligned}
\text{Axiom:} &\rightsquigarrow (e \parr e^\perp) \otimes (b \parr b^\perp) \otimes (c \parr c^\perp) \otimes (f \parr f^\perp) \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
\otimes\lhd 2 &\rightsquigarrow [(e^\perp \parr e) \lhd (b^\perp \parr b)] \otimes (c \parr c^\perp) \otimes (f \parr f^\perp) \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
\lhd \parr 4 &\rightsquigarrow [(e^\perp \lhd b^\perp) \parr (e \lhd b)] \otimes (c \parr c^\perp) \otimes (f \parr f^\perp) \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
\otimes \parr 3 &\rightsquigarrow [\{(e^\perp \lhd b^\perp) \otimes (c \parr c^\perp) \otimes (f \parr f^\perp)\} \parr (e \lhd b)] \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
2 \times \otimes\lhd 2 &\rightsquigarrow [\{((c \parr c^\perp) \lhd (e^\perp \parr b^\perp)) \lhd (f \parr f^\perp))\} \parr (e \lhd b)] \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
2 \times \lhd \parr 4 &\rightsquigarrow [(c \lhd b^\perp \lhd f) \parr (c^\perp \lhd e^\perp \lhd f^\perp) \parr (e \lhd b)] \otimes (a \parr a^\perp) \otimes (d \parr d^\perp) \\
\otimes \parr 3 &\rightsquigarrow [\{(a \parr a^\perp) \otimes (c \lhd b^\perp \lhd f)\} \parr (c^\perp \lhd e^\perp \lhd f^\perp) \parr (e \lhd b)] \otimes (d \parr d^\perp) \\
\otimes\lhd 3l &\rightsquigarrow [(\{(a \parr a^\perp) \otimes (c \lhd b^\perp)\} \lhd f) \parr (c^\perp \lhd e^\perp \lhd f^\perp) \parr (e \lhd b)] \otimes (d \parr d^\perp) \\
\otimes \parr 3 &\rightsquigarrow [(\{a^\perp \parr (a \otimes (c \lhd b^\perp))\} \lhd f) \parr (c^\perp \lhd e^\perp \lhd f^\perp) \parr (e \lhd b)] \otimes (d \parr d^\perp) \\
\lhd \parr 3 &\rightsquigarrow [(a \otimes (c \lhd b^\perp)) \parr (a^\perp \parr \lhd f) \parr (c^\perp \lhd e^\perp \lhd f^\perp) \parr (e \lhd b)] \otimes (d \parr d^\perp) \\
\otimes \parr 3 &\rightsquigarrow (a \otimes (c \lhd b^\perp)) \parr (a^\perp \lhd f) \parr \{(c^\perp \lhd e^\perp \lhd f^\perp) \otimes (d \parr d^\perp)\} \parr (e \lhd b) \\
\otimes \parr 3 &\rightsquigarrow (a \otimes (c \lhd b^\perp)) \parr (a^\perp \lhd f) \parr (c^\perp \lhd \{[(e^\perp \lhd f^\perp) \otimes d] \parr d^\perp\} \parr (e \lhd b) \\
\lhd \parr 3r &\rightsquigarrow (a \otimes (c \lhd b^\perp)) \parr (a^\perp \lhd f) \parr ((e^\perp \lhd f^\perp) \otimes d) \parr (c^\perp \lhd d^\perp) \parr (e \lhd b)
\end{aligned}
$$



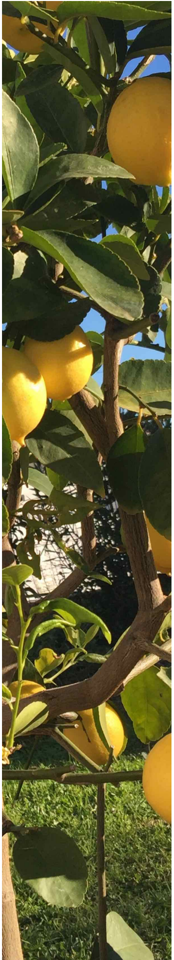
43. Rewriting Pomset Proof-Nets and SBV

Rewriting rules: the rules that preserve correctness.

A difference: in SBV there is a unit for all three connectives.

Unit should be defined as $1 = \varepsilon^{\perp} \parr \varepsilon$ that cannot be split with ε a variable that does not appear elsewhere.

The rule $a\uparrow$ is like removing an atomic cut, $a_i \otimes a_i^{\perp} \leadsto 1$ while $i\uparrow$ is like removing a complex cut $K \otimes K^{\perp} \leadsto 1$.



44. Deep Inference later on

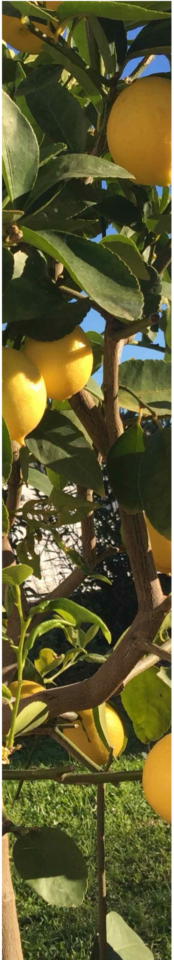
I did not really participate in subsequent development of Deep Inference, and we did not have much discussions, we rarely met.

I remember in 2011 Lutz Strassburger invited me to discuss pomset logic

and in 2017 Michel Parigot invited me to discuss pomset logic

Alessio was there as well as Sergei Slavnov who proposed a sequent calculus for pomset logic.

This lead me to work again on such questions.



45. Relation to Deep Inference

Starting with $\otimes_i (a_i \parr a_i^\perp)$

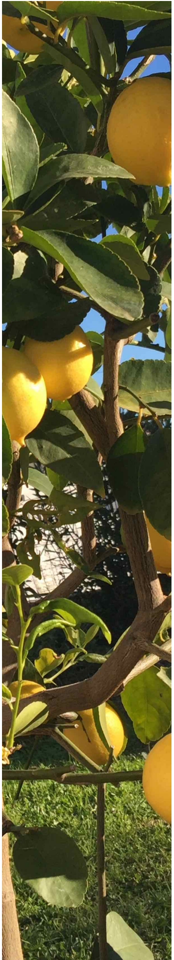
some rules handling 1 the common unit of $\otimes, \triangleleft, \parr$.

On can veiw BV on graphs rather than on terms, if one prefers.

Equivalent to SBV and you easily get SBV "cut" elimination (removal of $a \uparrow 1$) when $K \otimes K^\perp$ vanishes.

Rather easy proof using graph rewriting on proof nets and the graph-theoretical properties.

Christian Retoré Pomset Logic: the other approach to non commutativity in logic In C. Casadio and P. J. Scott (eds.), Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics, Outstanding Contributions to Logic 20,2021 pp. 299-345



46. Not derivable in Deep Inference but Derivable in Pomset Logic

Tito N'Guyen results (partly with Lutz Strassburger)

Conclusion **Joyeux anniversaire Christian !**

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas,
from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

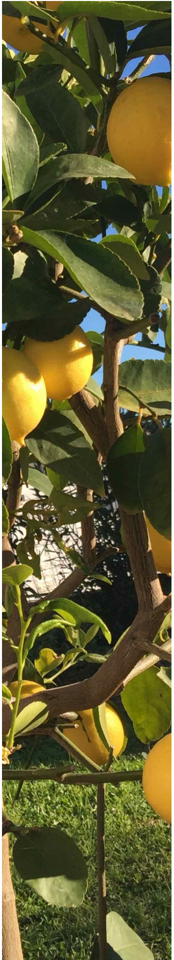
Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

  $$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

  Causally meaningful variant (K.–S.): $(((p^1)^\perp \lhd q^1) \otimes ((r^1)^\perp \lhd s^1)) \parr (((q^1)^\perp \lhd r^1) \otimes ((s^1)^\perp \lhd p^1))$

- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.



47. Slavnov's — complete but *ad hoc* — sequent calculus 2019, LMCS

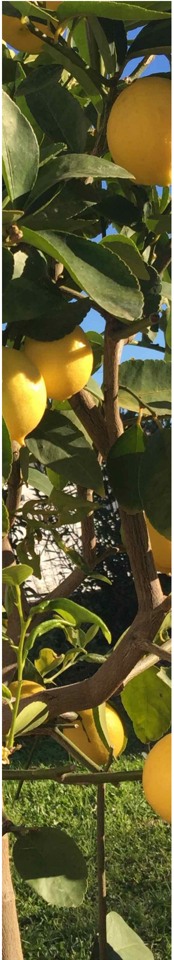
Very complex: if n conclusions, pairs of tuples of length k for all $k \leqslant n/2$.

Only unary rules but mix.

Intuition: independent alternate elementary paths between k conclusions and k other conclusions are known before the rule, and the rule should not make any alernate elementary path with them.

One very interesting idea : usual commutatives $\parr, \otimes$ plus a pair of dual non commutative connective: $\vec{\parr}$ and $\vec{\otimes}$, and $\triangleleft$ is a degenerate case, when both are equal.

Sergey Slavnov On noncommutative extensions of linear logic Logical Methods in Computer Science, Volume 15, Issue 3 (September 20, 2019)



48. Towards a sequent calculus for pomset logic using Alessio's work

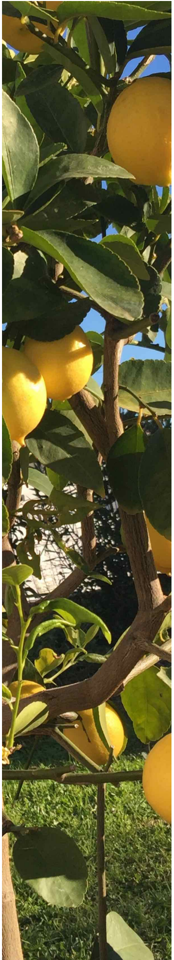
Not yet!

But some new ideas (like Slavnov $\vec{\parr}$ and $\vec{\otimes}$),

and some graph theoretical ideas as well.

However from 1991, there are often moments when I think i can solve this problem... without solving it yet, so i should not be too optimistic.

The brilliant developments of Deep Inference by Alessio, Lutz and their students should help.



49. Last discussions with Alessio

In 2020, I worked again on a self dual modality $\triangleleft\!\!|$, working with $\triangleleft$: allowing contraction w.r.t. $\triangleleft$ on both side! Up to now $\triangleleft\!\!|$ is only semantically defined in coherence spaces, and so far there is no syntax.

$\triangleleft\!\!| A \triangleleft \triangleleft\!\!| A$ is linearly isomorphic to $\triangleleft\!\!| A$

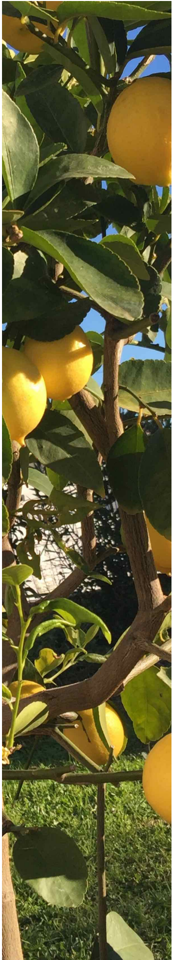
A is a retract of $\triangleleft\!\!| A$.

I spoke with Alessio in 2021: he already thought of some rules and meaning for this modality (talk, ENS Lyon, 2017)!

To me Alessio was a great scientist with DEEP ideas as opposed to most papers i review today.

Christian Retoré Une modalité autoduale pour le connecteur précède INRIA Research Report 2432. 1994

Christian Retoré Flag: a Self-Dual Modality for Non-Commutative Contraction and Duplication in the Category of Coherence Spaces In Proceedings LinearityTLLA 2020, EPTCS 353, 2021, pp. 157-174



50. Alessio, Paola and a bottle of Corton

I would like to present my deepest condoleances to Paola.

I remember a happy moment at the end of September 1997 when we moved from Nancy for Angers.

After packing and putting back our flat as we found it, Paola and Alessio invited my wife Amparo (deceased in 2022) me and our two children for dinner (it was our last evening in Nancy, our flat was already empty).

We were exhausted but i still remember this evening, no science, just discussing and laughing, and Paola — who knew i love wine — had brought a bottle of Corton, absolutely excellent, I still remember it! I was telling my wife: ”I know how tired you are, but you should nevertheless have one more sip of this Corton that Paola brought for us.”