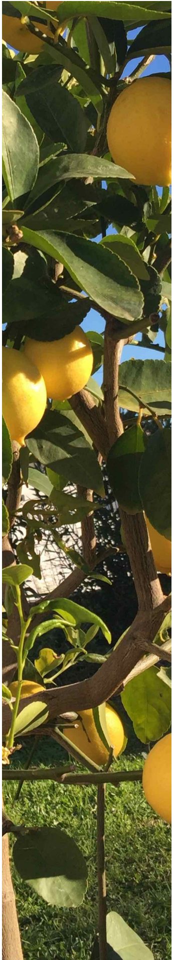
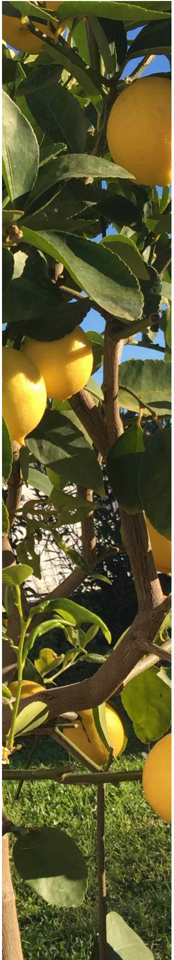


A closer look at the completeness of first order intuitionistic logic w.r.t. (pre)sheaves of classical models

Christian Retoré (LIRMM, Université de Montpellier)
with Jacques van de Wiele (Paris) 1987,
Ivano CIardelli (München) 2010-2011,
David Théret (Montpellier) 2016-2019,
...

GDR IM — GT LHC & SCALP
ENS Lyon 18 oct. 2019



Remarks

Motivation: natural language semantics, models of FO S4.

A beautiful subject — but not my main research area.

Quite difficult to know what has exactly been achieved on this question.

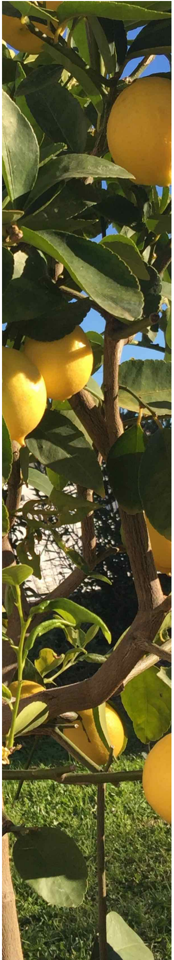
The presentation of Kripke-Joyal forcing and counter example is from a lecture by Jacques Van de Wiele in 1987. Special dedication to Laurent Regnier who attended as well.

The direct completeness proof is essentially due to Ivano Ciardelli (in TACL 2011 cf. reference at the end).

It has been re-worked and extended with David Théret since 2016.

Thanks to Alexis Saurin for suggesting a talk at this meeting.

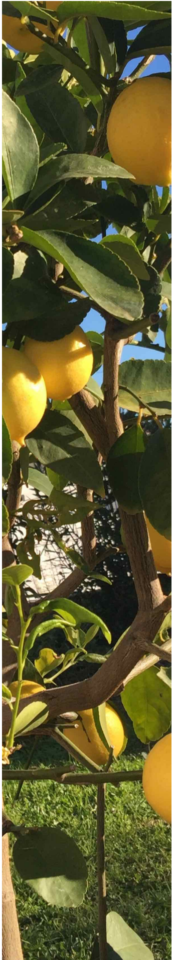
Thanks to the Topos & Logic group (Abdelkader Gouaich, Jean Malgoire, Nicolas Saby, David Theret) of the *Institut Montpelliérain Alexander Grothendieck*



A Logic?

formulas

proofs $\leftrightsquigarrow$ interpretations



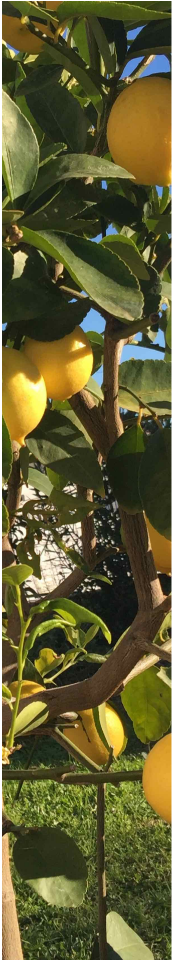
A.1. Formulas, proofs and models

Formulas of a given first-order logical language, say $\mathscr{L}$ can be true (or not) in a given $\mathscr{L}$- structure.

An $\mathscr{L}$- structure is simply a set M with an interpretation of constants in M and interpretation of n-ary predicates as n-ary relations on M, n-ary functions symbols as n-ary functions from M^n to M etc.

Soundness: what is provable is true in any $\mathscr{L}$-structure.

Completeness: what is true in every $\mathscr{L}$-structure is provable.



A.2. Morphisms of $\mathscr{L}$-structures

Let M_u and M_v be two $\mathscr{L}$ structures over the same language — interpretation in u or in v of function symbols (e.g. ϑ) and predicates (e.g. R) are denoted with a subscript u or v (e.g. R_u ϑ_u: interpretations in M_u and R_v ϑ_v: interpretations in M_v).

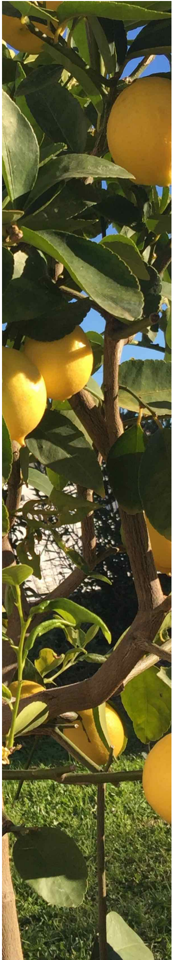
A map $\rho_{u\to v}$ from $|M_u|$ to $|M_v|$ is said to be a morphism of $\mathscr{L}$-structures when:

- For any k-ary function ϑ symbol of $\mathscr{L}$:

  $$\forall c_1, \ldots, c_k \in |M_u| \quad \rho_{u\to v}(\vartheta_u(c_1, \ldots, c_k)) = \vartheta_v(\rho_{u\to v}(c_1), \ldots, \rho_{u\to v}(c_k))$$

- For any n-ary predicate R of $\mathscr{L}$:

  $$\forall c_1, \ldots, c_n \in |M_u| \quad \textit{if}\ (c_1, \ldots, c_n) \in R_u \text{ then } (\rho_{u\to v}(c_1), \ldots, \rho_{u\to v}(c_k)) \in R_v$$

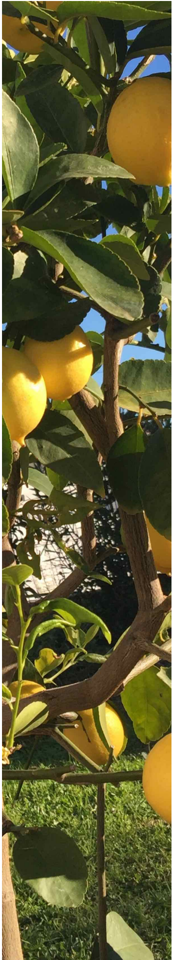


A.3. Presheaf semantics: models

A **presheaf model** M for $\mathscr{L}$ is a presheaf of first-order $\mathscr{L}-$structures over a Grothendieck site $(\mathscr{C}, \lhd)$ (or a topological space viewed as a poset for inclusion):

- for any object u an $\mathscr{L}$ structure M_u
- for any arrow $f : v \hookrightarrow u$ a morphism (cf. supra) of $\mathscr{L}$ structures $M(f) : M_u \to M_v$

satisfying the following extra conditions.
Separateness For any elements a, b of M_u,
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for all $i \in \mathscr{I}$ we have $M(f_i)(a) = M(f_i)(b)$,
then $a = b$.
Local character of atoms For any $n-$ary relation symbol R, for any tuple $(a_1, \ldots, a_n)$ from M_u
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that $\forall i \in \mathscr{I}$ one has $(M(f_i)(a_1), \ldots, M(f_i)(a_n)) \in R_{u_i}$,
then $(a_1, \ldots, a_n) \in R_u$.

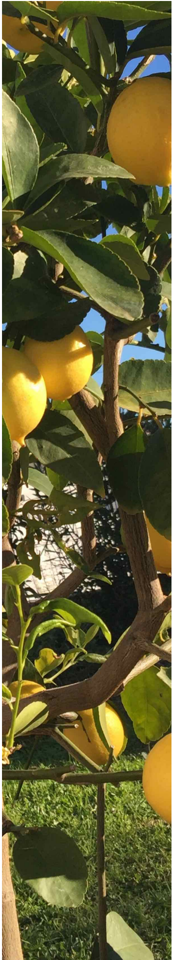


A.4. Presheaf semantics: Kripke-Joyal forcing — 1/4 assignments

Given a presheaf model M, and some open u, we inductively define for any formula F of $\mathscr{L}$ the relation $u \Vdash F$ ("meaning": F is true at u).

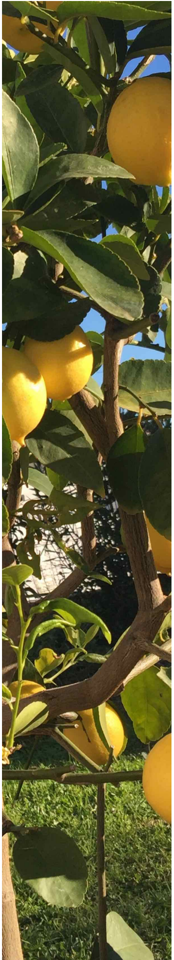
Assignment A usual, in order to define $u \Vdash F$, we need an assignment ν in M_u the free variables of F, and this is written $u \Vdash_\nu F$ with $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$ where the z_i are the free variables in F and $c_i \in |M_u|$.

As we shall see, $u \Vdash_\nu F$ can be defined from $v \Vdash_{\nu'} F'$ with $f : v \hookrightarrow u$ and with F' having free variables among those of F (plus possibly one free variable in the $\exists$ and $\forall$ cases, but its assignment will be defined when dealing with quantifiers). If $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$ we naturally define ν' by $\nu' = [z_1 \mapsto M(f)(c_1); \cdots; z_p \mapsto M(f)(c_p)]$ where $M(f)$ is the restriction $M(f) : |M_u| \to |M_v|$.



A.5. Presheaf semantics: Kripke-Joyal forcing — 2/4 atoms and conjunction

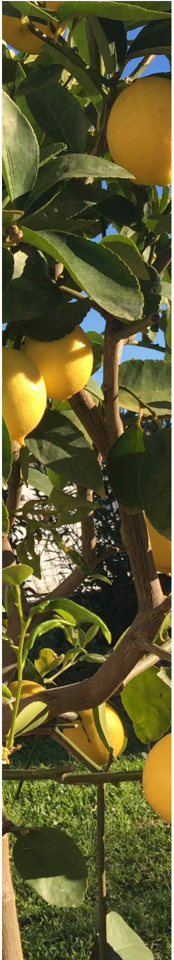
- $u \Vdash_v R(t_1, \ldots, t_n)$ iff $([t_1]_v, \ldots, [t_n]_v) \in R_u$.
- $u \Vdash_v t_1 = t_2$ iff $\quad [t_1]_v = [t_2]_v$.
- $u \Vdash_v \bot$ iff $u = \emptyset$ It is so, because the empty covering is a covering (with 0 open) of the empty open. Hence, because of the locality condition on atoms, the empty open forces all atomic formulas including $\bot$.
- $u \Vdash_v \varphi \wedge \psi$ iff $u \Vdash_v \varphi$ and $u \Vdash_v \psi$.



A.6. Presheaf semantics: Kripke-Joyal forcing — 3/4 disjunction and existential

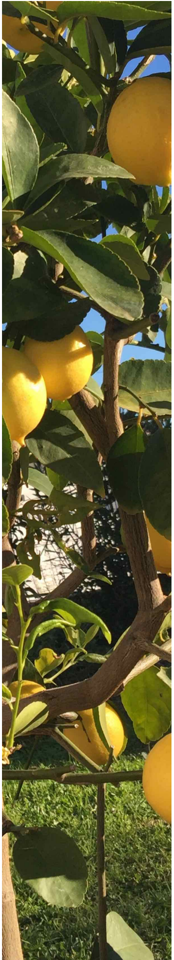
- $u \Vdash_v \varphi \vee \psi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for any $i \in \mathscr{I}$ we have $u_i \Vdash_{v_i} \varphi$ or $u_i \Vdash_{v_i} \psi$.
  Alternatively, $u \Vdash_v \varphi \vee \psi$ there exist two opens u_1, u_2 with $u_1 \cup u_2 = u$ $u_i \Vdash \varphi$ and $u_2 \Vdash \psi$.

- $u \Vdash_v \exists x \varphi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathscr{I}$ such that $u_i \Vdash_{v_i \cup [x \mapsto a_i]} \varphi$ for any index i.



A.7. Presheaf semantics: Kripke-Joyal forcing — 4/4 implication and universal

- $u \Vdash_{\nu} \varphi \to \psi$ iff for all $f : v \hookrightarrow u$, if $v \Vdash_{\nu_v} \varphi$ then $v \Vdash_{\nu_v} \psi$.
- $u \Vdash_{\nu} \neg\varphi$ iff for all $f : v \hookrightarrow u$, with $v \neq \emptyset$, $v \not\Vdash_{\nu_v} \varphi$. This is obtain from $\emptyset \Vdash \bot$ and $\to$ cases because $\neg\varphi = \varphi \to \bot$.
- $u \Vdash_{\nu} \forall x\varphi$ iff for all $f : v \hookrightarrow u$ and all $a \in M_v$, $v \Vdash_{\nu_v \cup [x \mapsto a]} \varphi$.

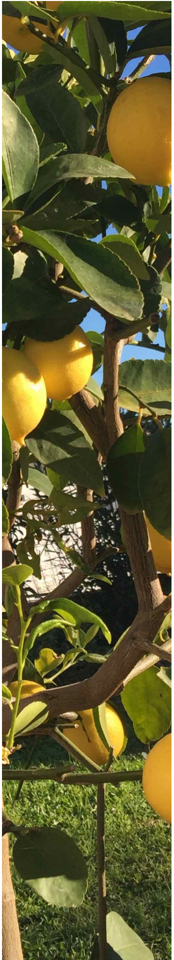


A.8. Validity

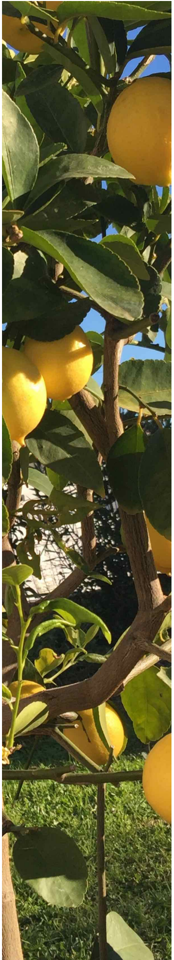
A formula F is said to be valid in a topological model in a presheaf model over a topological space $(X, O(X))$ or a pre-topology whenever

$$X \Vdash F$$

i.e. F is true at the global section.



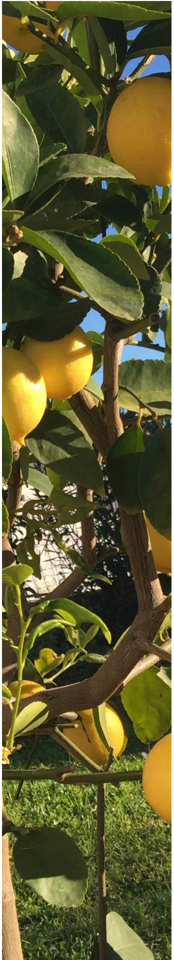
B An example



B.1. Language

Let us consider the language of ring theory:

- two constants $0, 1$
- two binary functions $+, \cdot$
- equality as the only predicate



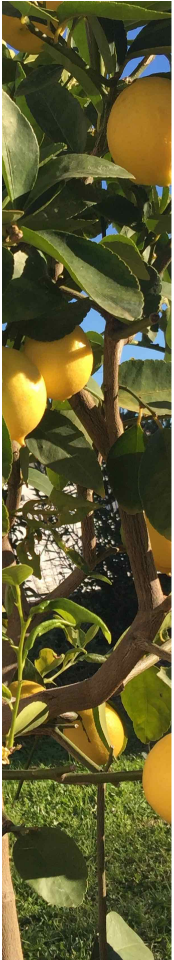
B.2. The (pre)sheaf of $\mathscr{L}$-structures

A presheaf model over the topological space $\mathbb{R}$ for this language is defined by $|M_u| = C(u,\mathbb{R})$ the continuous functions from u to $\mathbb{R}$ with $0_u(x) = 0$ and $1_u(x) = 1$ for all $x \in u$ $+_u$ pointwise addition $(f +_u g)(x) = f(x) + g(x)$, $\cdot_u$ pointwise multiplication $(f \cdot_u g)(x) = f(x) \cdot g(x)$.

The restriction $\rho_{u \to v} : |M_u| \to |M_v|$ morphism, when $v \hookrightarrow u$ is defined by: $\forall f \in C(u,\mathbb{R}) \forall x \in v$ $\rho_{u \to v}(f)(x) = f(x)$.

$\rho_{u \to v}$ is a morphism, because:

- $\rho_{u \to v}(0_u) = 0_v$,
- $\rho_{u \to v}(1_u) = 1_v$
- ("=" is the only predicate) $\forall f, g \in |M_u| = C(u,\mathbb{R})$ if $f = g$ in $C(u,\mathbb{R})$ then $\rho_{u \to v}(f) = \rho_{u \to v}(g)$ in $C(v,\mathbb{R})$.



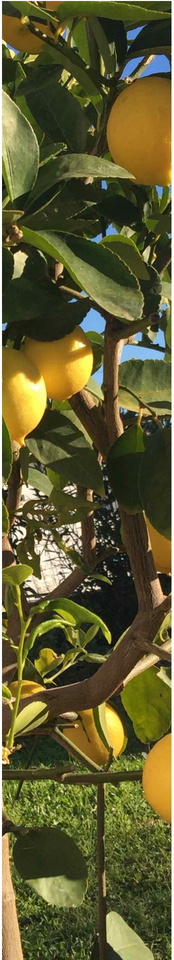
B.3. Locality and separateness conditions

Locality condition for atoms:

"=" is the only predicate so we just have to check that, given two elements a and b of M_u
if there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that $\forall i \in \mathscr{I}$ we have $(\rho_{u \to u_i}(a)) = \rho_{u \to u_i}(b))$ in $|M_{u_i}|$,
then $a = b$ in $|M_u|$. This is true, because two functions that are equal on each open of a covering of u are equal on u.

Separateness is exactly the locality condition for our unique predicate, i.e. the "=" predicate, which is interpreted as "=".

Remark: This presheaf is a sheaf: given a cover u_i of $\mathbb{R}$ and a family $f_i \in C(u_i, \mathbb{R})$ such that any two f_j and f_k agree on $u_j = u_k$ for all j, k there exists a unique f in $C(\mathbb{R}, \mathbb{R})$ such that $f\big|_{u_i} = f_i$.



B.4. A remark on $C(U,\mathbb{R})$ 1/3

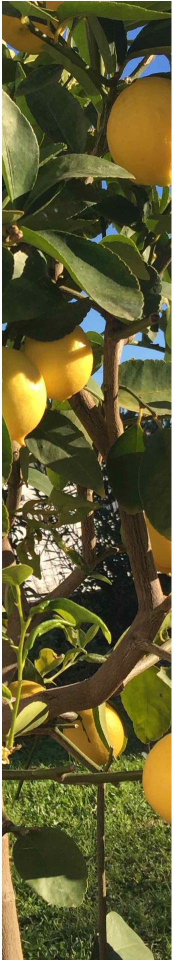
Given any non empty open subset $U \subset \mathbb{R}$ there exist

- an open subset $V =]a,b[\subset U$

- and a continuous function $\ell : V \mapsto \mathbb{R}$

such that $V \nVdash_{[x \mapsto \ell]} (x = 0) \vee \neg(x = 0)$ with ℓ:

$$\begin{array}{llcll} \ell : &]a,b[& \mapsto & \mathbb{R} & \\ & x & \mapsto & 0 & \text{if } x \leqslant (a+b)/2 \\ & x & \mapsto & x-(a+b)/2 & \text{if } x \geqslant (a+b/2) \end{array}$$



B.5. A remark on $C(U,\mathbb{R})$ 2/3

$$]a,b[\nVdash_{[x\mapsto\ell]} (x=0 \vee \neg(x=0)).$$

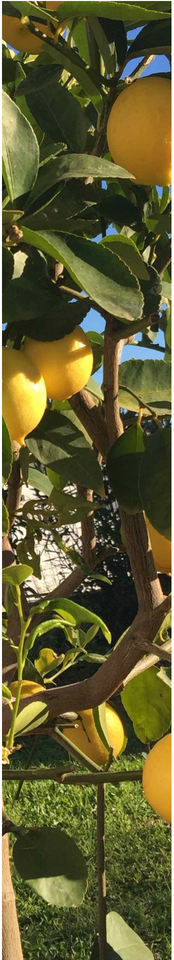
We proceed by contradiction (the meta logic is classical).

Let us assume that $]a,b[\Vdash_{[x\mapsto\ell]} (x=0 \vee \neg(x=0))$.

Then there exists open sets u_1, u_2 st. $u_1 \cup u_2 =]a,b[$, such that:

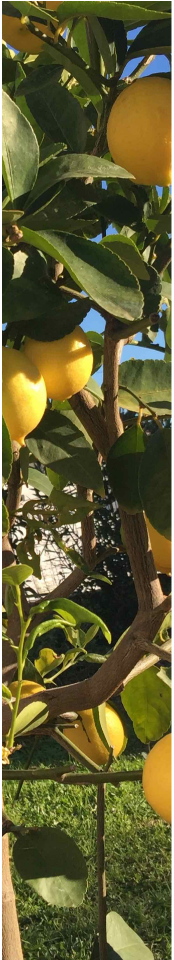
- $u_1 \Vdash_{[x\mapsto\ell_{u_1}]} x=0$ i.e. $\forall x_1 \in u_1 \quad \ell(x_1)=0$
- $u_2 \Vdash_{[x\mapsto\ell_{u_2}]} \neg(x=0)$ i.e. $\forall v_2 \subset u_2, v_2 \neq \emptyset \quad v_2 \nVdash_{[x\mapsto\ell_{u_2}]} \ell=0$ i.e. ℓ never is constantly 0 on a (non empty) sub open v_2 of u_2.

B.6. A remark on $C(U,\mathbb{R})$ 2/3



This is impossible because $(a+b)/2$ must be in u_1 or in u_2.

- If $(a+b)/2 \in u_1$ then ℓ should be constantly 0 on a neighbourhood of $(a+b)/2$, but it is false on the right side of $(a+b)/2$.
- If $(a+b)/2 \in u_2$ then ℓ should never be constantly 0 on any sub open of u_2 but if $(a+b)/2 \in u_2$ there are sub opens in u_2 on the left side of $(a+b)/2$ where ℓ is constantly 0.



B.7. A classically valid but intuitionistically non valid formula

$C(\mathbb{R},\mathbb{R})$ validates $\neg\forall x\ (x=0)\vee\neg(x=0)$ (*).

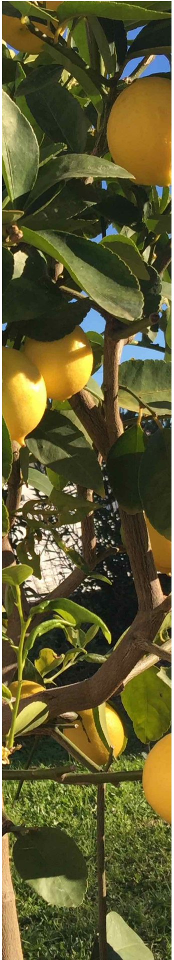
Indeed, according to Kripke-Joyal $\mathbb{R}\Vdash\neg\forall x\ (x=0)\vee\neg(x=0)$ means that for every non empty open $u\subset\mathbb{R}$, $u\not\Vdash\forall x\ (x=0)\vee\neg(x=0)$.

But $u\Vdash\forall x\ (x=0)\vee\neg(x=0)$ means that for every open $v\subset u$ and for every $f\in C(v,\mathbb{R})$ $v\Vdash_{[x\mapsto f]}(x=0)\vee\neg(x=0)$.

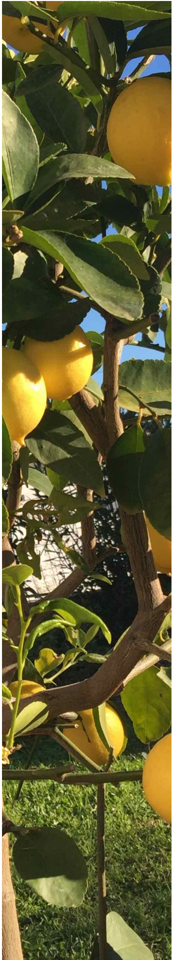
We precisely established supra (with ℓ)
that $u\not\Vdash\forall x\ (x=0)\vee\neg(x=0)$.

But $C(\mathbb{R},\mathbb{R})$ validates $\forall x\ \neg\neg((x=0)\vee\neg(x=0))$ (**)
— because $\vdash\neg\neg(C\vee\neg C)$ is provable for all C.

However in classical logic (*) is the negation of (**) !!!



C Completeness



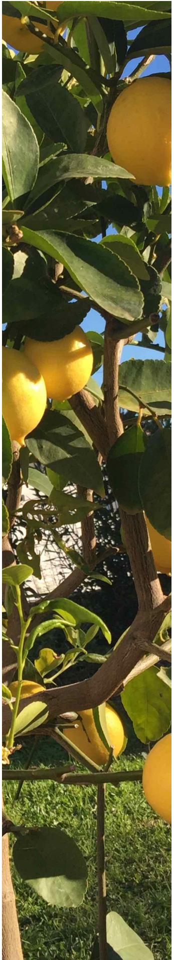
C.1. Statements

Soundness: F intuitionistically provable $\Rightarrow$ F true at any open of any topological interpretation.

Completeness: F true at a global section any topological interpretation $\Rightarrow$ F intuitionistically provable.

Two lemmas:

- (functoriality) if $F[c_1, ..., c_n]$ true at U then $F[{c_1}^V, ..., {c_n}^V]$true at any open $V \subset U$.
- (locality) The locality condition for atomic formula (cf. above) extends to *any* formula: if (u_i) covers u, for all i $u \Vdash_{x_k \mapsto c_k^{u_i}} F$ then $u \Vdash_{x_k \mapsto c} F$.



C.2. Proof of soundness

Induction on the proof height, looking at every possible last rule, e.g. in natural deduction. Below: $\vee_e$ case.

$$\frac{\Theta \vdash (A \vee B) \quad A, \Gamma \vdash C \quad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C} \vee_e$$

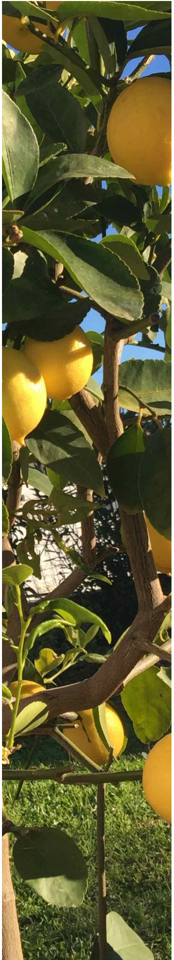
We have to show that $U \Vdash \Theta, \Gamma, \Delta$ then $U \Vdash C$.

If $U \Vdash \Theta$ by induction hypothesis, $U \Vdash A \vee B$. Hence, there exists a covering (U_i) such that for every i $U_i \Vdash A$ or $U_i \Vdash B$.

If $U_i \Vdash A$, because $U \Vdash \Gamma$ we have $U_i \Vdash \Gamma$ (functor property), and by induction hypothesis (proof of $A, \Gamma \vdash C$) $U_i \Vdash C$.

Similarly, if $U_i \Vdash B$, then $U_i \Vdash C$.

So for all i $U_i \Vdash C$ and by locality lemma $U \Vdash C$.



C.3. Canonical model construction: the underlying site

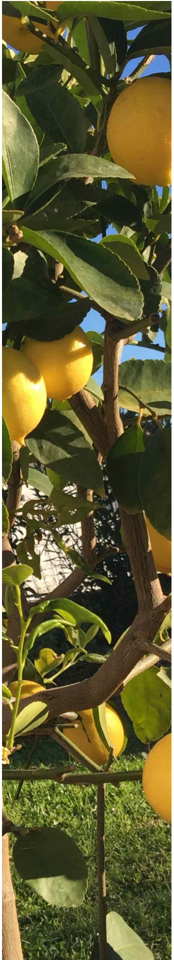
For a direct proof, we consider this particular ”syntactic” sheaf model.

Canonical site:

- **Category:** we take the Lindenbaum-Tarski algebra $\overline{\mathscr{L}}$
  - Objects: classes of provably equivalent formulas $\overline{\varphi}$.
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \iff \varphi \vdash \psi$
- **Grothendieck topology:** $\overline{\varphi} \lhd \{\psi_i\}_{i \in I}$ whenever

$$\forall \chi \left[\quad \varphi \vdash \chi \quad \mathit{iff} \quad (\forall i \in I \quad \psi_i \vdash \chi) \right]$$

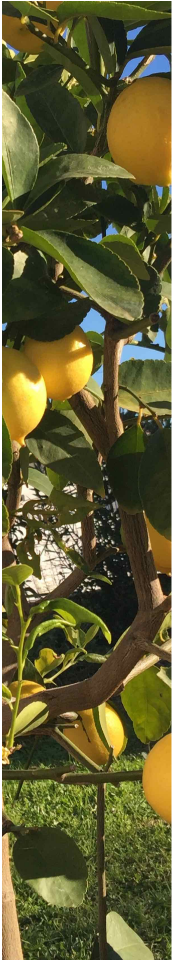
Think of the last line as $\varphi = \bigvee_i \psi_i$
(incorrect, because FOL formulae are finite!)



C.4. Properties of this site

The proposed site is actually a site
i.e. it enjoys the three properties.

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \vdash \varphi$ and $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ then $\psi \lhd \{\psi \wedge \varphi_i \mid i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$
   and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \mid k \in \mathscr{K}_i\}$,
   then $\varphi \lhd \{\psi_{i,k} \mid i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

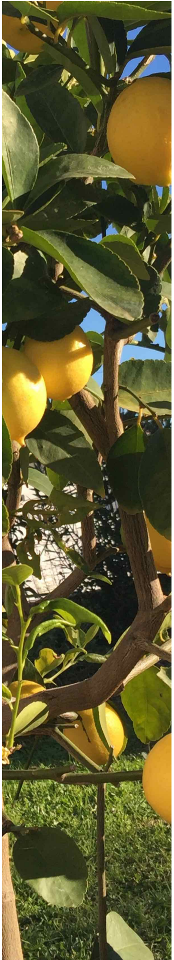


C.5. Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Canonical presheaf:

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$:
     set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^{\varphi}) = f(\vec{t})^{\varphi}$;
  3. Relation symbols: $\vec{t}^{\varphi} \in R_{\overline{\varphi}} \iff \varphi \vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \upharpoonright_{\overline{\varphi}} = t^\varphi$.

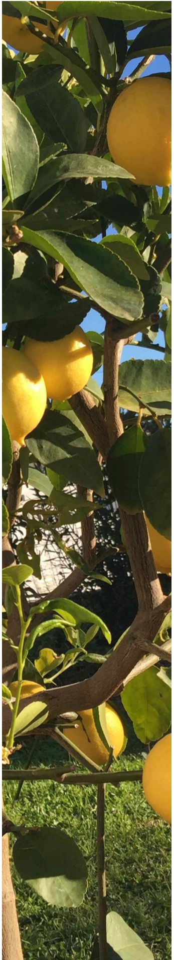


C.6. The canonical presheaf is well defined

The canonical presheaf is separated. If two elements have the same restrictions on each part of a cover, then they are equal.

The interpretation of atomic formulas is local. If an atomic formula holds on each part of a cover of U then it holds on U.

(Ivano Ciardelli claims that the glueing of compatible elements may not exists, i disagree, I think the canonical (separated) presheaf is actually a sheaf).



C.7. Method for the proof of completeness

$$\forall \psi \Big[\forall \varphi \; [\; \mathit{if} \quad \overline{\varphi} \Vdash \psi \quad \mathit{then} \quad \varphi \vdash \psi] \; \Big]$$

or without much additional effort

$$\forall \psi \Big[\forall \varphi [\overline{\varphi} \Vdash \psi \; \mathit{iff} \; \varphi \vdash \psi] \; \Big]$$

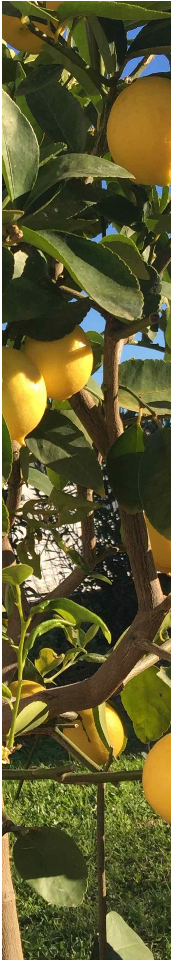
By induction on the formula ψ.

What is fun is that soundness mainly uses introduction rules while completeness mainly uses elimination rules.

The method and the construction can be parametrised by a context Γ for obtaining what is called *strong completeness*: if in every interpretation $u \Vdash \Gamma$ *entails* $u \Vdash X$ for any open u then (iff) $\Gamma \vdash X$.

The quotient on formulas is not really needed.

Having equality is not mandatory but pleasant.



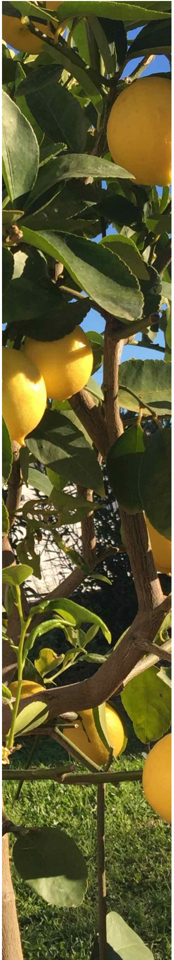
C.8. Sketch of completeness proof

Truth Lemma 1. For any formula φ and sentence ψ,

$$\overline{\varphi} \Vdash \psi \iff \varphi \vdash \psi$$

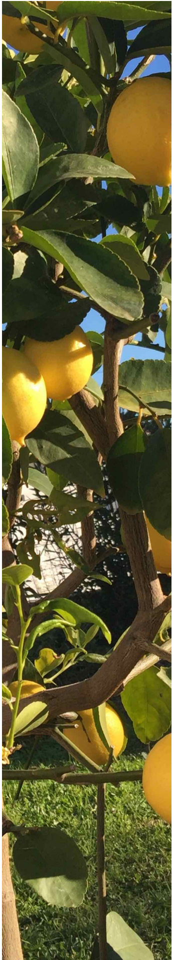
Proof By induction on ψ. The two directions of each inductive step amount to the introduction and elimination rules for the given logical constant.

Let us look at the case of the existential quantifier.



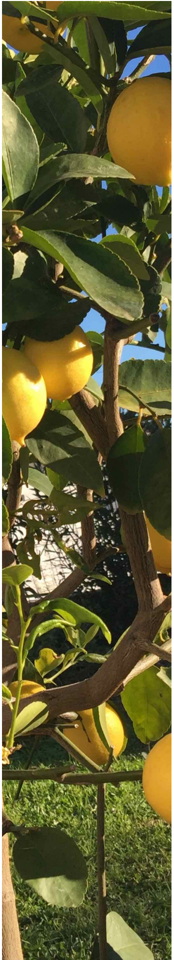
C.9. Completeness $\exists$ direction $\Rightarrow$

- Suppose $\overline{\varphi} \Vdash \exists x\, \psi(x)$.
- There is a family $\{\overline{\varphi_i} \,|\, i \in \mathscr{I}\}$ and elements $t_i^{\varphi_i} \in M_{\overline{\varphi_i}}$ such that $\overline{\varphi_i} \vdash_{[x \mapsto t_i^{\varphi_i}]} \psi(x)$ for all $i \in \mathscr{I}$.
- Since $[t] = t^{\varphi_i}$ for closed t at $\overline{\varphi_i}$, this is $\overline{\varphi_i} \Vdash \psi(t_i)$.
- By induction hypothesis amounts to $\varphi_i \vdash \psi(t_i)$.
- By rule $(\exists i)$, for any $i \in \mathscr{I}$ we have $\varphi_i \vdash \exists x\, \psi(x)$.
- Since $\overline{\varphi} \lhd \{\overline{\varphi_i} \,|, i \in \mathscr{I}\}$, by the meaning of $\lhd$ we have $\varphi \vdash \exists x\, \psi(x)$.



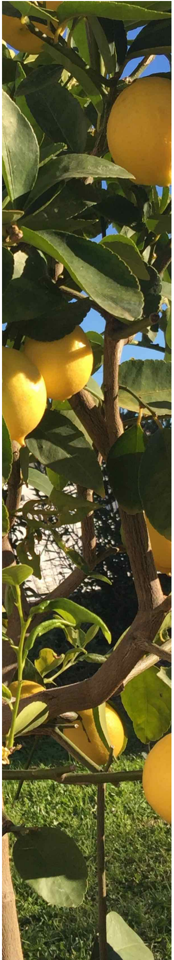
C.10. Completeness $\exists$ direction $\Leftarrow$

- Suppose $\varphi \vdash \exists x\, \psi(x)$.
- We must provide a covering of $\overline{\varphi}$ and local witnesses.
- For any constant c, define $\varphi_c = \varphi \wedge \psi(c)$.
- Since $\varphi_c \vdash \psi(c)$, by induction hypothesis $\overline{\varphi_c} \Vdash \psi(c)$.
- Since $[c] = c^{\varphi_c}$ at $\overline{\varphi_c}$, also $\overline{\varphi_c} \vdash_{[x \mapsto c^{\varphi_c}]} \psi(x)$, i.e. the element c^{φ_c} is a witness for the existential at $\overline{\varphi_c}$.
- It remains to be seen that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.



C.11. Completeness $\exists$ direction $\Leftarrow$, continued

- Suppose ξ is derivable from $\varphi \wedge \psi(c)$ for any constant c.
- Let c^* be a constant that occurs neither in φ nor in ξ.
- In particular, $\varphi \wedge \psi(c^*) \vdash \xi$, that is, $\varphi, \psi(c^*) \vdash \xi$.
- But since c^* occurs neither in φ nor in ξ, by the rule $(\exists e)$ we have $\varphi, \exists x \psi(x) \vdash \xi$.
- Thus by the assumption $\varphi \vdash \exists x \psi(x)$ we also have $\varphi \vdash \xi$.
- This shows that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.
- Hence we conclude $\overline{\varphi} \Vdash \exists x \psi(x)$.



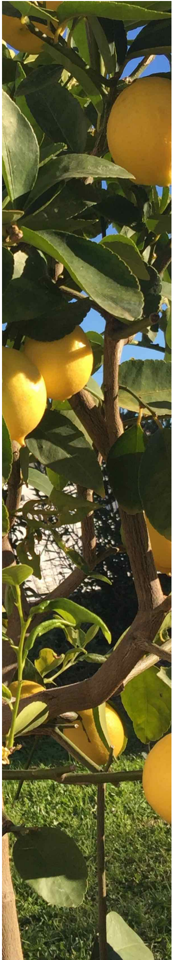
C.12. State of the art: hard to tell

Before 1995 : survey by Makkay and Reyes in 1995.

After 1995, other work in particular by Awodey.

Direct completeness via canonical presheaf: Ivano Ciardelli. A Canonical Model for Presheaf Semantics. Talk at Topology, Algebra and Categories in Logic (TACL) 2011, Jul 2011, Marseille, France. 2011.HAL Id: inria-00618862 https://hal.inria.fr/inria-00618862

Ongoing work with David Théret in Montpellier.



C.13. Future work

Connection to Ω sets of Dana Scott (roughly speaking, if i understand properly: one classical model, but the truth value of $P(a)$ varies in a Heyting algebra, like Boolean valued models) ?

Can we construct a canonical sheaf and not just separated presheaf e.g. with the sheaf completion method that basically simply formally adds the missing global sections? Or by imposing some additional locality condition on terms and equality?

(Pre)sheaves are particular kinds of Kripke models, conversely can any Kripke model be viewed as a pre(sheaf) with the order topology?

Is it possible to do so with a standard topology (instead of a pretopology / Grothendieck topology)?

Does it applies to first order S4?

Thank you for your attention.