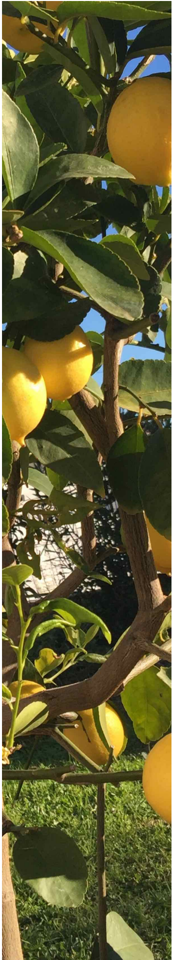
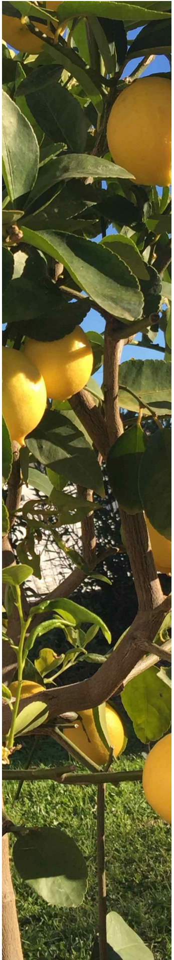


Completeness of first order intuitionistic logic with respect to (pre)sheaves of classical models

Christian Retoré (LIRMM, Université de Montpellier)
with Jacques van de Wiele (Paris) 1987,
Ivano CIardelli (München) 2011,
David Théret (Montpellier) 2016,
...

Stone duality, formal languages, and logic.
JAD, Nice, January 24, 2019



Remarks

A beautiful subject - not my main research area.

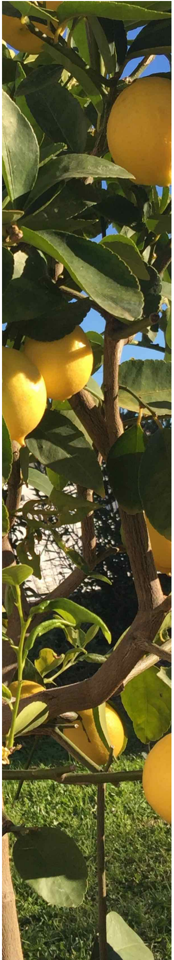
The presentation of Kripke-Joyal forcing and counter example is from a lecture by Jacques Van de Wiele in 1987.

The direct completeness proof is essentially due to
Ivano Ciardelli (in TACL 2011 cf. reference at the end).

It has been re-worked with David Théret in 2016.

Thanks to Frédéric Patras for inviting me and to Célia Borlido.

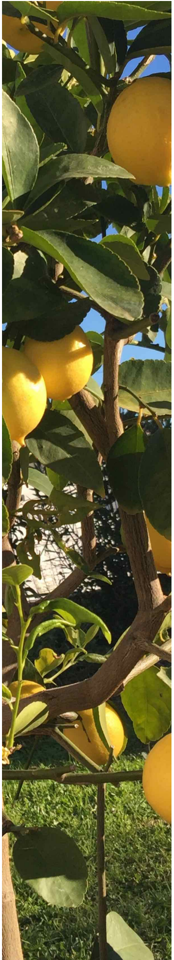
Thanks to the Topos & Logic group
(Jean Malgoire, Nicolas Saby, David Theret)
of the Institut Montpelliérain Alexander Grothendieck
(+ Abdelkader Gouaich, LIRMM)



A Logic?

formulas

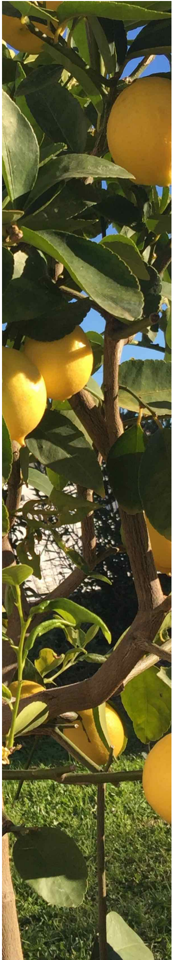
proofs ⇜⇝ interpretations



A.1. Formulas, proofs and models

Formulas of a given logical language,
can be true (or not) in a given model :
wellformed expressions of a logical language
have a meaning the language is interpreted.

A proof may prove a formula of a given language,
or prove a formula of a given language
from assumptions (or axioms) of the same language.



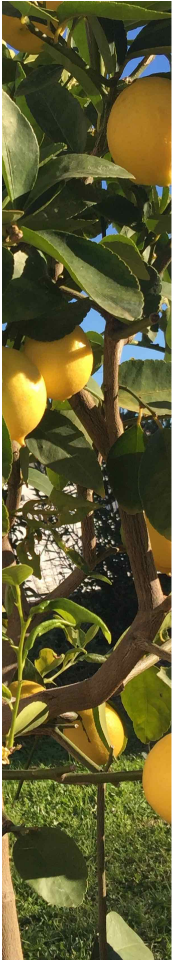
A.2. Usual / classical models of first order logic

One is given a language $\mathscr{L}$,
e.g. constants $(0, 1)$, functions $(+, *)$, and predicates $(\leqslant)$.

One is given a set $|M|$ (non empty).

Constants are interpreted by elements of $|M|$,
n-ary functions symbols by n-ary applications from $|M|^n$ to $|M|$, and n-ary predicates by parts of $|M|^n$. ($\mathscr{L}$-structure)

Logical connectives and quantifiers are interpreted intuitively (Tarskian truth: ”$\wedge$” means ”and”, ”$\forall$” means ”for all” etc.).

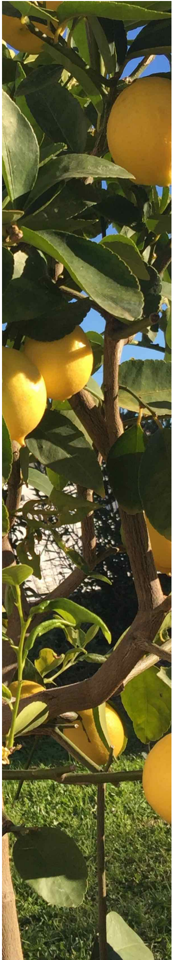


A.3. Soundness

any provable formula is true for every interpretation

or:

when T entails F then any model that satisfies T satisfies F



A.4. Completeness

Completeness (a word that often encompass soundness):

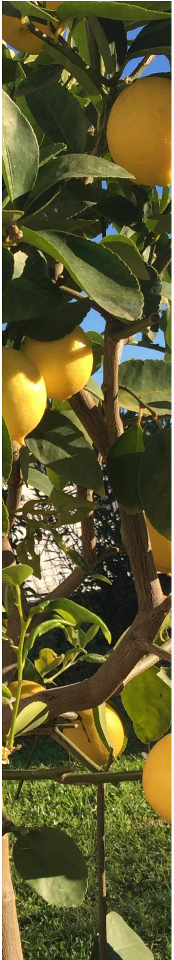
a formula that is true in every interpretation is derivable

or

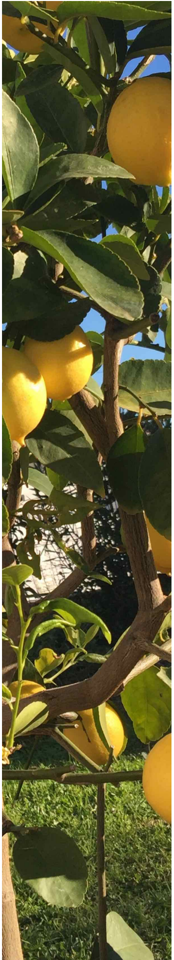
a formula F that is true in every model of T
is a logical consequence of T

e.g.
a formula F of ring theory is true in any ring
if and only if
F is provable from the axioms of ring theory

Soundness, completeness (and compacity)
are typical fo first order logic (as opposed to higher order logic).



B Intuitionistic logic



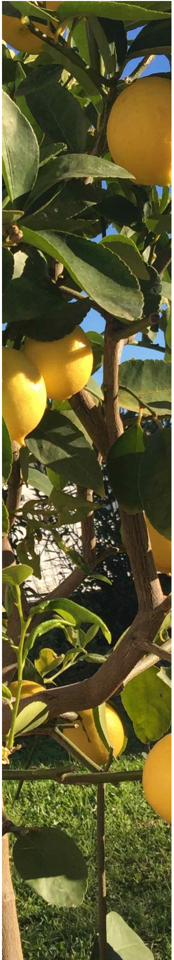
B.1. Intutionistic logic vs. classical logic (the usual logic of mathematics)

Absence of Tertium no Datur, $A \vee \neg A$ does not always hold.

Disjunctive statemeents are stronger.

Existential statements are stronger.

Proof have a constructive meaning,
algorithms can be extracted from proofs.



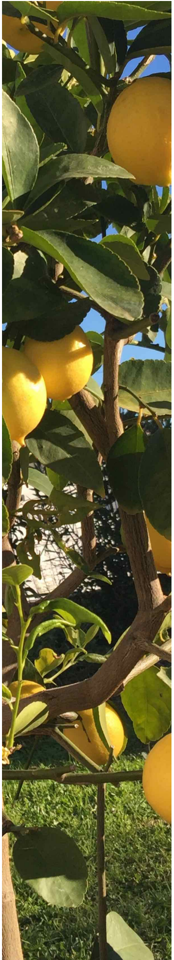
B.2. Rules of intuitionist logic: structures

Structural rules

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\, E_g$$

$$\frac{\Delta \vdash C}{A, \Delta \vdash C}\, A_g$$

$$\frac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}\, C_g$$

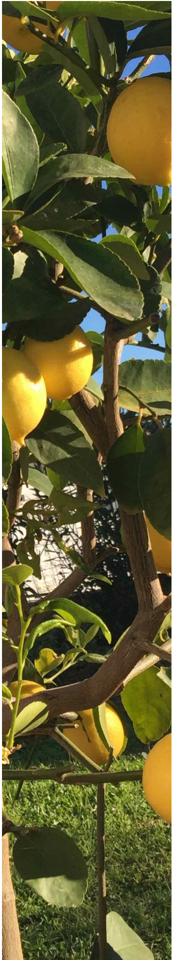


B.3. Rules of intuitionistic logic: connectives

Axioms are $A \vdash A$ (if A then A...) for every A.

Negation $\neg A$ is just a short hand for $A \Rightarrow \bot$.

$\dfrac{\Theta \vdash (A \wedge B)}{\Theta \vdash A}\wedge_e \quad \dfrac{\Theta \vdash (A \wedge B)}{\Theta \vdash B}\wedge_e$	$\dfrac{\Theta \vdash A \quad \Delta \vdash B}{\Theta, \Delta \vdash (A \wedge B)}\wedge_d$
$\dfrac{\Theta \vdash (A \vee B) \quad A, \Gamma \vdash C \quad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C}\vee_e$	$\dfrac{\Theta \vdash A}{\Theta \vdash (A \vee B)}\vee_d \quad \dfrac{\Theta \vdash B}{\Theta \vdash (A \vee B)}\vee_d$
$\dfrac{\Theta \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma, \Theta \vdash B}\Rightarrow_e$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash (A \Rightarrow B)}\Rightarrow_d$
$\dfrac{\Gamma \vdash \bot}{\Gamma \vdash \mathrm{C}}\bot_e$	

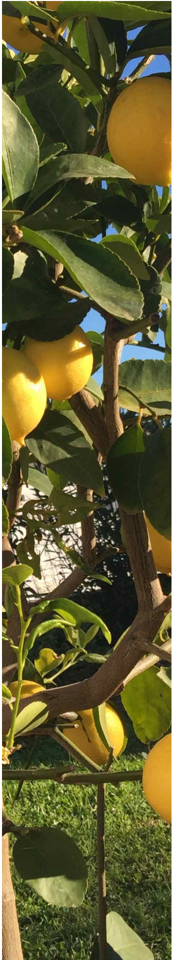


B.4. Rules of intuitionistic logic: connectives

Axioms are $A \vdash A$ (if A then A...) for every A.

Negation $\neg A$ is just a short hand for $A \Rightarrow \bot$.

$\dfrac{\Theta \vdash \forall x A}{\Theta \vdash A[x := t]} \wedge_e *$	$\dfrac{\Theta \vdash A}{\Theta \vdash \forall x A} \forall_d *$
$\dfrac{\Theta \vdash \exists x A \quad A, \Gamma \vdash C}{\Theta, \Gamma \vdash C} \exists_e *$	$\dfrac{\Theta \vdash A[t]}{\Theta \vdash \exists x A[x]} \vee_d$



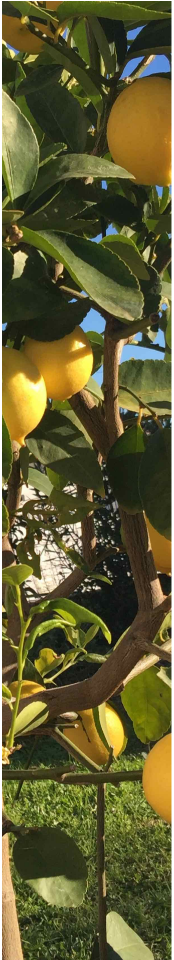
B.5. Differences

$A \vee \neg A$ does not hold for any A.
$\neg\neg B$ does not entail B.
However $\neg\neg(C \vee \neg C)$ hlods for any C.

$$\dfrac{\dfrac{[\neg(A\vee\neg A)]^2 \quad \dfrac{\dfrac{\dfrac{[\neg(A\vee\neg A)]^2 \quad \dfrac{[A]^1}{A\vee\neg A}\,\vee_i}{\bot}\,\to_e}{\neg A}\,\to_i{}^1}{A\vee\neg A}\,\vee_i}{\bot}\,\to_e}{\neg\neg(A\vee\neg A)}\,\to_i{}^2$$

- tertium non datur
- reductio ad absurdum
- Pierce law $((p \to q) \to p) \to p$

are inutionistically equivalent.



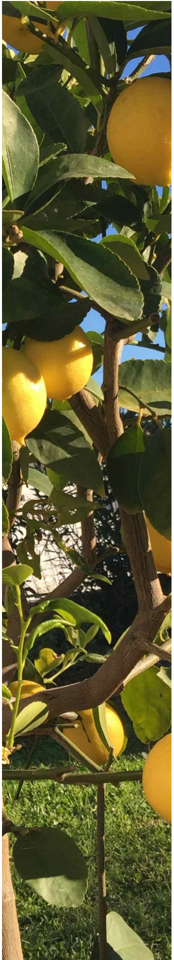
B.6. Relations to classical logic

All ”classical” proofs are valid intuitionistically.

Conversely, $\Vdash^{LK} F$ iff $\Vdash^{LJ} F^{\neg\neg}$ (Gödel, Glivenko, Kolmogorov).

$$
\begin{aligned}
\bot^{\neg\neg} &= \bot \\
a^{\neg\neg} &= \neg\neg a \\
(A \wedge B)^{\neg\neg} &= A^{\neg\neg} \wedge B^{\neg\neg} \\
(A \rightarrow B)^{\neg\neg} &= A^{\neg\neg} \rightarrow B^{\neg\neg} \\
(\forall x. A)^{\neg\neg} &= \forall x. A^{\neg\neg} \\
(A \vee B)^{\neg\neg} &= \neg\neg(A^{\neg\neg} \vee B^{\neg\neg}) \\
(\exists x. A)^{\neg\neg} &= \neg\neg\exists x. A^{\neg\neg}
\end{aligned}
$$

Richard Moot Christian Retoré Classical logic and intuitionistic logic: equivalent formulations in natural deduction, Gödel-Kolmogorov-Glivenko translation — complete proofs of well-known results that are not available elsewhere arXiv:1602.07608



B.7. Existential differences

$\neg\forall x.\neg P(x)$ does not entail $\exists x P(x)$.

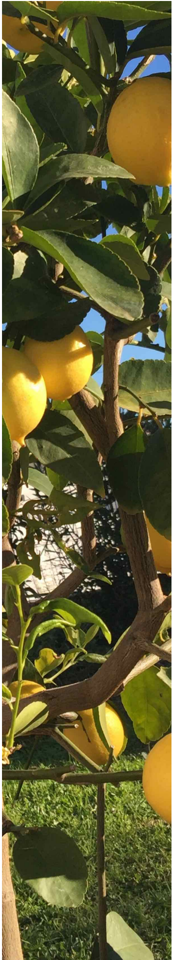
A normal intuitionistic proof of $\exists x P(x)$ is a proof of $P(t)$ for some term t.

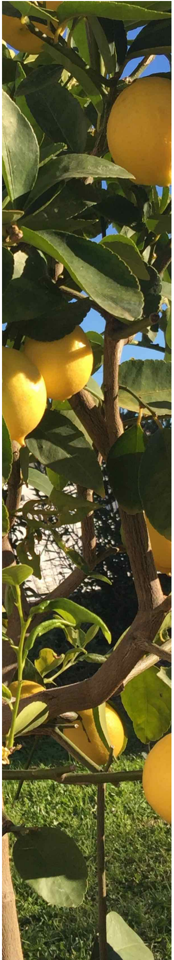
From a proof of $\forall x \exists y\ P(x, y)$ one may extract a function that computes from any x a term $t(x)$ such that $P(x, T(x))$. Extraction of certified functional programs from formal proofs of their specification.

An example, in the language of rings: $\forall x.((x = 0) \vee \neg(x = 0))$ is not provable and their are concrete counter models.

$[\neg\forall x.((x = 0) \vee \neg(x = 0))] \rightarrow [\neg\forall x.\neg\neg((x = 0) \vee \neg(x = 0))]$
is also non provable.

C (Pre)sheaf semantics a.k.a. topological models

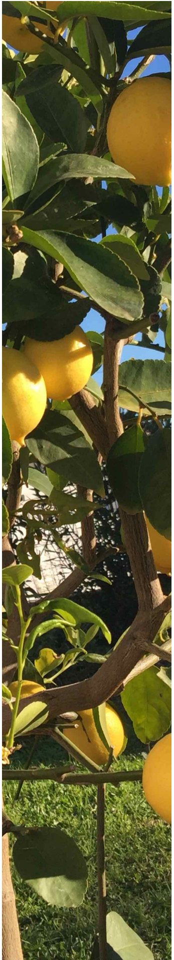




C.1. First order language

A first order language $\mathscr{L}$ is defined by

- a collection of predicates (also called relational symbols), each of them endowed with an arity There might be a binary predicate, ”=”.
- a collection of functions (also called function symbols) each of them endowed with an arity — this collection may include functions of arity 0, which are called constants.

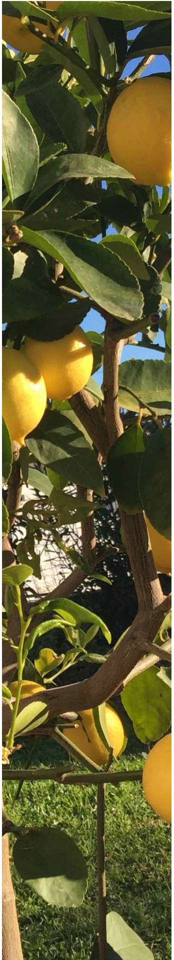


C.2. $\mathscr{L}$-terms

Terms of $\mathscr{L}$ are defined as usual from an at least countable set of variables:

- variables are terms;
- if ϑ is a k-ary function symbols and if $t_1, \dots, t_k$ are k terms $\vartheta(t_1, \dots, t_k)$ is a term as well — hence, constants, which are 0-ary functions are terms.

A term without variables is said to be a closed term.



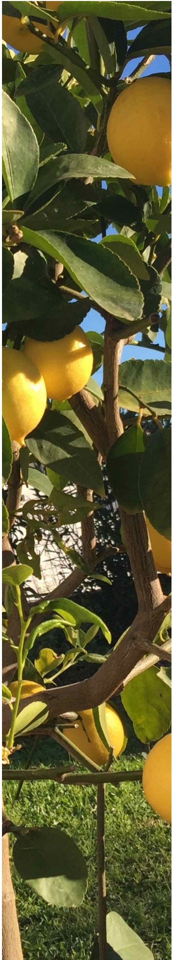
C.3. $\mathscr{L}$-formulas

Given an n-ary predicate R of $\mathscr{L}$, and n terms $t_1, ..., t_n$ of $\mathscr{L}$,

$R(t_1, ..., t_n)$ is an atomic $\mathscr{L}$-formula

Formulas of $\mathscr{L}$ are defined as follows:

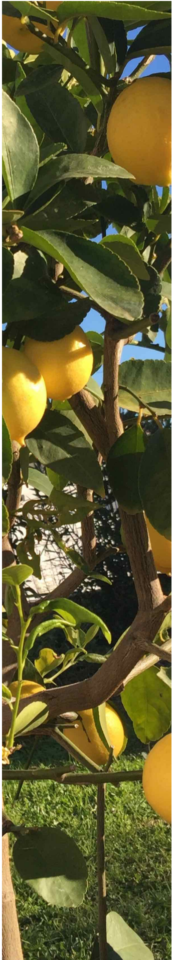
- atomic $\mathscr{L}$-formulas are formulas;
- if F is an $\mathscr{L}$-formula, then $\neg F$ is an $\mathscr{L}$-formula;
- if F and G are $\mathscr{L}$-formulas, $F \wedge G$, $F \vee G$ and $F \rightarrow G$ are $\mathscr{L}$-formulas.
- if F is an $\mathscr{L}$-formula, and if x is a variable $\forall x\ F$ and $\exists x F$ are $\mathscr{L}$-formulas.



C.4. $\mathscr{L}$-formulas: bound and free variables

Bound and free occurrences of variables are defined as expected:

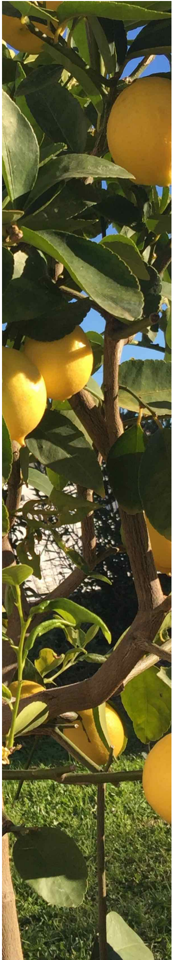
- any occurrence of a variable in an atomic formula is free;
- an occurrence of a variable in $F \wedge G$, $F \vee G$ and $F \rightarrow G$ is free (resp. bound) if and only if this occurrence is free (resp. bound) in the subformula F or in G in which the occurence is.
- an occurrence of a variable in $\neg F$ is free (resp. bound) if and only if tis is free (resp. bound) in the subformula F
- the free occurrences of x in F are bound by $\exists x$ in $\exists x F$ and they are bound by $\forall x$ in $\forall x F$; occurrences of variables other than in $\exists x F$ or $\forall x F$ are free (resp. bound) iff they are free (resp. bound) in F.



C.5. $\mathscr{L}$-structure

Given a first order language $\mathscr{L}$ an $\mathscr{L}$-structure (or a model) M_u is a non empty set $|M_u|$ and an interpretation of the symbols in the language:

- if ϑ is an n-ary function the interpretation ϑ_u of ϑ in M_u is a k-ary (total) function $\vartheta_u : |M_u|^k \mapsto |M_u|$ — in particular a constant a is interpreted as an element a_u of $|M_u|$ (a function from $\{*\}$ to $|M_u|$).
- if R is an n-ary predicate, the interpretation R_u of R in M_u is an n-ary relation R_u on $|M_u|$ i.e. $R_u \subset |M_u|^n$. If there is the equality "=" predicate, it is necessarily interpreted by equality in $|M_u|$ i.e. $=_u$ is $\{(x, x) | x \in |M_u|\}$.



C.6. Morphisms of $\mathscr{L}$-structures

Let M_u and M_v be two $\mathscr{L}$ structures over the same language — interpretation in u or in v of function symbols (e.g. ϑ) and predicates (e.g. R) are denoted with a subscript u or v (e.g. R_u ϑ_u: interpretations in M_u and R_v ϑ_v: interpretations in M_v).

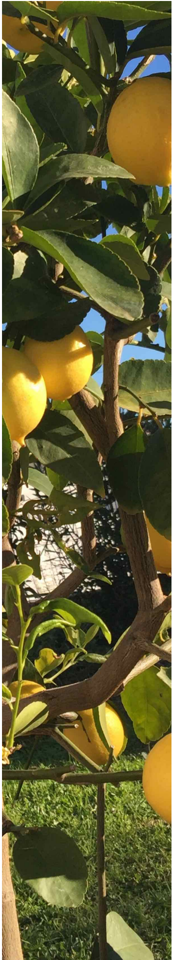
A map $\rho_{u\to v}$ from $|M_u|$ to $|M_v|$ is said to be a morphism of $\mathscr{L}$-structures when:

- For any k-ary function ϑ symbol of $\mathscr{L}$:

  $\forall c_1, \ldots, c_k \in |M_u| \qquad \rho_{u\to v}(\vartheta_u(c_1, \ldots, c_k)) = \vartheta_v(\rho_{u\to v}(c_1), \ldots, \rho_{u\to v}(c_k))$

- For any n-ary predicate R of $\mathscr{L}$:

  $\forall c_1, \ldots, c_n \in |M_u| \qquad$ *if* $(c_1, \ldots, c_n) \in R_u$ then $(\rho_{u\to v}(c_1), \ldots, \rho_{u\to v}(c_k)) \in R_v$



C.7. Presheaf semantics: models

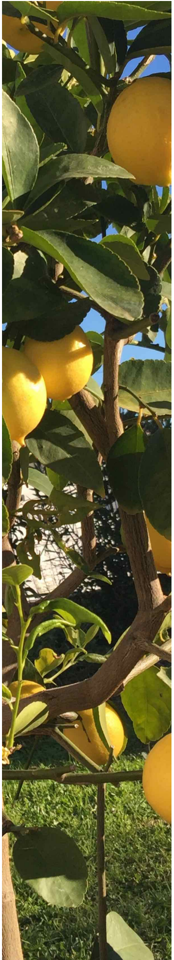
A **presheaf model** M for $\mathscr{L}$ is a presheaf of first-order $\mathscr{L}-$structures over a Grothendieck site $(\mathscr{C}, \lhd)$ (or a topological space viewed as a poset for inclusion):

- for any object u an $\mathscr{L}$ structure M_u
- for any arrow $f : v \hookrightarrow u$ a morphism (cf. supra) of $\mathscr{L}$ structures $M(f) : M_u \to M_v$

satisfying the following extra conditions.

Separateness For any elements a, b of M_u, **if** there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for all $i \in \mathscr{I}$ we have $M(f_i)(a) = M(f_i)(b)$, **then** $a = b$.

Local character of atoms For any $n-$ary relation symbol R, for any tuple $(a_1, \ldots, a_n)$ from M_u **if** there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that $\forall i \in \mathscr{I}$ one has $(M(f_i)(a_1), \ldots, M(f_i)(a_n)) \in R_{u_i}$, **then** $(a_1, \ldots, a_n) \in R_u$.

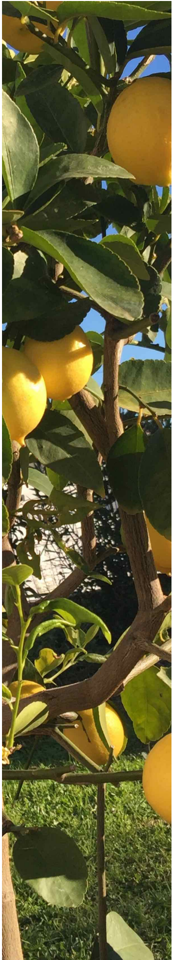


C.8. Presheaf semantics: Kripke-Joyal forcing — 1/4 assignments

Given a presheaf model M, and some open u, we inductively define for any formula F of $\mathscr{L}$ the relation $u \Vdash F$ ("meaning": F is true at u).

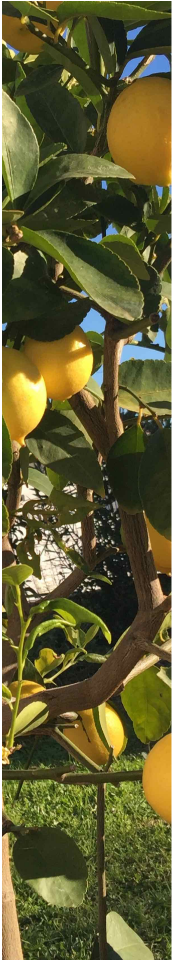
Assignment A usual, in order to define $u \Vdash F$, we need an assignment ν in M_u the free variables of F, and this is written $u \Vdash_\nu F$ with $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$ where the z_i are the free variables in F and $c_i \in |M_u|$.

As we shall see, $u \Vdash_\nu F$ can be defined from $v \Vdash_{\nu'} F'$ with $f : v \hookrightarrow u$ and with F' having free variables among those of F (plus possibly one free variable in the $\exists$ and $\forall$ cases, but its assignment will defined when dealing with quantifiers). If $\nu = [z_1 \mapsto c_1; \cdots; z_p \mapsto c_p]$ we naturally define ν' by $\nu' = [z_1 \mapsto M(f)(c_1); \cdots; z_p \mapsto M(f)(c_p)]$ where $M(f)$ is the restriction $M(f) : |M_u| \to |M_v|$.



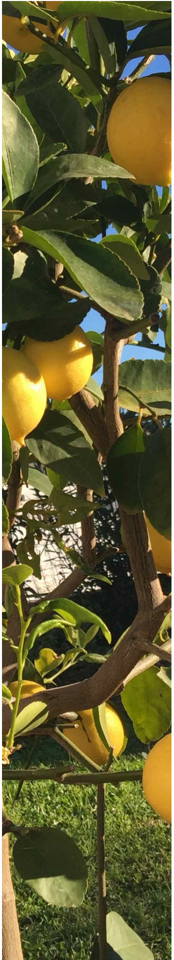
C.9. Presheaf semantics: Kripke-Joyal forcing — 2/4 atoms and conjunction

- $u \Vdash_v R(t_1, \ldots, t_n)$ iff $([t_1]_v, \ldots, [t_n]_v) \in R_u$.
- $u \Vdash_v t_1 = t_2$ iff $[t_1]_v = [t_2]_v$.
- $u \Vdash_v \perp$ iff $u = \emptyset$ It is so, because the empty covering is a covering (with 0 open) of the empty open. Hence, because of the locality condition on atoms, the empty open forces all atomic formulas including $\perp$.
- $u \Vdash_v \varphi \wedge \psi$ iff $u \Vdash_v \varphi$ and $u \Vdash_v \psi$.



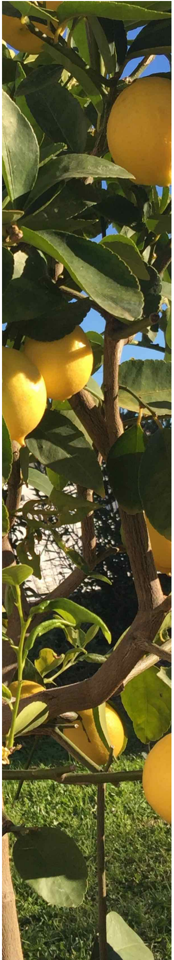
C.10. Presheaf semantics: Kripke-Joyal forcing — 3/4 disjunction and existential

- $u \Vdash_{\nu} \varphi \vee \psi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that for any $i \in \mathscr{I}$ we have $u_i \Vdash_{\nu_i} \varphi$ or $u_i \Vdash_{\nu_i} \psi$.
- $u \Vdash_{\nu} \exists x \varphi$ iff there exists a covering family $\{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathscr{I}$ such that $u_i \Vdash_{\nu_i \cup [x \mapsto a_i]} \varphi$ for any index i.



C.11. Presheaf semantics: Kripke-Joyal forcing — 4/4 implication and universal

- $u \Vdash_{\nu} \varphi \to \psi$ iff for all $f : v \hookrightarrow u$, if $v \Vdash_{\nu_v} \varphi$ then $v \Vdash_{\nu_v} \psi$.
- $u \Vdash_{\nu} \neg\varphi$ iff for all $f : v \hookrightarrow u$, with $v \neq \emptyset$, $v \nVdash_{\nu_v} \varphi$. This is obtain from $\emptyset \Vdash \bot$ and $\to$ cases because $\neg\varphi = \varphi \to \bot$.
- $u \Vdash_{\nu} \forall x \varphi$ iff for all $f : v \hookrightarrow u$ and all $a \in M_v$, $v \Vdash_{\nu_v \cup [x \mapsto a]} \varphi$.

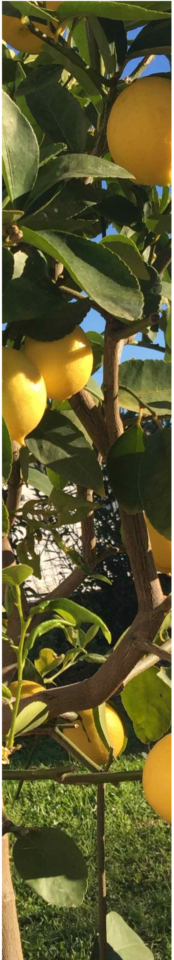


C.12. Validity

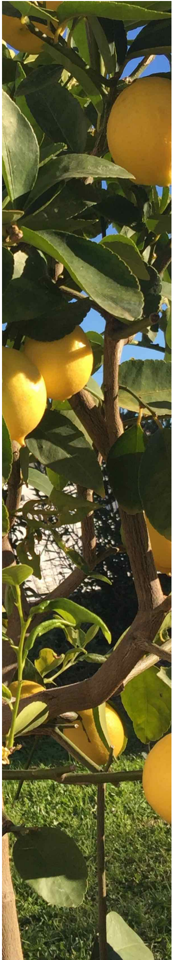
A formula F is said to be valid in a topological model in a presheaf model over a topological space $(X, O(X))$ or a pre-topology whenever

$$X \Vdash F$$

i.e. F is true at the global section.



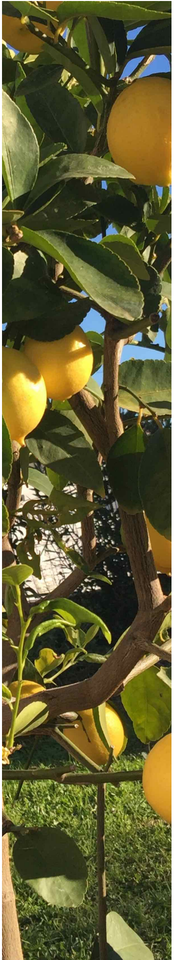
D An example



D.1. Language

Let us consider the language of ring theory:

- two constants $0, 1$
- two binary functions $+, \cdot$
- equality as the only predicate



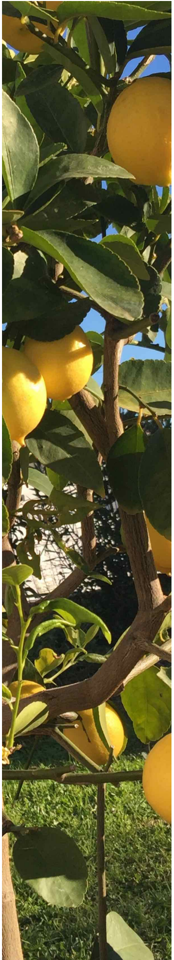
D.2. The (pre)sheaf of $\mathscr{L}$-structures

A presheaf model over the topological space $\mathbb{R}$ for this language is defined by $|M_u| = C(u,\mathbb{R})$ the continuous functions from u to $\mathbb{R}$ with $0_u(x) = 0$ and $1_u(x) = 1$ for all $x \in u$ $+_u$ pointwise addition $(f +_u g)(x) = f(x) + g(x)$, $\cdot_u$ pointwise multiplication $(f \cdot_u g)(x) = f(x) \cdot g(x)$.

The restriction $\rho_{u\to v} : |M_u| \to |M_v|$ morphism, when $v \hookrightarrow u$ is defined by: $\forall f \in C(u,\mathbb{R}) \forall x \in v\ \rho_{u\to v}(f)(x) = f(x)$.

$\rho_{u\to v}$ is a morphism, because:

- $\rho_{u\to v}(0_u) = 0_v$,
- $\rho_{u\to v}(1_u) = 1_v$
- ("=" is the only predicate) $\forall f, g \in |M_u| = C(u,\mathbb{R})$ if $f = g$ in $C(u,\mathbb{R})$ then $\rho_{u\to v}(f) = \rho_{u\to v}(g)$ in $C(v,\mathbb{R})$.

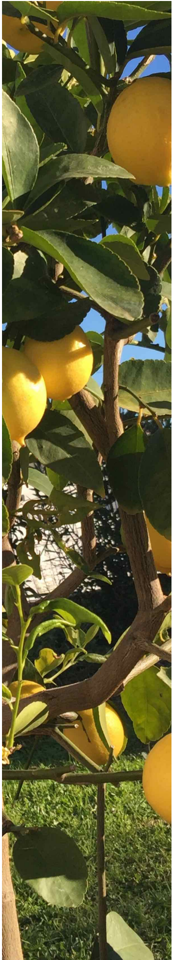


D.3. Locality and separateness conditions

Locality condition for atoms:

"=" is the only predicate so we just have to check that, given two elements a and b of M_u if we have a covering **if** there is a cover $u \lhd \{f_i : u_i \hookrightarrow u \,|\, i \in \mathscr{I}\}$ such that **if** $\forall i \in \mathscr{I}$ we have $(\rho_{u \to u_i}(a)) = \rho_{u \to u_i}(b))$ in $|M_{u_i}|$, **then** $a = b$ in $|M_u|$. This is true, because two functions that are equal on each open of a covering of u are equal on u.

Separateness is exactly the locality condition for our unique predicate, i.e.e the "=" predicate, which is interpreted as "=".



D.4. A remark on $C(U,\mathbb{R})$ 1/2

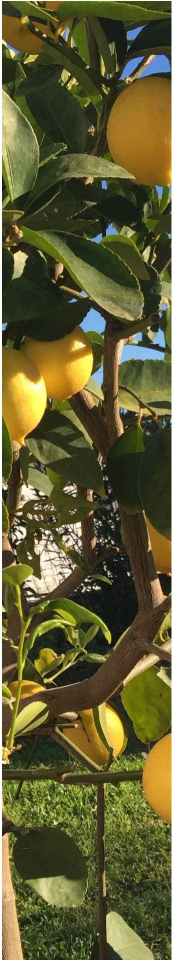
Given any non empty open subset $U \subset \mathbb{R}$ there exist

- an open subset $V =]a,b[\subset U$

- and a continuous function $\ell : V \mapsto \mathbb{R}$

such that $V \not\Vdash_{[x \mapsto \ell]} (x = 0) \vee \neg(x = 0)$ with ℓ:

$$\begin{array}{llcll} \ell : &]a,b[& \mapsto & \mathbb{R} & \\ & x & \mapsto & 0 & \text{if } x \leqslant (a+b)/2 \\ & x & \mapsto & x-(a+b)/2 & \text{if } x \geqslant (a+b/2) \end{array}$$



D.5. A remark on $C(U,\mathbb{R})$ 2/2

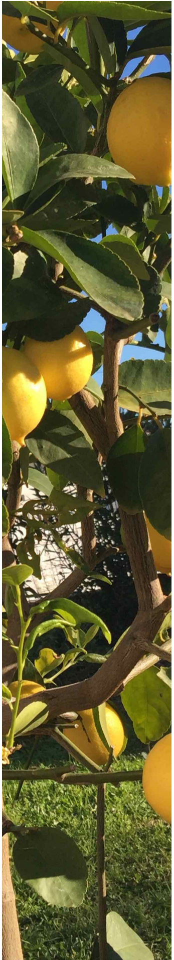
$]a,b[\nVdash_{[x\mapsto\ell]} (x=0\vee\neg(x=0))$.

We proceed by contradiction (the meta logic is classical). Let us assume that $]a,b[\Vdash_{[x\mapsto\ell]} (x=0\vee\neg(x=0))$. Then there exists open sets u_1,u_2 st. $u_1\cup u_2=]a,b[$, such that:

- $u_1 \Vdash_{[x\mapsto\ell_{u_1}]} x=0$ i.e. $\forall x_1\in u_1\quad \ell(x_1)=0$
- $u_2 \Vdash_{[x\mapsto\ell_{u_2}]} \neg(x=0)$ i.e. $\forall v_2\subset u_2\quad v_2\nVdash_{[x\mapsto\ell_{u_2}]} \ell=0$ i.e. ℓ never is constantly 0 on a sub open v_2 of u_2.

This is impossible because $(a+b)/2$ must be in u_1 or in u_2.

- If $(a+b)/2\in u_1$ then ℓ should be constantly 0 on a neighbourhood of $(a+b)/2$, but it is false on the right side of $(a+b)/2$.
- If $(a+b)/2\in u_2$ then ℓ should never be constantly 0 on any sub open of u_2 but if $(a+b)/2\in u_2$ there are sub opens in u_2 on the left side of $(a+b)/2$ where ℓ is constantly 0.

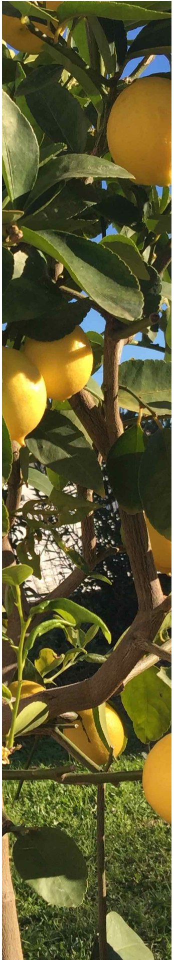


D.6. A classically valid but intuitionistically non valid formula

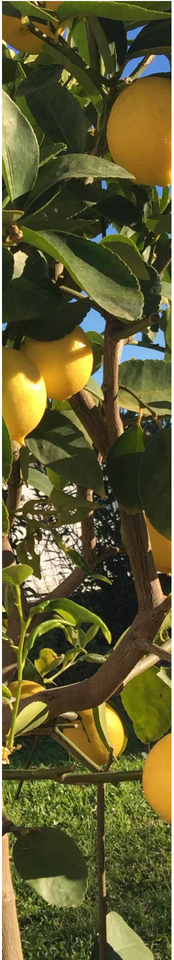
$C(\mathbb{R},\mathbb{R})$ validates $\neg\forall x\ (x=0)\vee\neg(x=0)$ (*). Indeed, according to Kripke-Joyal $\mathbb{R}\Vdash\neg\forall x\ (x=0)\vee\neg(x=0)$ means that for every non empty open $u\subset\mathbb{R}$, $u\nVdash\forall x\ (x=0)\vee\neg(x=0)$. But $u\Vdash\forall x\ (x=0)\vee\neg(x=0)$ means that for every open $v\subset u$ and for every $f\in C(v,\mathbb{R})$ $v\Vdash_{[x\mapsto f]}(x=0)\vee\neg(x=0)$. We precisely established supra (with ℓ) that $u\nVdash\forall x\ (x=0)\vee\neg(x=0)$.

But $C(\mathbb{R},\mathbb{R})$ validates $\forall f\ \neg\neg((x=0)\vee\neg(x=0))$ (**) — we formally proved supra that $\neg\neg(C\vee\neg C)$ for all C.

However in classical logic (*) is the negation of (**) !!!



E Completeness



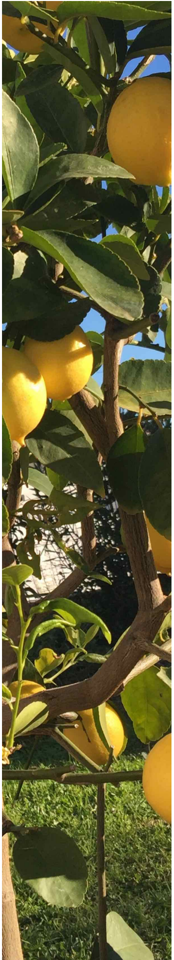
E.1. Statements

Soundness: F intuitionistically provable $\Rightarrow$ F true at any open of any topological interpretation.

Completeness: F true at a global section any topological interpretation $\Rightarrow$ F intuitionisticallyprovable.

Two lemmas:

- (functoriality) $F[c_1, ..., c_n]$ true at $U \Rightarrow F[{c_1}^V, ..., {c_n}^V]$true at any open $V \subset U$.
- (locality) The locality condition for atomic formula (cf. above) extends to *any* formula.



E.2. Proof of soundness

Induction on the proof height, looking at every possible last rule, e.g. in natural deduction. Below: $\vee_e$ case.

$$\frac{\Theta \vdash (A \vee B) \quad A, \Gamma \vdash C \quad B, \Delta \vdash C}{\Theta, \Gamma, \Delta \vdash C} \vee_e$$

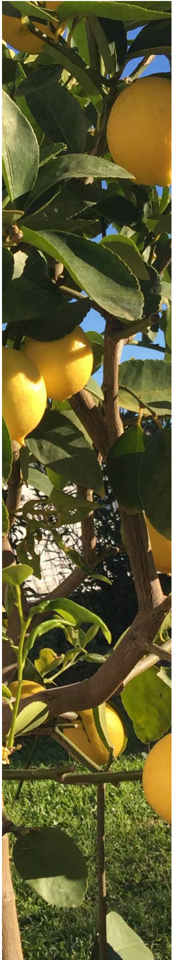
We have to show that $U \Vdash \Theta, \Gamma, \Delta$ then $U \Vdash C$.

If $U \Vdash \Theta$ by induction hypothesis, $U \Vdash A \vee B$. Hence, there exists a covering (U_i) such that for every i $U_i \Vdash A$ or $U_i \Vdash B$.

If $U_i \Vdash A$, because $U \Vdash \Gamma$ we have $U_i \Vdash \Gamma$ (functor property), and by induction hypothesis (proof of $A, \Gamma \vdash C$) $U_i \Vdash C$.

Similarly, if $U_i \Vdash B$, then $U_i \Vdash C$.

So for all i $U_i \Vdash C$ and by locality lemma $U \Vdash C$.



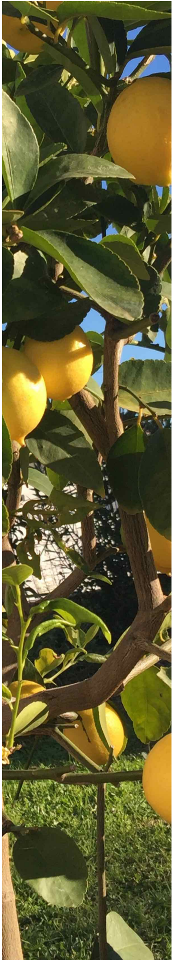
E.3. Completeness for presheaf semantics

If every presheaf models satisfies φ
then φ is provable in **intuitionistic** logic.

Usually established by:

- equivalence with $\Omega-$models;
- construction of a canonical Kripke model.

Here: canonical model (separated presheaf) in which F valid means F intuitionistically provable.



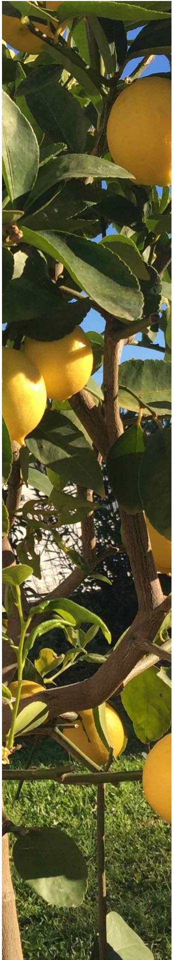
E.4. Canonical model construction: the underlying site

Canonical site:

- **Category:** we take the Lindenbaum-Tarski algebra $\overline{\mathscr{L}}$
  - Objects: classes of provably equivalent formulas $\overline{\varphi}$.
  - Arrows: $\overline{\varphi} \leq \overline{\psi} \iff \varphi \Vdash \psi$
- **Grothendieck topology:** $\overline{\varphi} \lhd \{\psi_i\}_{i \in I}$ whenever

$$\forall \chi \left[\quad \varphi \Vdash \chi \quad \textit{iff} \quad (\forall i \in I \quad \psi_i \Vdash \chi) \right]$$

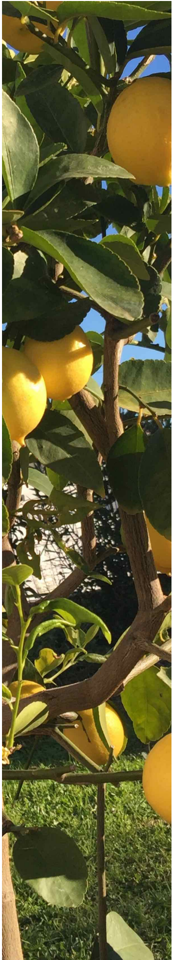
Think of the last line as $\varphi = \bigvee_i \psi_i$
(incorrect, because FOL formulae are finite!)



E.5. Properties of this site

The proposed site is actually a site
i.e. it enjoys the three properties.

1. $\varphi \lhd \{\varphi\}$;
2. if $\psi \Vdash \varphi$ and $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ then $\psi \lhd \{\psi \wedge \varphi_i \mid i \in \mathscr{I}\}$;
3. if $\varphi \lhd \{\varphi_i \mid i \in \mathscr{I}\}$ and if for each $i \in \mathscr{I}$, $\varphi_i \lhd \{\psi_{i,k} \mid k \in \mathscr{K}_i\}$, then $\varphi \lhd \{\psi_{i,k} \mid i \in \mathscr{I}, k \in \mathscr{K}_i\}$.

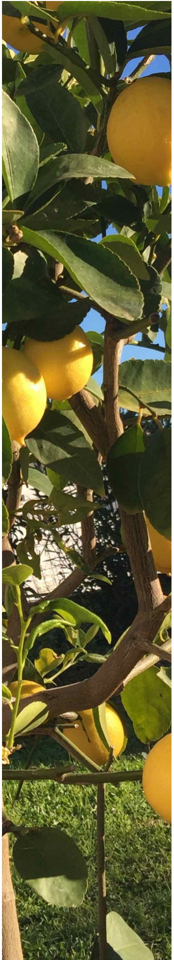


E.6. Canonical model construction: the presheaf

- Put $t \equiv_\varphi t'$ in case $\varphi \Vdash t = t'$.
- Denote by t^φ the equivalence class of t modulo $\equiv_\varphi$.

Canonical presheaf:

- Model $M_{\overline{\varphi}}$:
  1. Universe $|M_{\overline{\varphi}}|$:
     set of equivalence classes t^φ of closed terms;
  2. Function symbols: $f_{\overline{\varphi}}(\vec{t}^\varphi) = f(\vec{t})^\varphi$;
  3. Relation symbols: $\vec{t}^\varphi \in R_{\overline{\varphi}} \iff \varphi \Vdash R(\vec{t})$.
- Restriction. If $t^\psi \in M_{\overline{\psi}}$ and $\overline{\varphi} \leq \overline{\psi}$, put $t^\psi \restriction_{\overline{\varphi}} = t^\varphi$.



E.7. The canonical presheaf is well defined

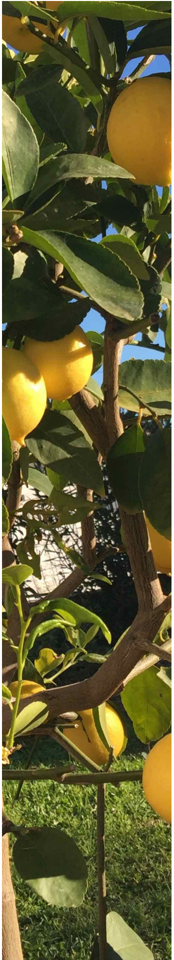
The canonical presheaf is separated. If two elements have the same restictions on each part of a cover, then they are equal.

The interpretation of atomic formulas is local. If an atomic formula holds on each part of a cover of U then it holds on U.

Observe that it is not a sheaf
(the glueing of compatible elements may not exist).

$\varphi_0 : c_1 = c_2$ $\varphi_1 : c_0 = c_2$ $\varphi_2 : c_0 = c_1$ covers $\varphi = \varphi_0 \vee \varphi_1 \vee \varphi_2$. On intersection, all the three constants are identical. However one cannot find (in the absence of additional properties of the c_i) three elements that restricts properly on each of the three φ_i.

E.8. Method for the proof of completeness



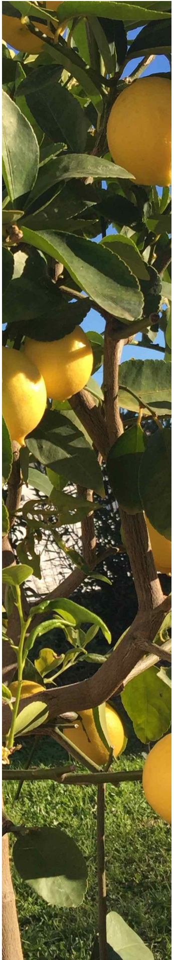
$$\forall \psi \left[\forall \varphi \left[\textit{ if } \quad \overline{\varphi} \Vdash \psi \quad \textit{then} \quad \varphi \Vdash \psi \right] \right]$$

By induction on the formula ψ.

It is also possible to prove directly:

$$\forall \psi \left[\forall \varphi [\overline{\varphi} \Vdash \psi \textit{ iff } \varphi \Vdash \psi] \right]$$

What is fun is that soundness mainly uses introduction rules while completeness mainly uses elimination rules.



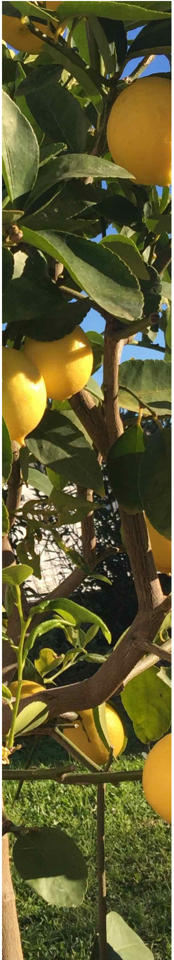
E.9. Completeness roof sketch

Truth Lemma 1. For any formula φ and sentence ψ,

$$\overline{\varphi} \Vdash \psi \iff \varphi \Vdash \psi$$

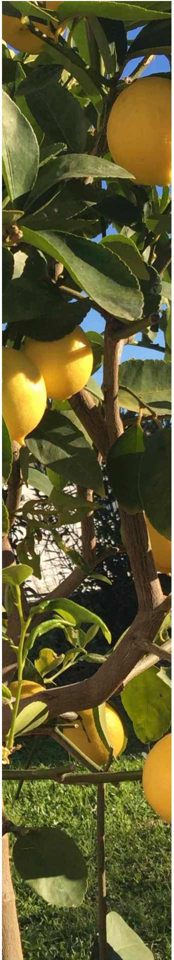
Proof By induction on ψ. The two directions of each inductive step amount to the introduction and elimination rules for the given logical constant.

Let us look at the case of the existential quantifier.



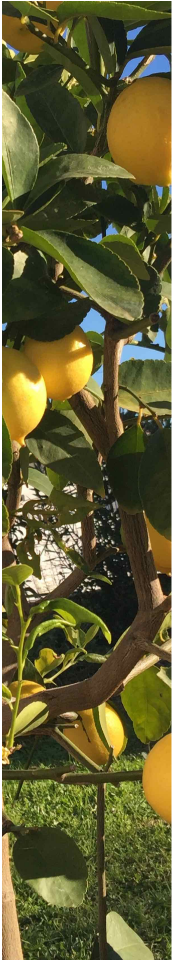
E.10. Completeness $\exists$ direction $\Rightarrow$

- Suppose $\overline{\varphi} \Vdash \exists x \psi(x)$.
- There is a family $\{\overline{\varphi_i} \mid i \in \mathscr{I}\}$ and elts $t_i^{\varphi_i} \in M_{\overline{\varphi_i}}$ such that $\overline{\varphi_i} \Vdash_{[x \mapsto t_i^{\varphi_i}]} \psi(x)$ for all $i \in \mathscr{I}$.
- Since $[t] = t^{\varphi_i}$ for closed t at $\overline{\varphi_i}$, this is $\overline{\varphi_i} \Vdash \psi(t_i)$.
- By induction hypothesis amounts to $\varphi_i \Vdash \psi(t_i)$.
- By rule $(\exists i)$, for any $i \in \mathscr{I}$ we have $\varphi_i \Vdash \exists x \psi(x)$.
- Since $\overline{\varphi} \lhd \{\overline{\varphi_i} |, i \in \mathscr{I}\}$, by the meaning of $\lhd$ we have $\varphi \Vdash \exists x \psi(x)$.



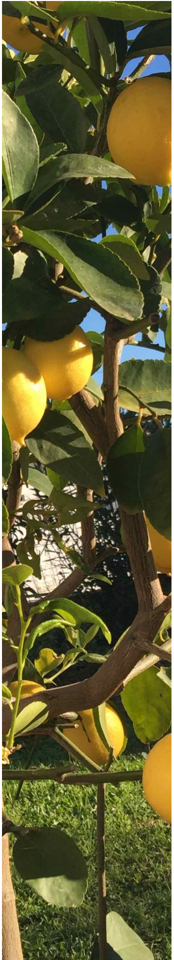
E.11. Completeness $\exists$ direction$\Leftarrow$

- Suppose $\varphi \Vdash \exists x\, \psi(x)$.
- We must provide a covering of $\overline{\varphi}$ and local witnesses.
- For any constant c, define $\varphi_c = \varphi \wedge \psi(c)$.
- Since $\varphi_c \Vdash \psi(c)$, by induction hypothesis $\overline{\varphi_c} \Vdash \psi(c)$.
- Since $[c] = c^{\varphi_c}$ at $\overline{\varphi_c}$, also $\overline{\varphi_c} \Vdash_{[x \mapsto c^{\varphi_c}]} \psi(x)$, i.e. the element c^{φ_c} is a witness for the existential at $\overline{\varphi_c}$.
- It remains to be seen that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.



E.12. Completeness $\exists$ direction$\Leftarrow$, continued

- Suppose ξ is derivable from $\varphi \wedge \psi(c)$ for any constant c.
- Let c^* be a constant that occurs neither in φ nor in ξ.
- In particular, $\varphi \wedge \psi(c^*) \Vdash \xi$, that is, $\varphi, \psi(c^*) \Vdash \xi$.
- But since c^* occurs neither in φ nor in ξ, by the rule $(\exists e)$ we have $\varphi, \exists x \psi(x) \Vdash \xi$.
- Thus by the assumption $\varphi \Vdash \exists x \psi(x)$ we also have $\varphi \Vdash \xi$.
- This shows that $\overline{\varphi} \lhd \{\overline{\varphi_c} \mid c \text{ a constant}\}$.
- Hence we conclude $\overline{\varphi} \Vdash \exists x \psi(x)$.



E.13. Variants

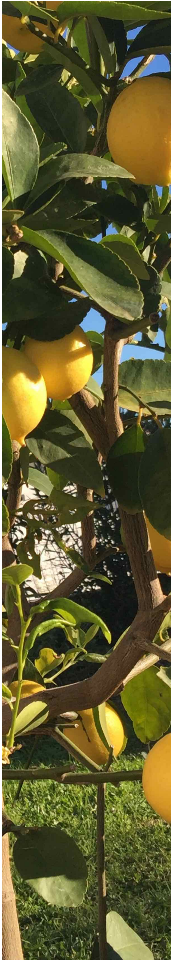
The method and the construction can be parametrised by a context Γ for obtaining what is called *strong completeness*:

The quotient on formula is not really needed.

Equality is not mandatory but pleasant.

Terms and constants can be eliminated in FOL with equality.

If there are no constants, one should add a denumerable set of constants. Indeed some proof steps need a *fresh constant*.



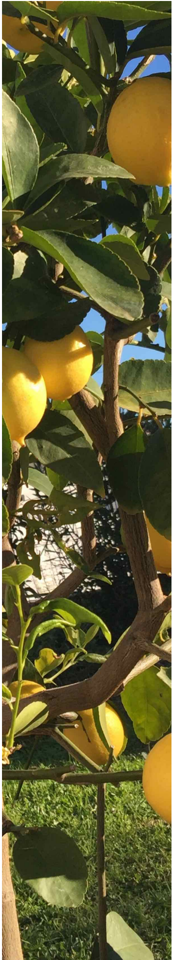
E.14. State of the art: hard to tell

Before 1995 : survey by Makkay and Reyes in 1995.

After 1995, other work in particular by Awodey.

Direct completeness via canonical presheaf: Ivano Ciardelli. A Canonical Model for Presheaf Semantics. Talk at Topology, Algebra and Categories in Logic (TACL) 2011, Jul 2011, Marseille, France. 2011.HAL Id: inria-00618862 https://hal.inria.fr/inria-00618862

Ongoing work with David Théret in Montpellier.



E.15. Future work

Can we construct a canonical sheaf and not just separated presheaf e.g. with the sheaf completion method that basically simply formally adds the missing global sections? Or by imposing some additional locality condition on terms and equality?

Is it possible to do so with a real topology (instead of a bare pretopology)?

Does it applies to first order S4?

Thank you for your attention.