For Antonio’s birthday

The sequentiality connective of pomset logic, from its denotational semantics to its proof-theoretical syntax

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A non commutative extension of classical linear logic

with a sequentiality operator

$\triangleleft$ before, sequential, ...
précède ($\rightarrow$ formal grammars)

Coherence Spaces

Definition 15. *A coherence space A is a set $|A|$ (possibly infinite) called the* web *of A whose elements are called* tokens, *endowed with a binary reflexive and symmetric relation called* coherence *on $|A| \times |A|$ noted $\alpha \frown\!\!\!\!\smile \alpha'[A]$ or simply $\alpha \frown\!\!\!\!\smile \alpha'$ when A is clear.*

The following notations are common and useful:

$\alpha \frown \alpha'[A]$ iff $\alpha \frown\!\!\!\!\smile \alpha'[A]$ and $\alpha \neq \alpha'$ strictly coherent

$\alpha \asymp \alpha'[A]$ iff $\alpha \not\frown\!\!\!\!\smile \alpha'[A]$ or $\alpha = \alpha'$

$\alpha \smile \alpha'[A]$ iff $\alpha \not\frown\!\!\!\!\smile \alpha'[A]$ and $\alpha \neq \alpha'$

objects: cliques

set of pairwise coherent tokens

Linear morphisms

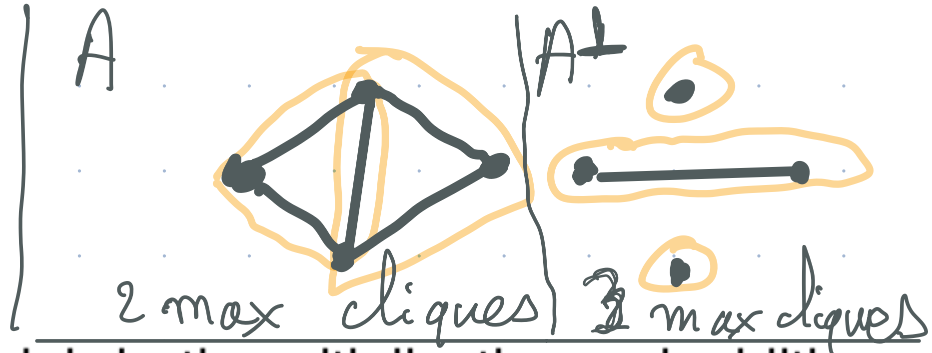
Definition 16. *A linear morphism F from A to B is a morphism mapping cliques of A to cliques of B such that:*

- *For all $x \in A$ if $(x' \subset x)$ then $F(x') \subset F(x)$*

- *For every family $(x_i)_{i \in I}$ of pairwise compatible cliques — that is to say $(x_i \cup x_j) \in A$ holds for all $i, j \in I$ — $F(\cup_{i \in I} x_i) = \cup_{i \in I} F(x_i)$.*[7]

- *For all $x, x' \in A$ if $(x \cup x') \in A$ then $F(x \cap x') = F(x) \cap F(x')$ – thsi last condition is called* stability.

Negation



Negation is a unary connective which is both multiplicative and additive:

$$|A^{\perp}| = |A| \text{ and } \alpha \smile' \alpha[A^{\perp}] \text{ iff } \alpha \asymp \alpha'[A]$$

Multiplicatives $|A * B| = |A| \times |B|$

The covariant connectives:

$A * B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	NE?
$=$	$\smile$	$=$	$\frown$
$\frown$	SW?	$\frown$	$\frown$

otherwise use $(\ )^{\perp}$

Covariant binary connectives

If one wants $*$ to be commutative, there are only two possibilities, namely $NE = SW = \frown$ ($\parr$) and $NE = SW = \smile$ ($\otimes$).

$A \parr B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

and

$A \otimes B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

However if we do not ask for the connective $*$ to be commutative we have a third connective $A \triangleleft B$ and a fourth connective $A \triangleright B$ which is simply $B \triangleleft A$.

$A \triangleleft B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

and

$A \triangleright B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

Properties

$\boxed{\text{Self dual}}$ $(A \lhd B)^{\perp} \equiv A^{\perp} \lhd B^{\perp}$

NO SWAP

Associative

$$(A \lhd B) \lhd C \equiv A \lhd (B \lhd C)$$

Non commutative

$$(A \lhd B) \not\equiv (B \lhd A)$$

In between $\otimes$ and $⅋$

$$\begin{tikzcd}
 & A \lhd B \arrow[dr] & \\
A \otimes B \arrow[ur] \arrow[dr] & & A ⅋ B \\
 & B \lhd A \arrow[ur] &
\end{tikzcd}$$

Coherence wrt a partial order

$(\alpha_1, \ldots, \alpha_n) \frown (\alpha'_1, \ldots, \alpha'_n)[T[A_1, \ldots, A_n]$ are strictly coherent whenever: there exist i such that $\underbrace{\alpha_i \frown \alpha'_i}_{A_i}$ and for every $j > i$ one has $\underbrace{\alpha_j = \alpha'_j}_{A_j}$.

When the order is Series Parallel

$T[A_1, \ldots A_n]$ can be written with ⅋ (par) ◁ (series)

The design of
an appropriate syntax

Semantical guidelines

- Interpretable ;)
- Extends MLL

- Implements

$$\begin{tikzcd}
 & B < A \arrow[dr] & \\
A \otimes B \arrow[ur] \arrow[dr] & & A ⅋ B \\
 & A < B \arrow[ur] &
\end{tikzcd}$$

- Cut elimination

IF $\Pi \xrightarrow{\beta} \Pi'$ **THEN** $[\![\Pi]\!] = [\![\Pi']\!]$

A simple sequent calculus

Sequents : Series Parallel partial orders of formulas $\otimes \lhd \parr$

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma \mathbin{\widehat{\lhd}} \Delta}\ \mathit{dimix} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma'}\ \text{entropy}(\Gamma' \text{ sub sp order of } \Gamma)$$

$$\frac{}{\vdash a, a^{\perp}}$$

$$\frac{\vdash A \mathbin{\widehat{\parr}} \Gamma \qquad \vdash B \mathbin{\widehat{\parr}} \Delta}{\vdash \Gamma \mathbin{\widehat{\parr}} (A \otimes B) \mathbin{\widehat{\parr}} \Delta}\ \otimes \text{ / cut when } A = B^{\perp}$$

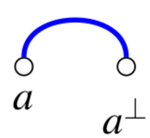
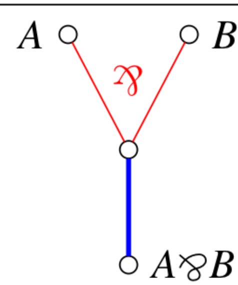
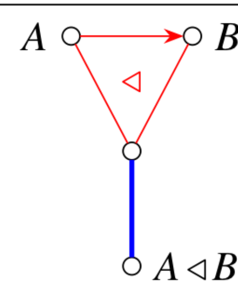
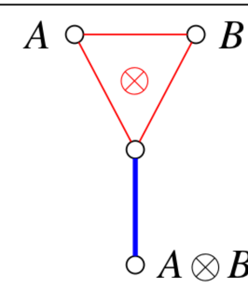
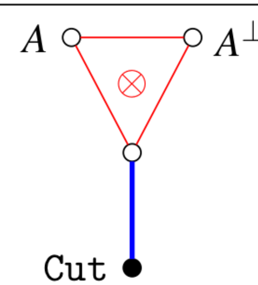
$$\frac{\vdash \Gamma[A \mathbin{\widehat{\parr}} B]}{\vdash \Gamma[A \parr B]}\ \parr \text{ when } A \overset{..}{\sim} B \qquad\qquad \frac{\vdash \Gamma[A \mathbin{\widehat{\lhd}} B]}{\vdash \Gamma[A \lhd B]}\ \lhd \text{ when } A \overset{\rightarrow}{\sim} B$$

Figure 7: Sequent calculus on SP pomset or formulas; called SP-pomset sequent calculus

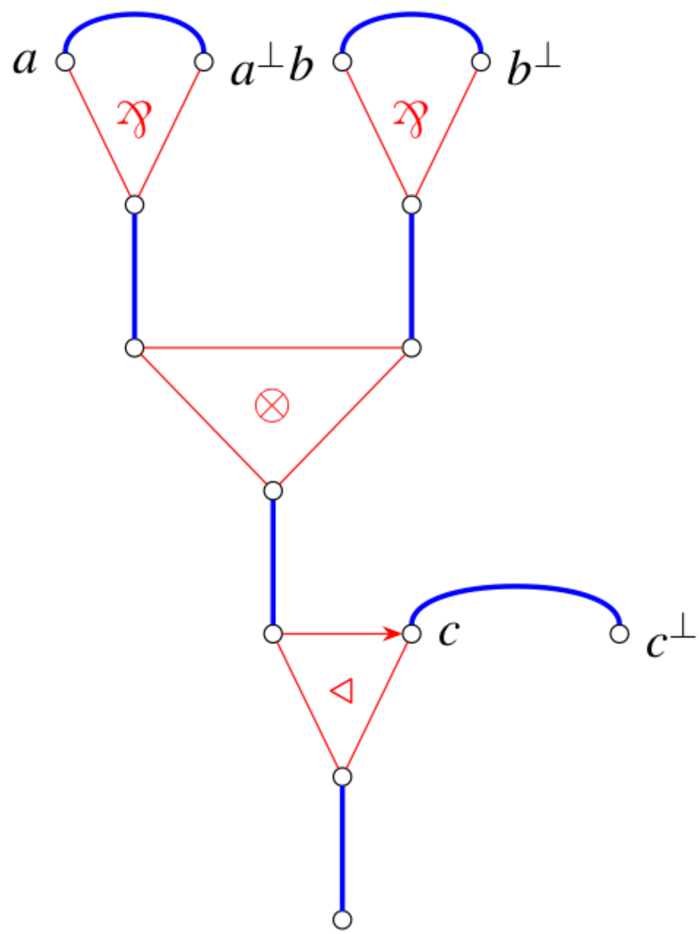
An example of a proof

$$
\dfrac{
  \dfrac{
    \dfrac{\vdash \{a, a^{\perp}\}}{\vdash a \parr a^{\perp}}
    \qquad
    \dfrac{\vdash \{b, b^{\perp}\}}{\vdash b \parr b^{\perp}}
  }{
    \vdash (a \parr a^{\perp}) \otimes (b \parr b^{\perp})
  }
  \qquad
  \vdash c, c^{\perp}
}{
  \dfrac{\vdash \langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); \{c, c^{\perp}\} \rangle}{\vdash \{\langle (a \parr a^{\perp}) \otimes (b \parr b^{\perp}); c \rangle, c^{\perp}\}}\ \textit{entropy}
}\ \textit{dimix}
$$

Proof net (with links)

	Axiom	Par $\parr$	Before $\triangleleft$	Times $\otimes$	Cut
Premisses	None	A and B	A and B	A and B	K and $K^{\perp}$
RnB link					
Conclusion(s)	a and $a^{\perp}$	$A \parr B$	$A \triangleleft B$	$A \otimes B$	None

The previous sequent calculus proof



Correctness criterion

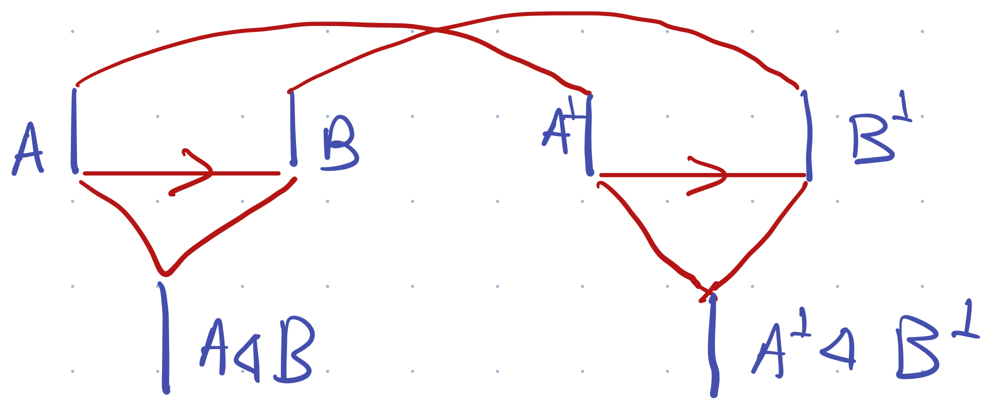
No Alternate Elementary circuit (= directed cycle)

because of the shape of the links

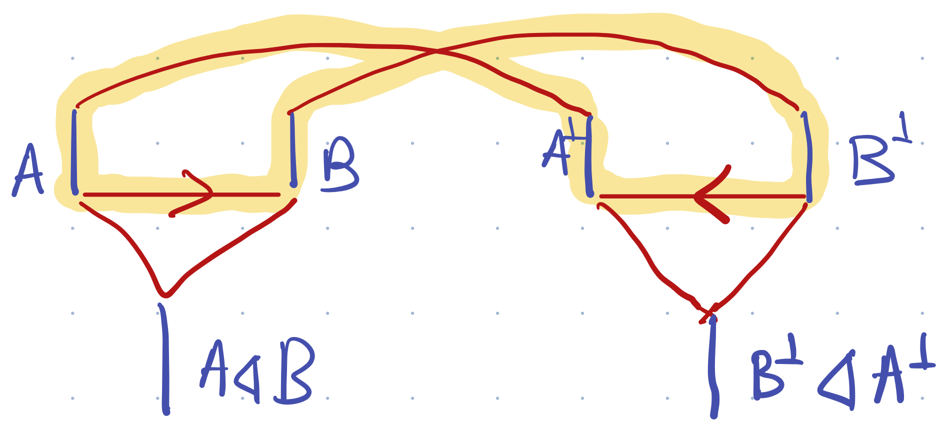
$\Updownarrow$

Every Alternate Elementary Cycle contains a chord

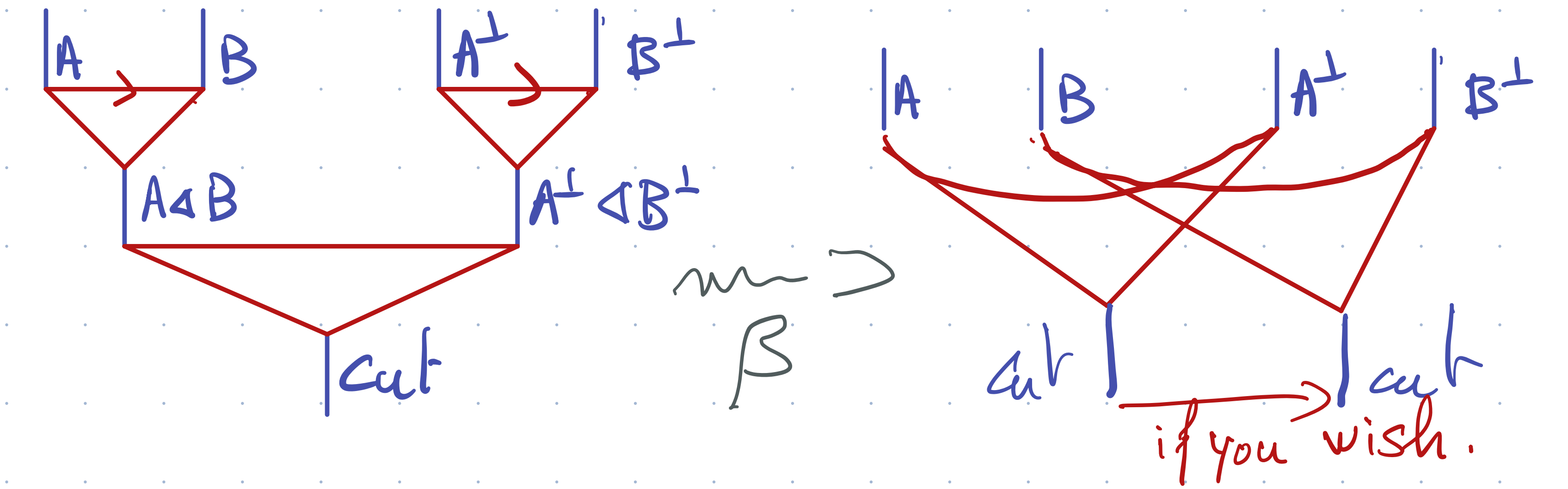
CORRECT:

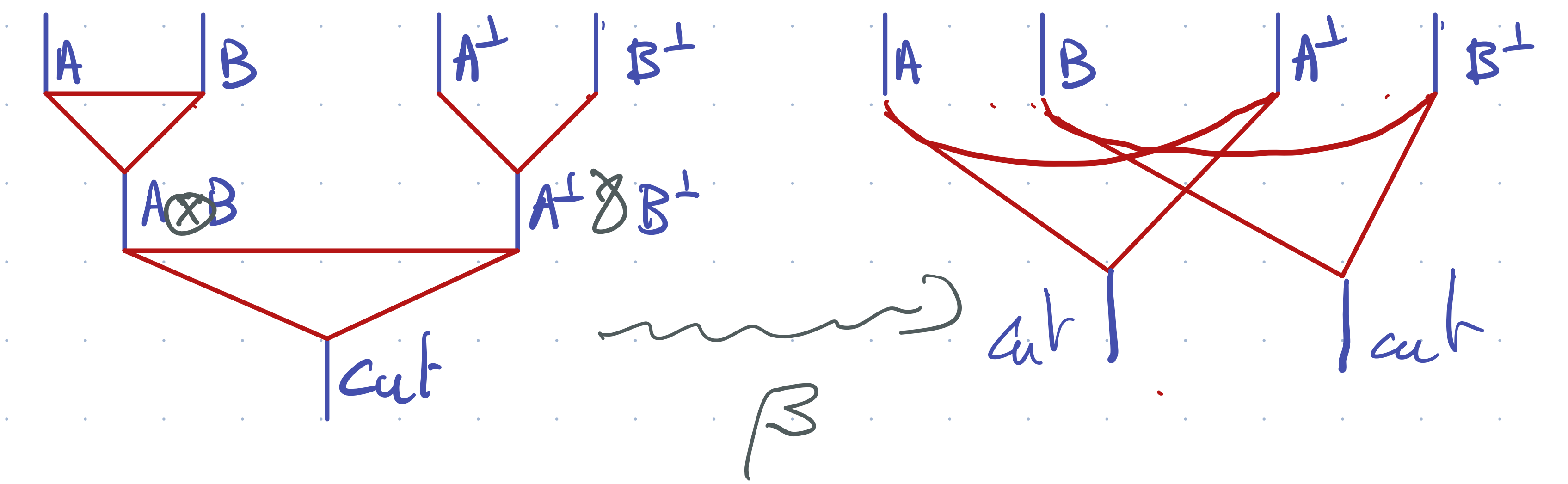


NOT CORRECT

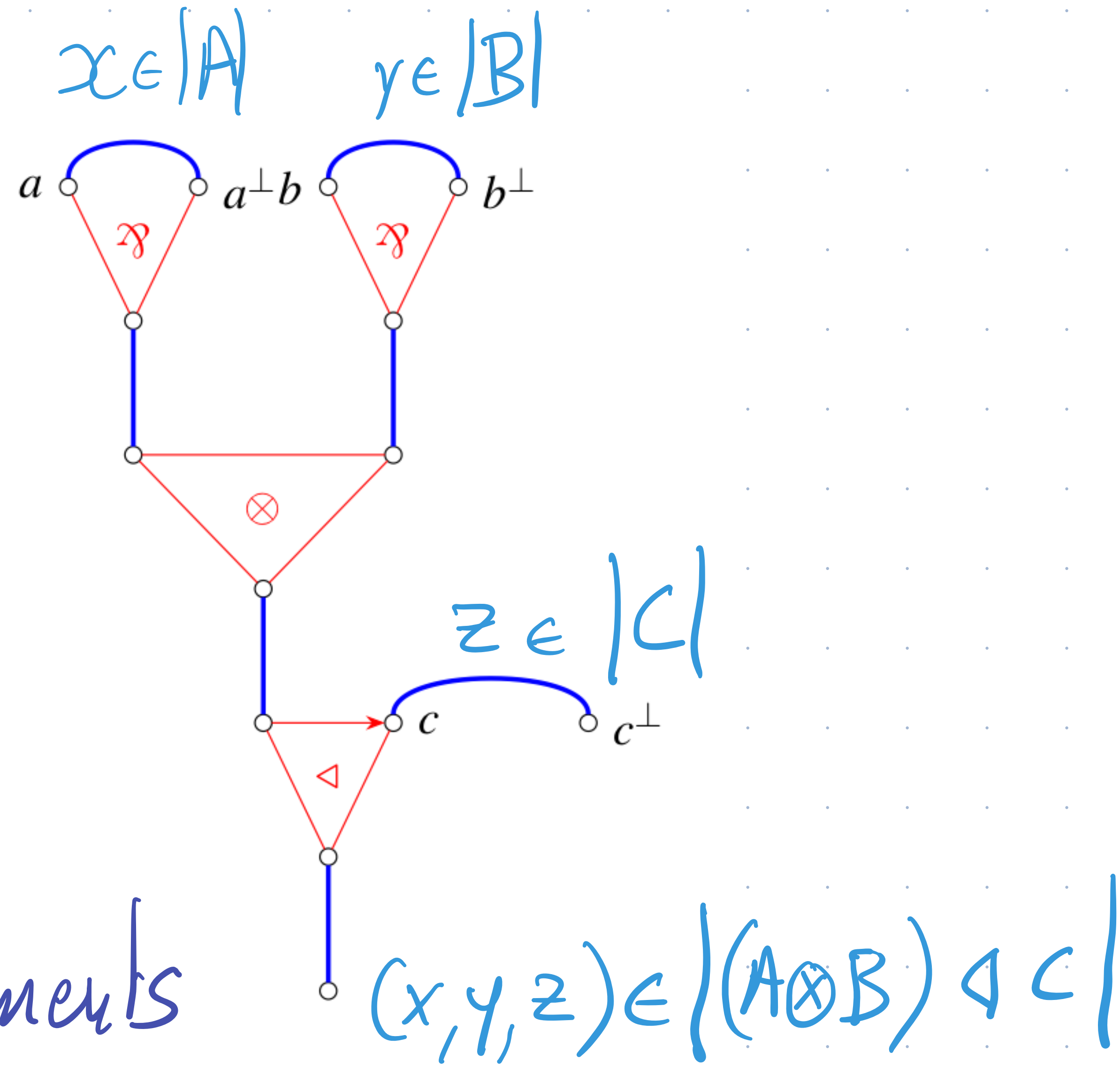


Cut - elimination


A
B
A⊥
B⊥
A◁B
A⊥◁B⊥
cut
β
A
B
A⊥
B⊥
cut
cut
if you wish.


A
B
A⊥
B⊥
A⊗B
A⊥⅋B⊥
cut
β
A
B
A⊥
B⊥
cut
cut

Semantics : experiments



$[\![\pi]\!]$ the results of all experiments

SEMANTICS

whenever π correct
conclusion C
results of experiments
clique of the
corresponding coherence space C

It's better than that:

syntactic criterion (no AE cycle)

$$\Updownarrow$$

semantic criterion

$[\![\pi]\!]$ is a clique of C

with cuts:

Π = results of all SUCCEEDING experiments

succeedings:

$(A \lhd (B \otimes C)) \otimes D = K \qquad K^\perp = (A^\perp \lhd (B^\perp \otimes C^\perp)) \otimes D^\perp$

$((x,(y,z)),u) \qquad ((x',(y',z')),u')$

same tuples on K and K'

$x = x' \quad y = y' \quad z = z' \quad u = u'$

DENOTATIONAL semantics

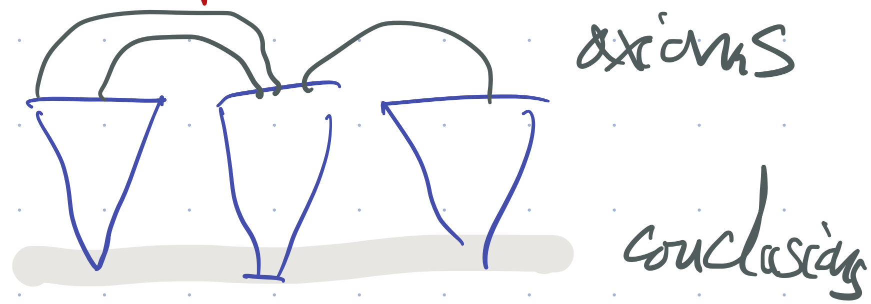
whenever $\pi \xrightarrow{\beta} \pi'$

$$[\![\pi]\!] = [\![\pi']\!]$$

[the results of the succeeding experiments are preserved under cut elimination

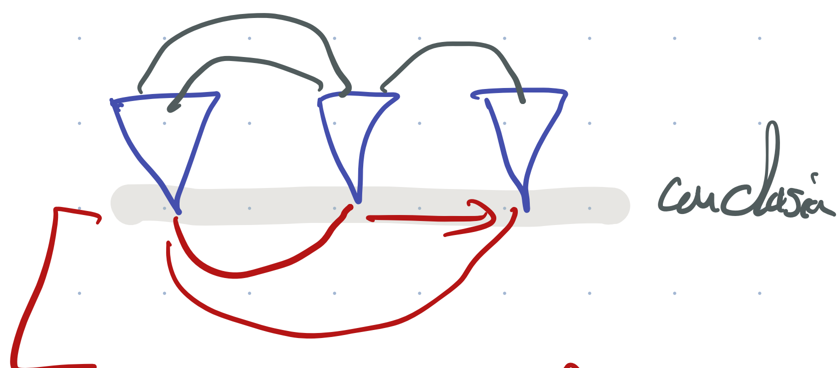
Folding / unfolding
Generalised proof net

Proof net :



Generalised proof net

direct cograph



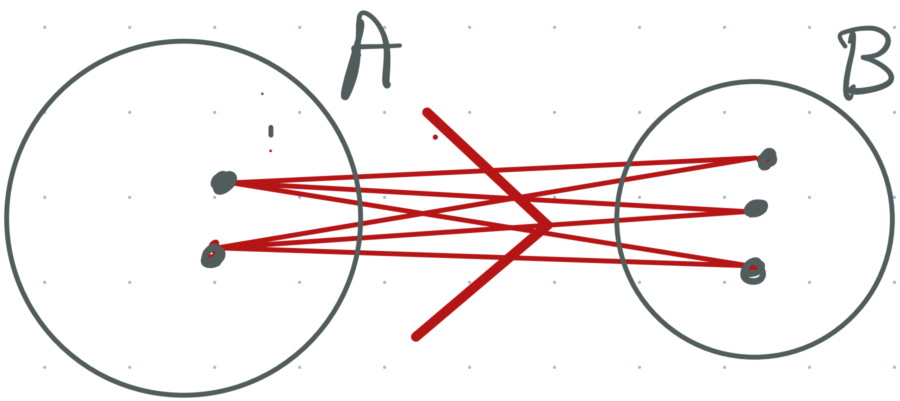
$c_1 \otimes (c_2 \lhd c_3)$

Directed cographs :

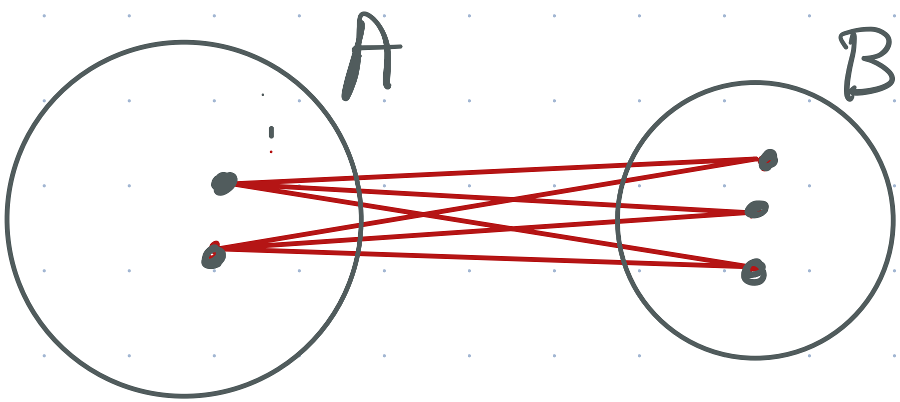
inductive class of directed graphs

$\bullet\ a$

$A ⅋ B$



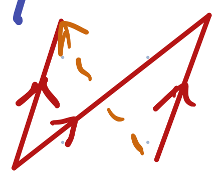
$A \lhd B$



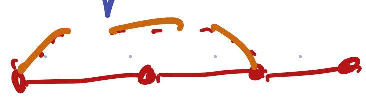
$A \otimes B$

Characterisation

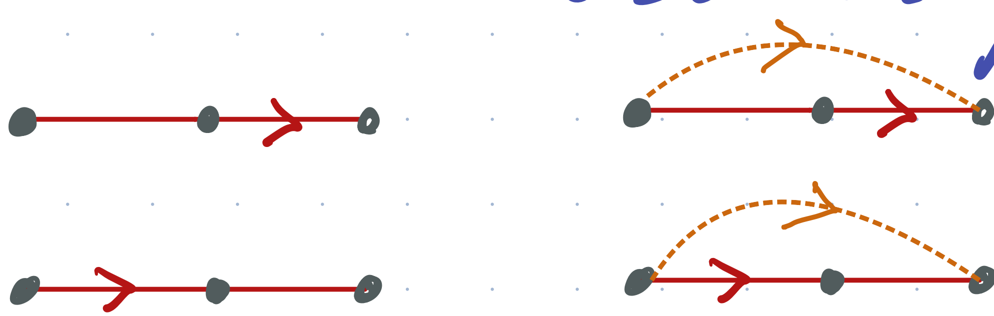
Directed part: SP order (N free)



Symmetric part: cograph (P4 free)



+ Weak transitivity:



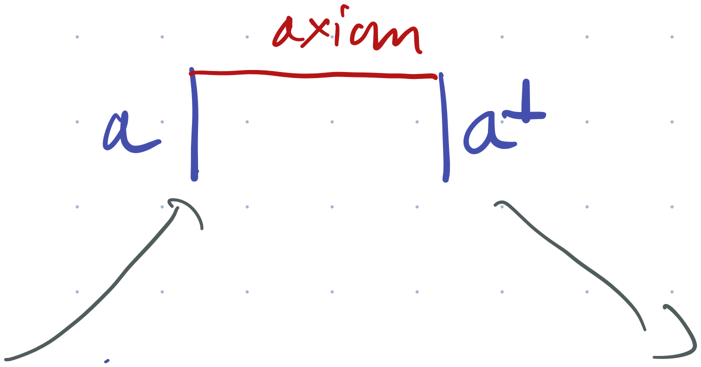
From Proof Nets with links

To

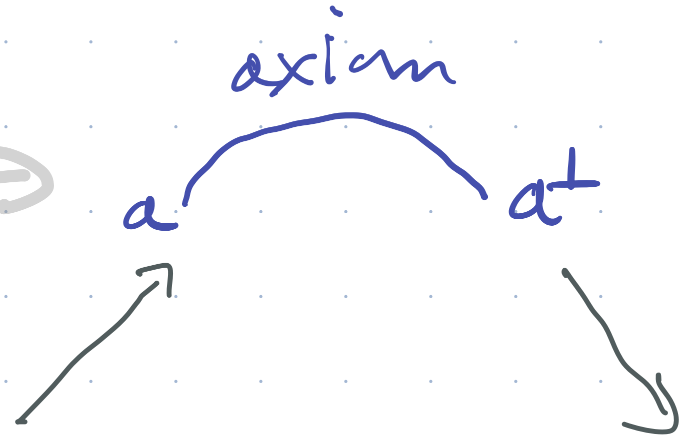
Handsome proof nets

Fold / Unfold preserve the criterion:

every AE circuit contains a chord

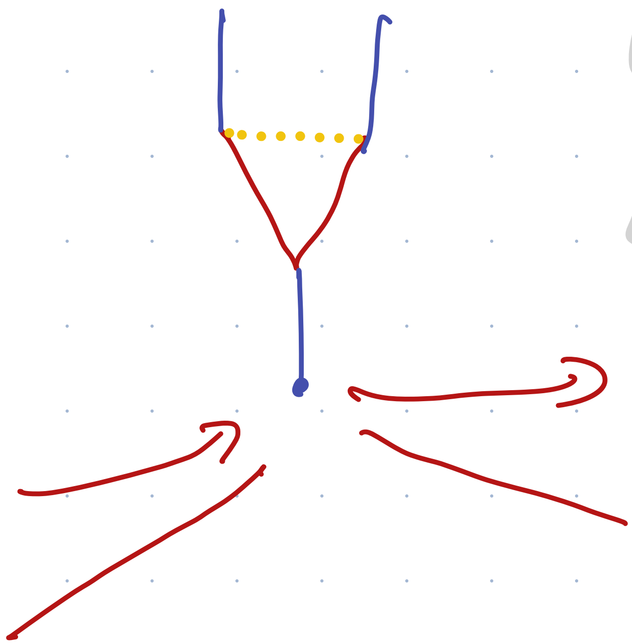


unfold

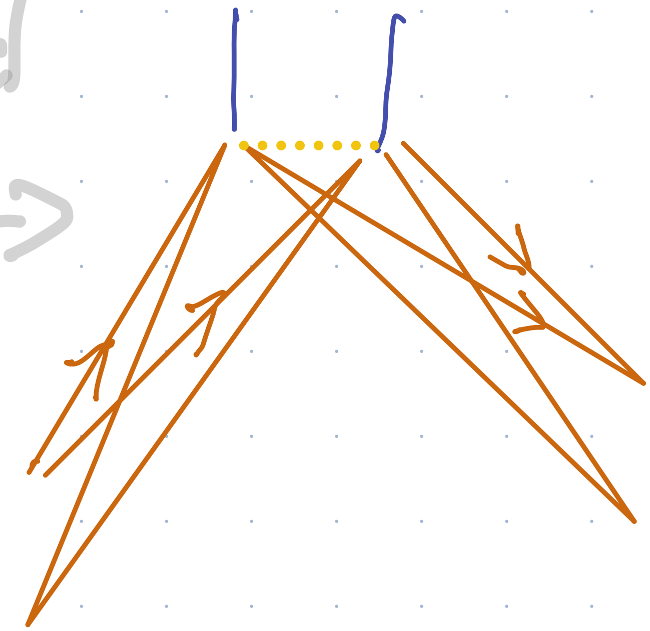


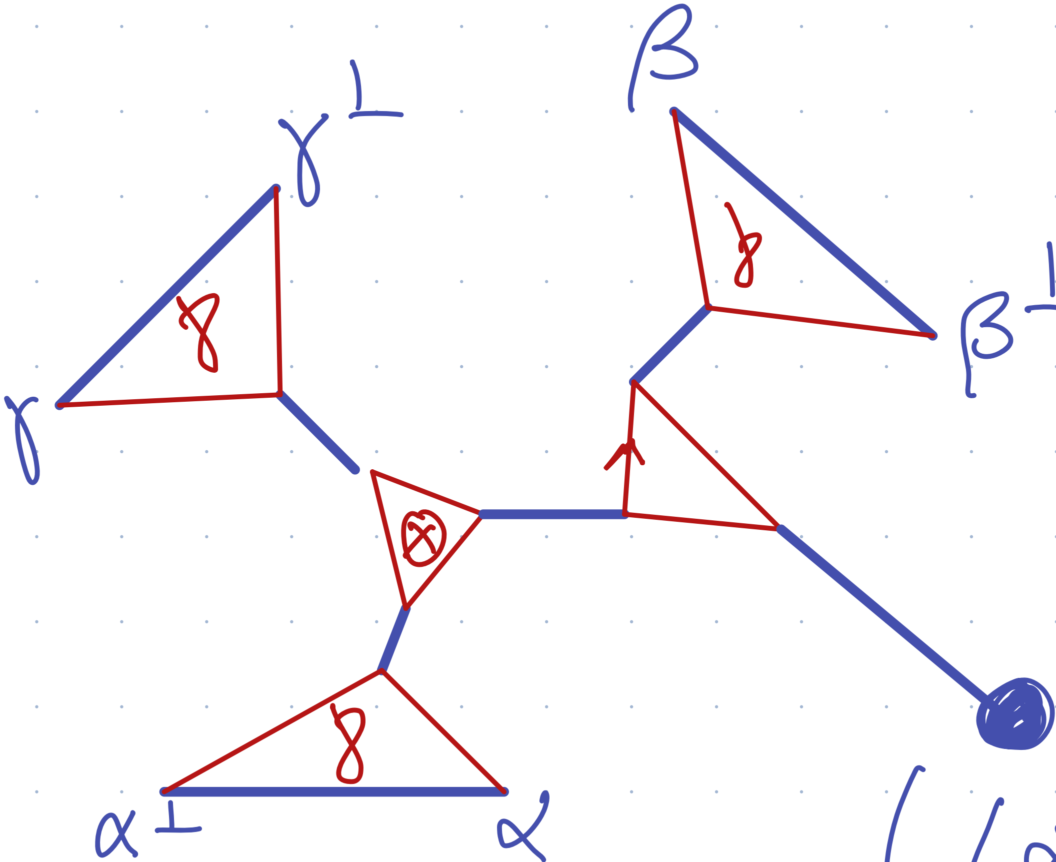
same AE paths

$\cdots\cdots$ is $\longrightarrow$ — (nothing)



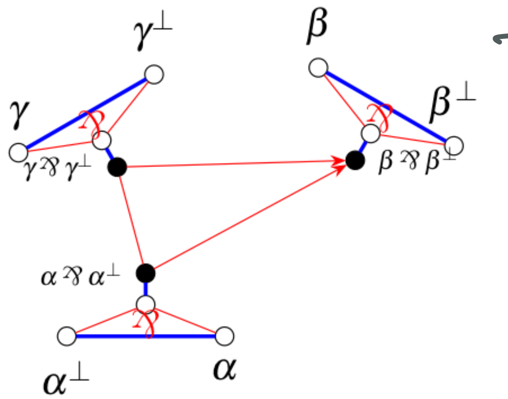
unfold



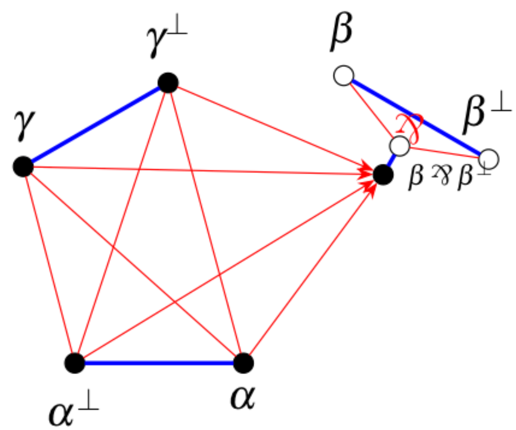


1 conclusion

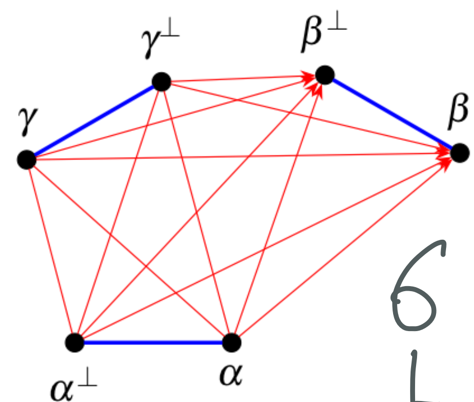
$$((\alpha^\perp \parr \alpha) \otimes (\gamma \parr \gamma^\perp)) \triangleleft (\beta \parr \beta^\perp)$$



3 conclusions



5 conclusions

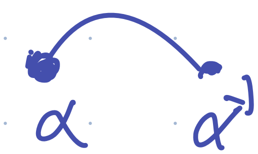


6 conclusions
= atoms

Handsome proof nets.

No link(s)

Atoms: $\alpha, \alpha^{\perp}, \beta, \beta^{\perp}, \ldots$ vertices

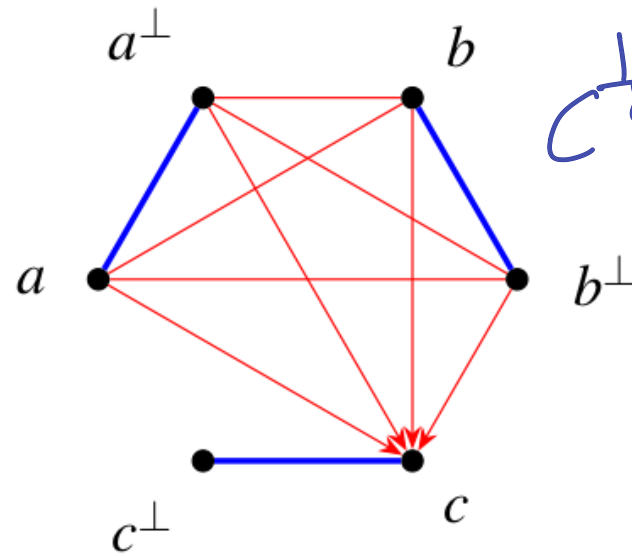
Axioms: blue edges 

Formula = directed cograph

Correct: every alternate elementary circuit contains a chord

(a) Two incorrect handsome proof structures (chordless æ-circuit: $a, c^\perp, c, a^\perp, a$ in both cases)

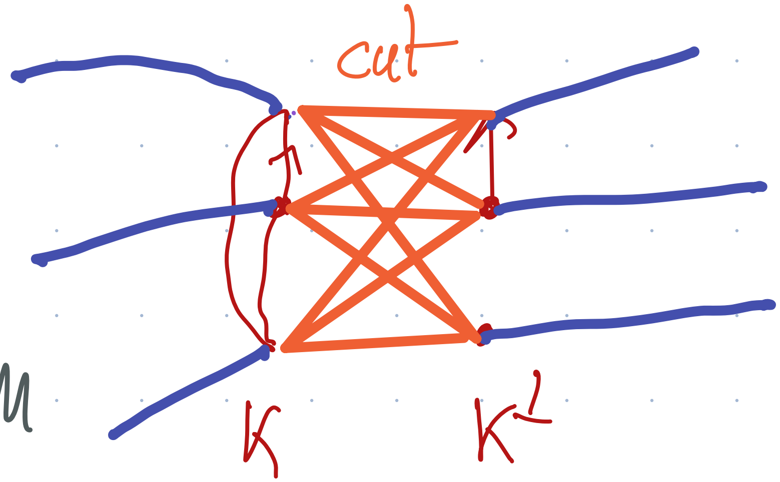
(b) Two correct handsome proof structures (i.e. two correct hansome proof nets)

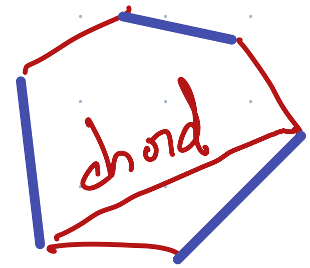


$$c^\perp \parr \big((a \parr a^\perp) \otimes (b \parr b^\perp) \lhd c\big)$$

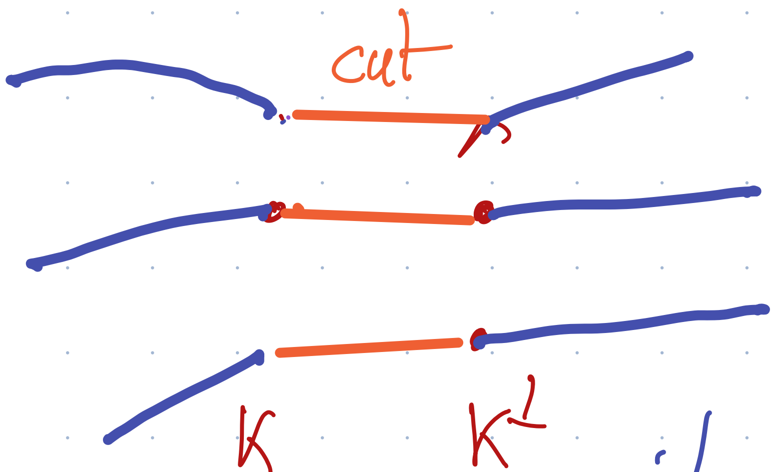
Fine for – semantics
– cut elimination
(Girard's turbo cut elimination)

Turbo cut elimination





preserves correctness:



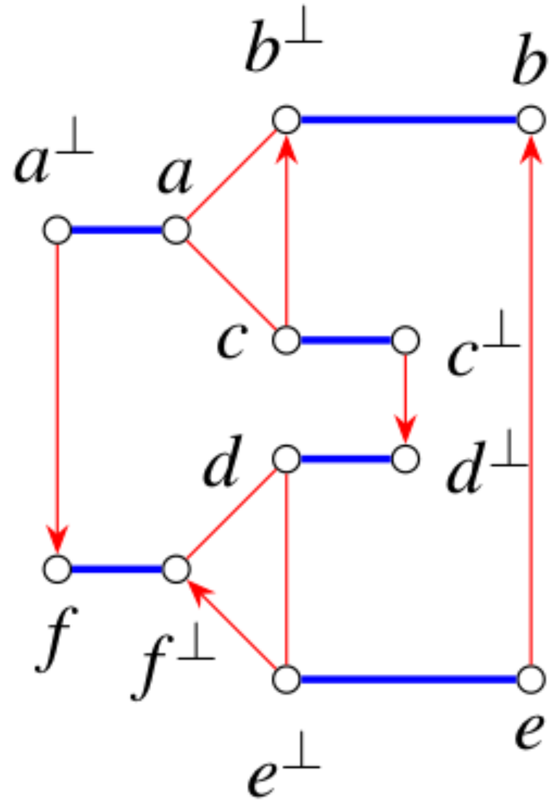
every AE cycle contains a chord

(I defined the rewritings to prove this)

Proof net calculus: perfect
(matches semantics)

Sequent calculus ?
- simple ones correct
not complete
- Slavnov 2019 ?
very complicated, correct
complete ?

A correct pomset proofnet without any corresponding sequent calculus proof. (Straßburger)



Provable in BV

REWRiTiNG FoR BU

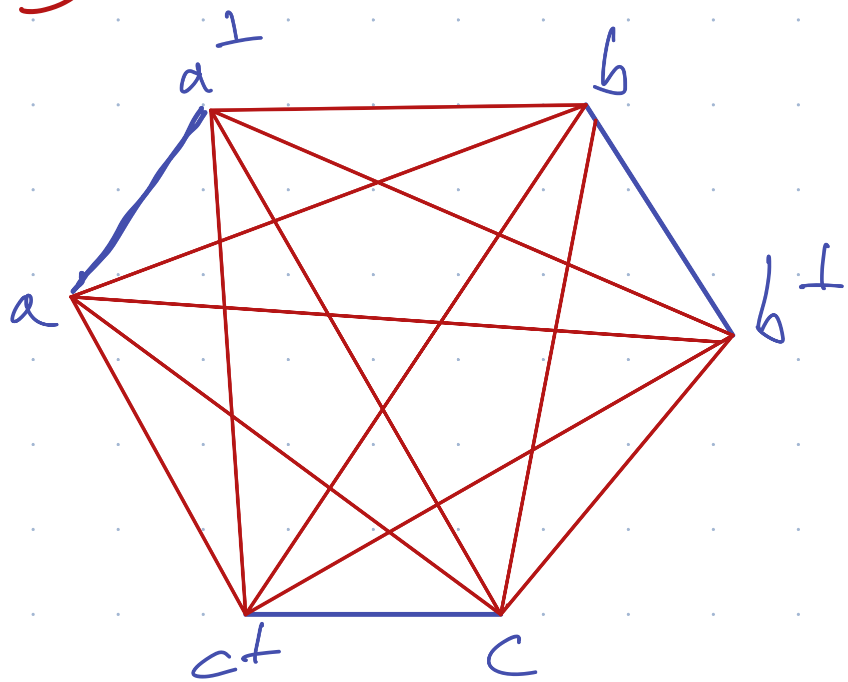
$(X \circ Y) \,\Box\, (U \circ V)$

$\rightsquigarrow$

$(X \,\Box\, U) \circ (Y \,\Box\, V)$

	rule name	*dicograph*	$\rightsquigarrow$	*dicograph'*
☠	$\widehat{\otimes}\widehat{\parr}4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\otimes}} V)$
	$\otimes\parr3$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} Y$
	$\otimes\parr2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$
	$\otimes\lhd4$	$(X \mathbin{\widehat{\lhd}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\lhd}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\lhd}} (Y \mathbin{\widehat{\otimes}} V)$
	$\otimes\lhd3l$	$(X \mathbin{\widehat{\lhd}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\lhd}} Y$
	$\otimes\lhd3r$	$Y \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\lhd}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\lhd}} (Y \mathbin{\widehat{\otimes}} V)$
	$\otimes\lhd2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\lhd}} Y$
	$\lhd\parr4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\lhd}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\lhd}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\lhd}} V)$
	$\lhd\parr3l$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\lhd}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\lhd}} U) \mathbin{\widehat{\parr}} Y$
	$\lhd\parr3r$	$Y \mathbin{\widehat{\lhd}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\lhd}} V)$
	$\lhd\parr2$	$Y \mathbin{\widehat{\lhd}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$

Axioms



$$(a \parr a^{\perp}) \otimes (b \parr b^{\perp}) \otimes (c \parr c^{\perp})$$

Guglielmi Straßburger

(S)BV

axioms

+ all correct rewritings

Pomset Handsome Proof Nets
+ rewriting

|||

(S)BV

$\longrightarrow$ proof of $a\uparrow$ admissibility
(kind of cut-elimination)

Example (s)BV derivation

$$\begin{array}{c} 1 \\ \big\downarrow \\ (\alpha \parr \alpha^{\perp}) \end{array} \qquad \begin{array}{c} (\alpha \otimes \alpha^{\perp}) \\ \big\downarrow \text{cut} \\ 1 \end{array}$$

$$\begin{aligned}
\text{Axiom} &\rightsquigarrow \mathbf{1} \\
a\!\downarrow &\rightsquigarrow (e^{\perp} \parr e) \\
\mathbf{1}a\!\downarrow &\rightsquigarrow (e^{\perp} \parr e) \otimes (b^{\perp} \parr b) \\
\otimes\!\lhd 2 &\rightsquigarrow (e^{\perp} \parr e) \lhd (b^{\perp} \parr b) \\
\lhd\!\parr 4 &\rightsquigarrow (e^{\perp} \lhd b^{\perp}) \parr (e \lhd b) \\
(\mathbf{1}a\!\downarrow) \times 2 &\rightsquigarrow ((c \parr c^{\perp}) \otimes (e^{\perp} \parr b^{\perp}) \otimes (f \parr f^{\perp})) \parr (e \lhd b) \\
\otimes\!\lhd 2x2 &\rightsquigarrow ((c \parr c^{\perp}) \lhd (e^{\perp} \parr b^{\perp}) \lhd (f \parr f^{\perp})) \parr (e \lhd b) \\
\lhd\!\parr 4x2 &\rightsquigarrow (c \lhd b^{\perp} \lhd f) \parr (c^{\perp} \lhd e^{\perp} \lhd f^{\perp}) \parr (e \lhd b) \\
\mathbf{1}a\!\downarrow &\rightsquigarrow (((c \lhd b^{\perp}) \otimes (a \parr a^{\perp})) \lhd f) \parr (c^{\perp} \lhd e^{\perp} \lhd f^{\perp}) \parr (e \lhd b) \\
\otimes\!\parr 3 &\rightsquigarrow (((c \lhd b^{\perp}) \otimes a) \parr a^{\perp}) \lhd f) \parr (c^{\perp} \lhd e^{\perp} \lhd f^{\perp}) \parr (e \lhd b) \\
\lhd\!\parr 3 &\rightsquigarrow ((c \lhd b^{\perp}) \otimes a) \parr (a^{\perp} \lhd f) \parr (c^{\perp} \lhd e^{\perp} \lhd f^{\perp}) \parr (e \lhd b) \\
\mathbf{1}a\!\downarrow &\rightsquigarrow ((c \lhd b^{\perp}) \otimes a) \parr (a^{\perp} \lhd f) \parr (c^{\perp} \lhd ((e^{\perp} \lhd f^{\perp}) \otimes (d \parr d^{\perp})) \parr (e \lhd b) \\
\otimes\!\parr 3 &\rightsquigarrow ((c \lhd b^{\perp}) \otimes a) \parr (a^{\perp} \lhd f) \parr \{c^{\perp} \lhd [((e^{\perp} \lhd f^{\perp}) \otimes d) \parr d^{\perp}]\} \parr (e \lhd b) \\
\lhd\!\parr 3 &\rightsquigarrow ((c \lhd b^{\perp}) \otimes a) \parr (a^{\perp} \lhd f) \parr (((e^{\perp} \lhd f^{\perp}) \otimes d) \parr (c \lhd d^{\perp}) \parr (e \lhd b)
\end{aligned}$$

Working with graphs (not terms)

much easier proof of

$1\uparrow$ removing

(kind of cut elimination)

$$\frac{a \otimes a^{\perp}}{1} \qquad 1\uparrow$$

What is known;

standard sequent calculi $\nvDash$ (S)BV proofs $\nvDash$ pomset proofnets

---

skovmov sequent calculus ?

- good idea $(A \triangleleft B)^{\perp} = A^{\perp} \blacktriangleleft B^{\perp}$

  pomset: $\triangleleft = \blacktriangleleft$

- (for the rest too complicated: + conclusions
  R_1 R_t relation on tuples of conclusions

Nguyen Straßburger 2021

Some pomset proofnets have no proof in (S)BV

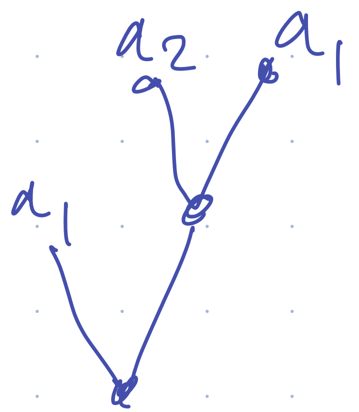
A self dual modality $\natural$

$$\natural A \overset{\text{iso}}{\multimapboth} \natural A < \natural A$$

$$A \;\substack{\multimapinv \\ \multimap}\; \natural A$$

retract of A

continuous functions $2^{\omega} \to |A|$



$f \frown g$

$$\exists w \quad f(w) \frown g(w)$$

$$\forall w' > w \quad f(w) = f(w')$$

No syntax so far.
Some ideas by Guglielmi

Happy birthday Antonio!
(I leave the only anecdote(s)
I know for tonight)