POMSET LOGIC

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Bath Seminar on
Mathematical Foundations of Computation
2022 June 21

Coherence spaces / linear maps

categorical model + interpretation

Proof d $A \vdash B$: $\overset{\text{linear}}{\text{map}}$ $[\![d]\!] : [\![A]\!] \multimap [\![B]\!]$

$d \overset{\text{cut-elim}}{\rightsquigarrow} d'$ $\qquad [\![d]\!] = [\![d']\!]$

no A $\qquad\qquad \vdash B$

$1 \xrightarrow{\quad} B$

f = object of B

$$\begin{tikzcd}
{[\![A]\!]} \arrow[r, "d"] \arrow[r, bend right=50, "d'"'] \arrow[r, phantom, "\|"] & {[\![B]\!]}
\end{tikzcd}$$

Coherence spaces

notation / connective / tables

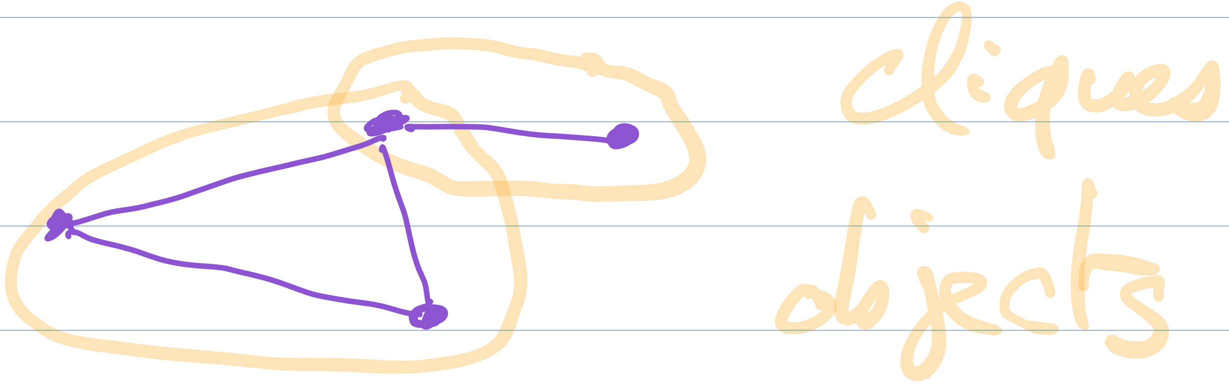
Definition 6 A coherence space A is a set $|A|$ (possibly infinite) called the *web* of A whose elements are called *tokens*, endowed with a binary reflexive and symmetric relation called *coherence* on $|A| \times |A|$ noted $\alpha \supset \alpha'[A]$ or simply $\alpha \supset \alpha'$ when A is clear.

The following notations are common and useful:

negation:

$\alpha \frown \alpha'[A]$ iff $\alpha \smile \alpha'[A]$

$\alpha \frown \alpha'[A]$ iff $\alpha \supset \alpha'[A]$ and $\alpha \neq \alpha'$
$\alpha \asymp \alpha'[A]$ iff $\alpha \not\supset \alpha'[A]$ or $\alpha = \alpha'$
$\alpha \smile \alpha'[A]$ iff $\alpha \not\supset \alpha'[A]$ and $\alpha \neq \alpha'$



$A^{\perp\perp} = A$

Meaning full object, interpretation of proofs:

CLIQUES

Definition 7 A linear morphism F from A to B is a morphism mapping cliques of A to cliques of B such that:

- For all $x \in A$ if $(x' \subset x)$ then $F(x') \subset F(x)$
- For every family $(x_i)_{i \in I}$ of pairwise compatible cliques — that is to say $(x_i \cup x_j) \in A$ holds for all $i, j \in I$ — $F(\cup_{i \in I} x_i) = \cup_{i \in I} F(x_i)$.[7]
- For all $x, x' \in A$ if $(x \cup x') \in A$ then $F(x \cap x') = F(x) \cap F(x')$ – thsi last condition is called *stability*.

$$\mathrm{LinHom}(A,B) \simeq [\![A \multimap B]\!] = [\![A^{\perp} \parr B]\!]$$

cf afterwards

$$B^{A} = [A \to B]$$

multiplicative

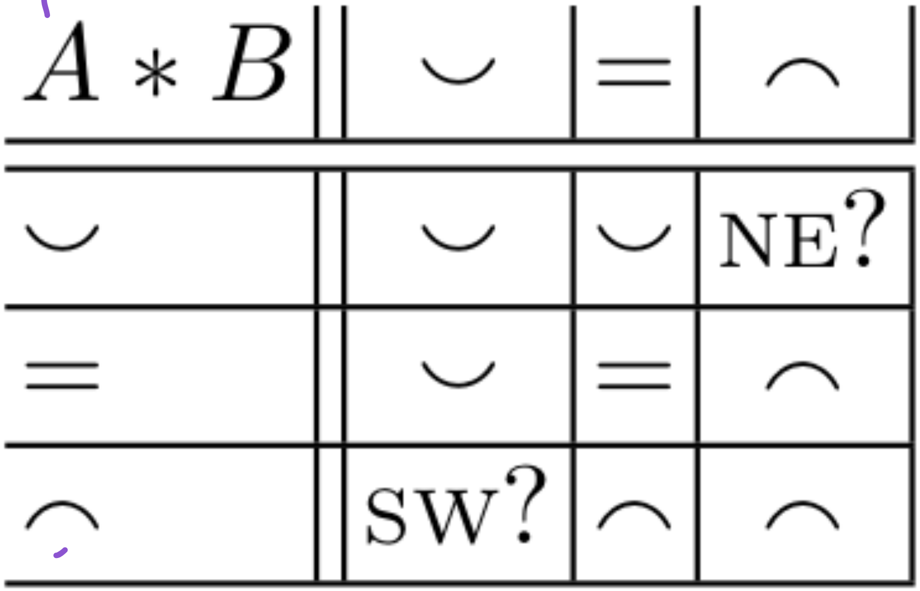
covariant binary connectives

$|A * B| = |A| \times |B|$

$(\alpha,\beta) \overset{?}{\asymp} (\alpha',\beta')$

$\alpha, \alpha' \in |A|$

$\beta, \beta' \in |B|$



$A * B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	NE?
$=$	$\smile$	$=$	$\frown$
$\frown$	SW?	$\frown$	$\frown$

$\neg A \vee B$ $\qquad X \vee Y$

Coherence spaces

notation / connective / tables

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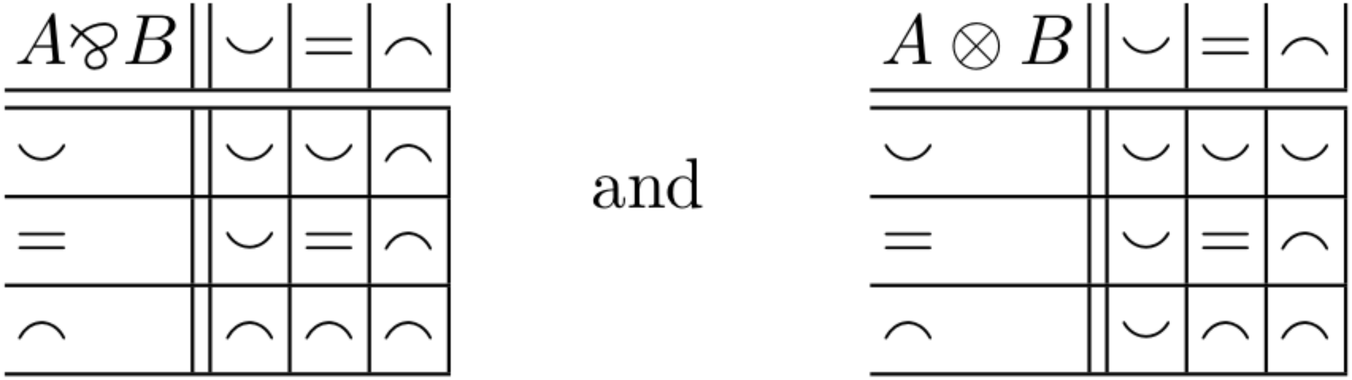
covariant binary connectives

$A * B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	NE?
$=$	$\smile$	$=$	$\frown$
$\frown$	SW?	$\frown$	$\frown$

Commutative :



$A \parr B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

and

$A \otimes B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

non commutative

$A \lhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

and

$A \rhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

Properties (in coherence spaces)

self dual

$$(A < B)^{\perp} \cong A^{\perp} < B^{\perp}$$ **NO SWAP!**

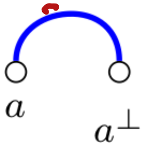
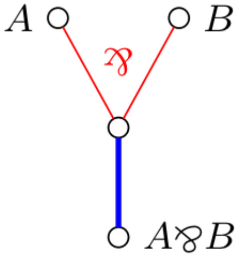
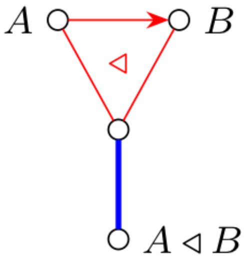
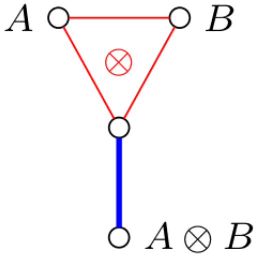
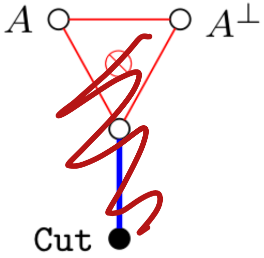
associative

$$A < (B < C) \cong (A < B) < C$$

Proof nets with links

as Red n' Blue graphs

Regular n' Bold

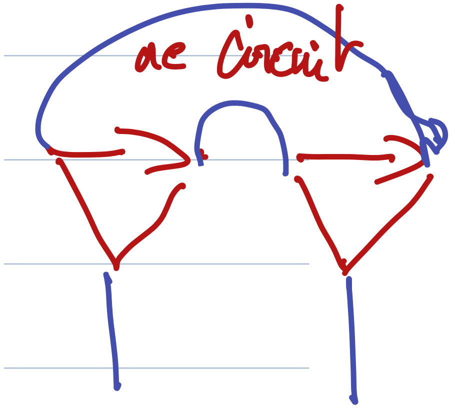
	Axiom	Par ⅋	Before ◁	Times ⊗	Cut
Premisses	None	A and B	A and B	A and B	K and $K^\perp$
RnB link					
Conclusion(s)	a and $a^\perp$	$A⅋B$	$A \lhd B$	$A \otimes B$	None

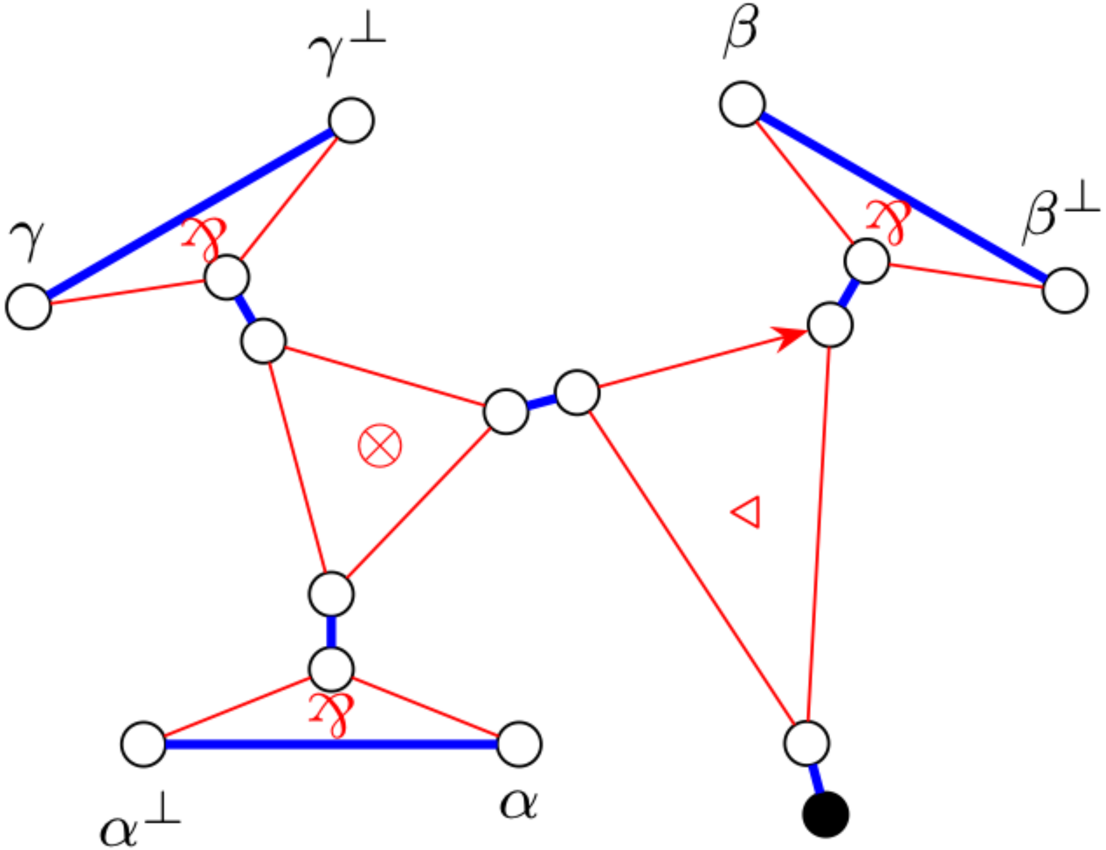
| blue formulas

| connective

$\overline{| a \quad | a^\perp}^{ax}$

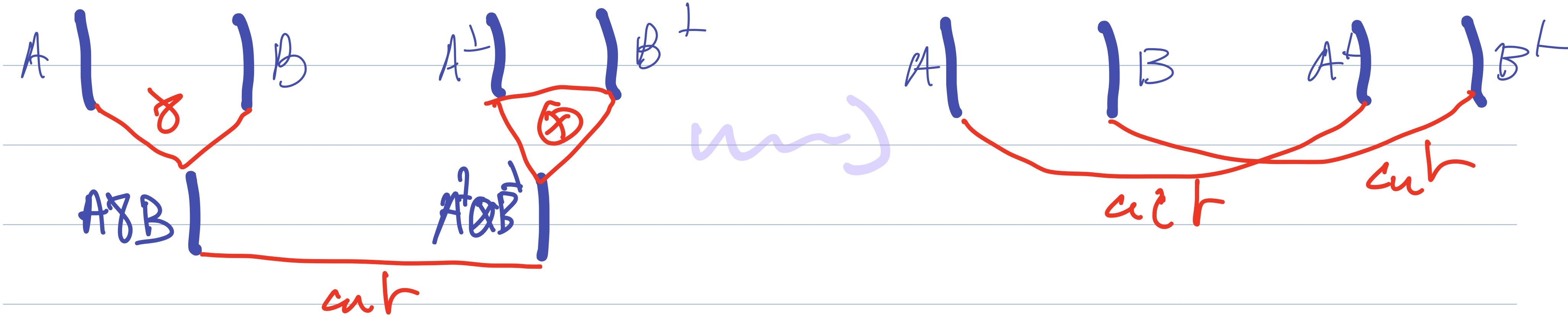
Examples correct / incorrect

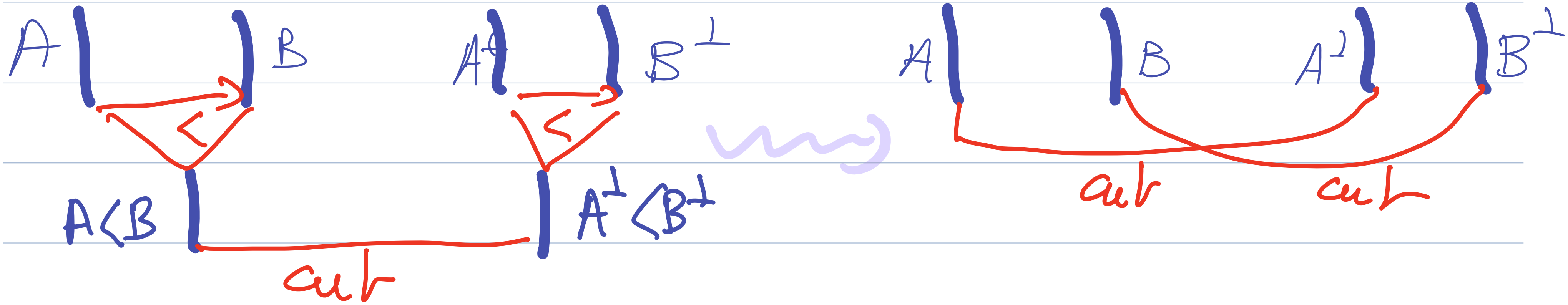




$((\alpha \parr \alpha^\perp) \otimes (\gamma \parr \gamma^\perp)) \lhd (\beta \parr \beta^\perp)$

Cut elimination





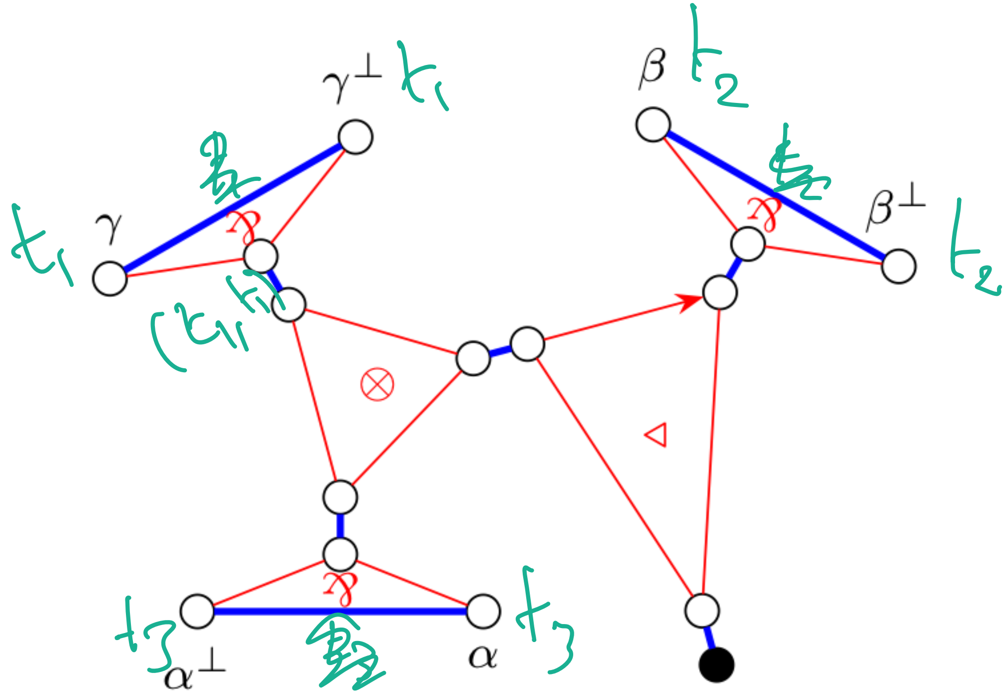
criterion is preserved
under cut elimination

Retoré 1997

Interpretation (experiments)

α, β, γ : coherence spaces

$|\alpha|, |\beta|, |\gamma|$



X: $((\alpha \parr \alpha^\perp) \otimes (\gamma \parr \gamma^\perp)) \triangleleft (\beta \parr \beta^\perp)$

Tokens

$t_1, t_3 \in |\alpha|$ $\quad$ $t_2 \in |\beta|$

$t_1', t_3' \in |\alpha|$ $\quad$ $t_2 \in |\beta|$

Result:

$(((t_3, t_3)(t_1, t_1)), (t_2, t_2))$

$\subset [X]$

$(((t_3', t_3')(t_1', t_1')), (t_2', t_2'))$

Theorem:

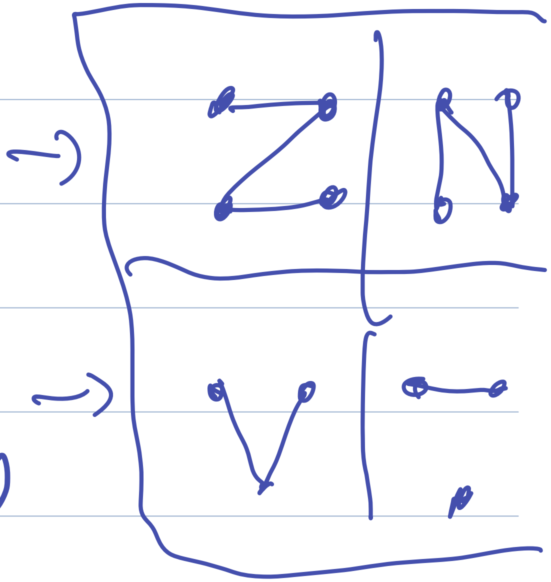
(no ae cycle)

syntactic correctness

⇕

semantic correctness

(experiments make a clique)

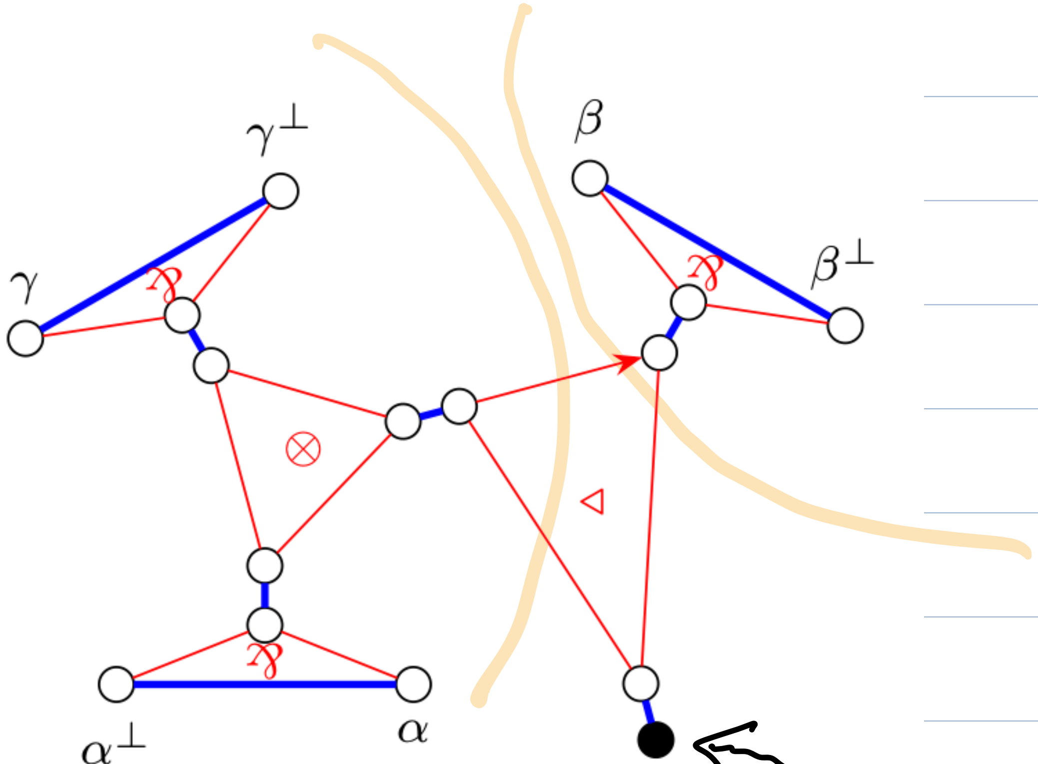


Retoré 1994

Fold / Unfold

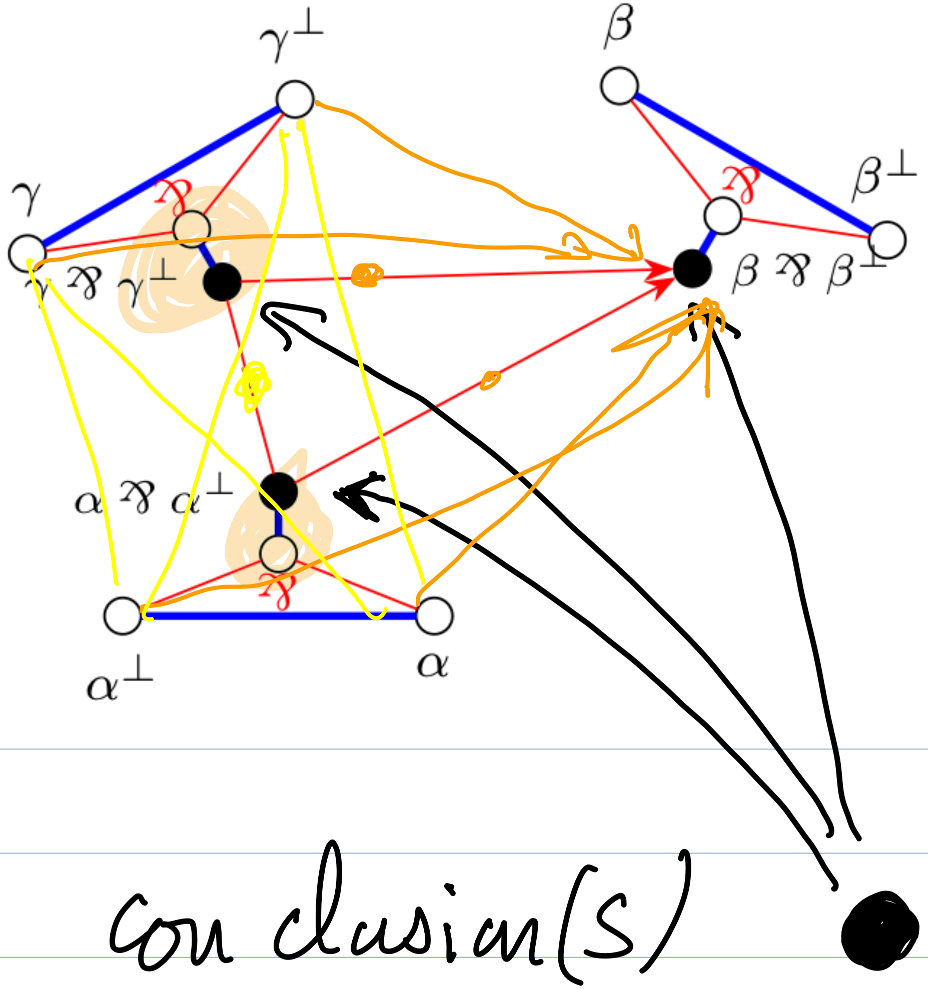
1998

proof net
with links:



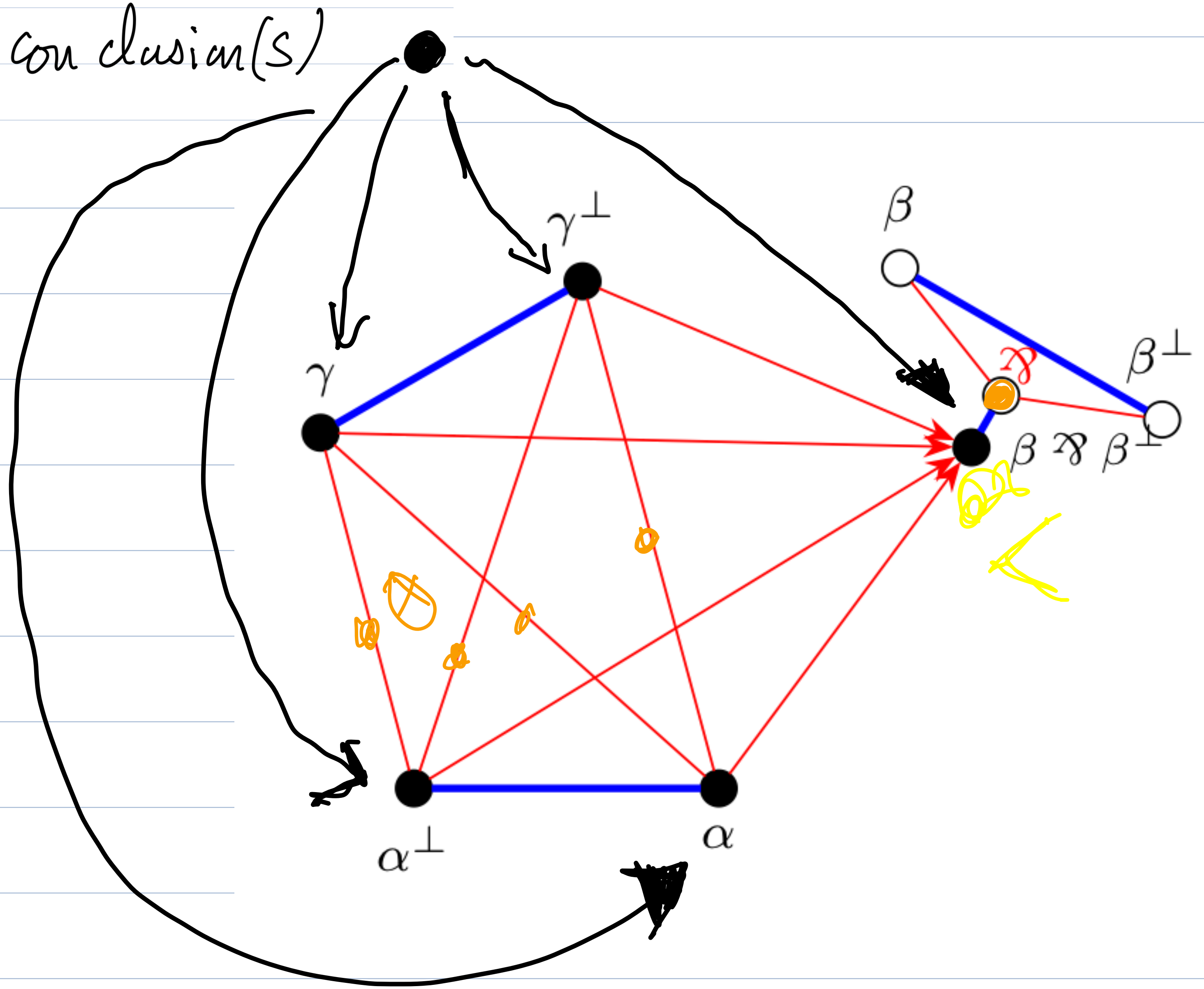
$((\alpha ⅋ \alpha^\perp) \otimes (\gamma ⅋ \gamma^\perp)) \lhd (\beta ⅋ \beta^\perp)$

conclusion(s) ●



conclusion(s)

5 conclusion(s)

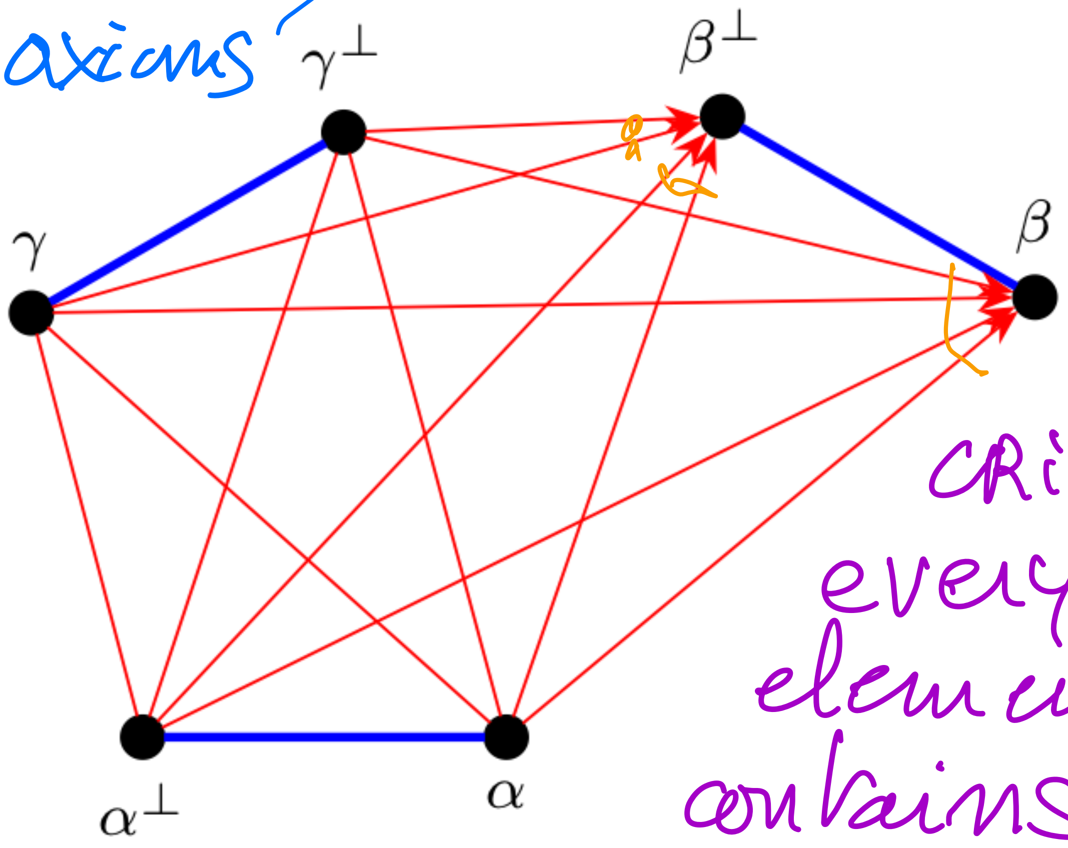


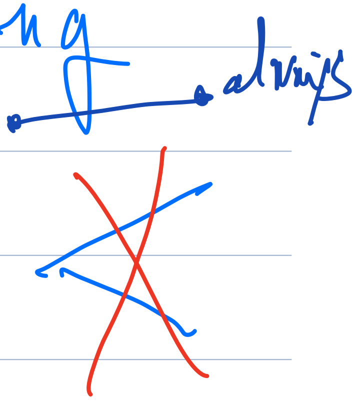
HANDSOME PROOF NETS

vertices : atoms

blue edges: axioms → perfect matching

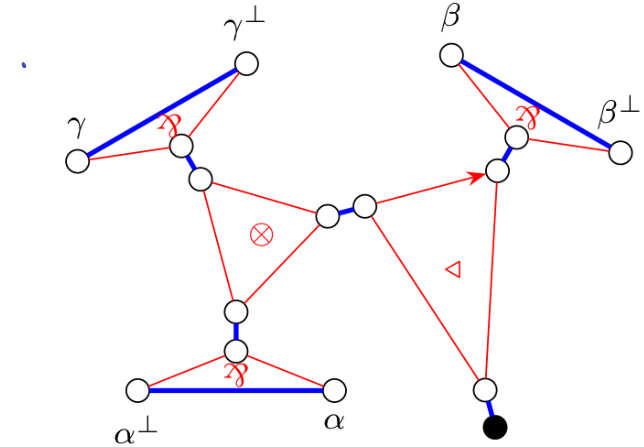
red edges: formula = directed cograph





CRITERION
every alternate elementary circuit contains a chord

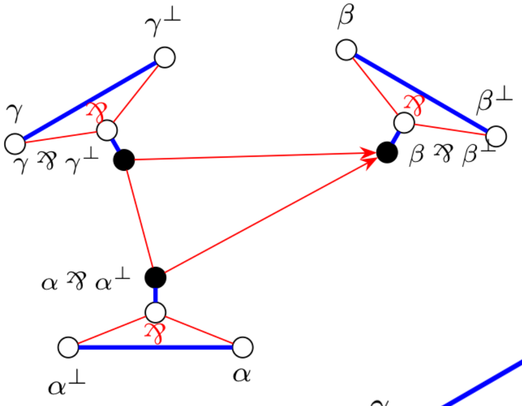
every atom is a conclusion

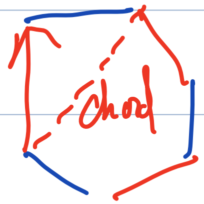


$((\alpha \parr \alpha^\perp) \otimes (\gamma \parr \gamma^\perp)) \lhd (\beta \parr \beta^\perp)$

criterion: every ae circuit contains a chord.

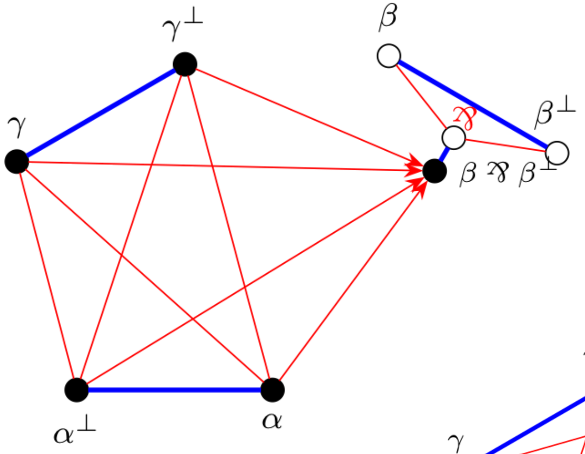
(pn w/ links no ae circuit)

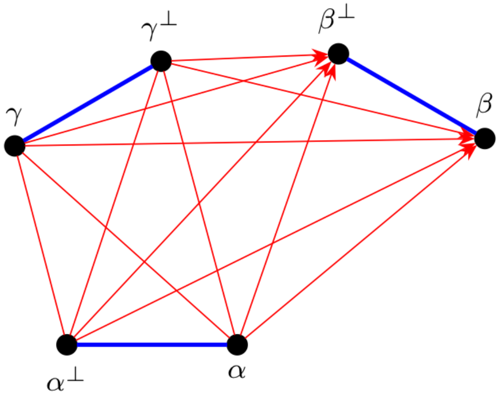




Theorem

fold and unfold both preserve the correctness criterion





Correct / incorrect proof structure

(a) Two incorrect handsome proof structures (chordless æ-circuit: $a, c^\perp, c, a^\perp, a$ in both cases)

(b) Two correct handsome proof structures (i.e. two correct hansome proof nets)

Formulas as graphs

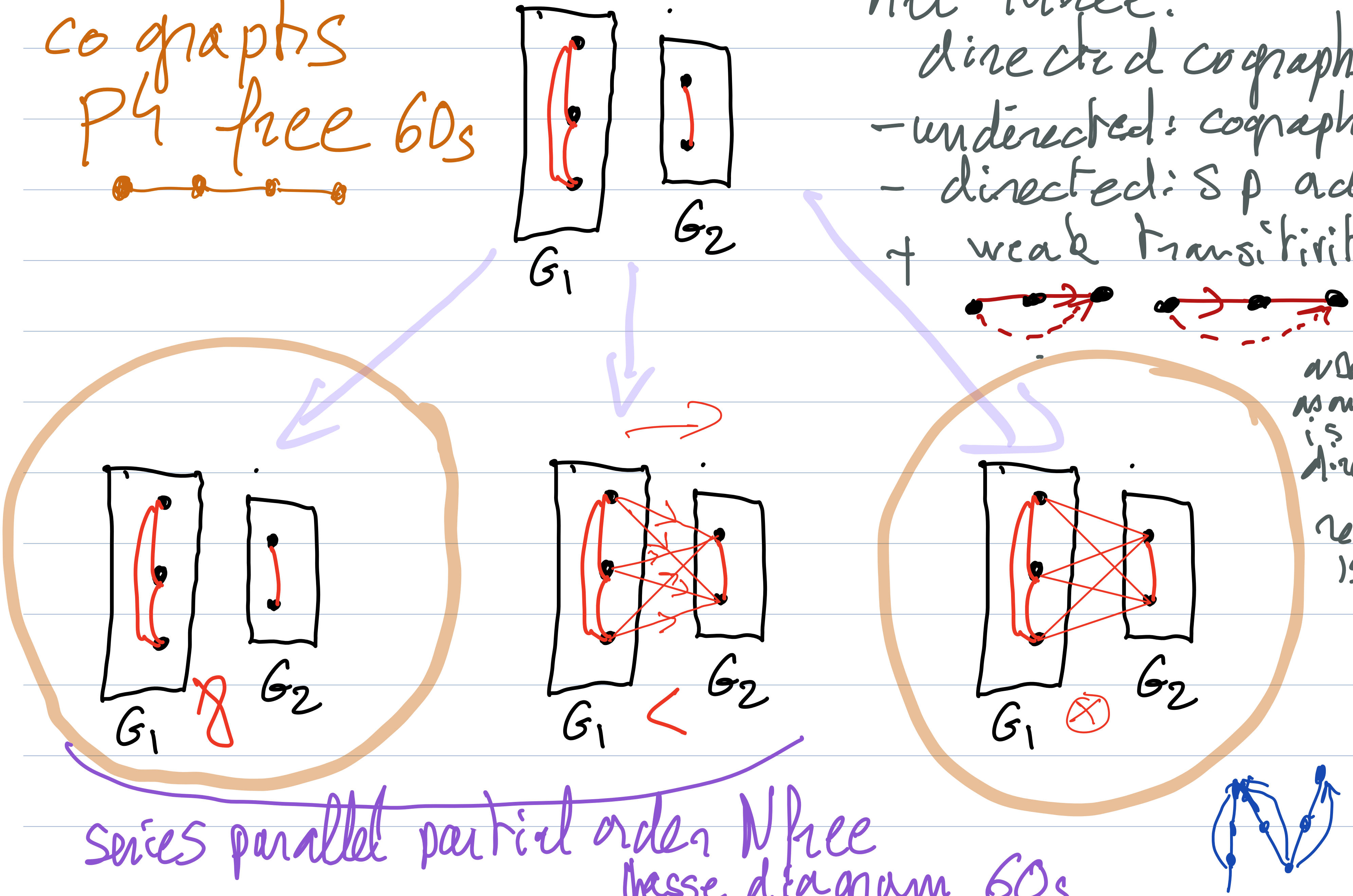
- of inductive family of graphs
- characterised by absence of some configurations

- identify formulas:
  - up to associativity of ⅋ < ⊗
  - up to commutativity of ⅋ ⊗

classes of graphs (cf. Gao 2016)

cographs
P4 free 60s

All three:
directed cographs
- undirected: cograph
- directed: s p order
+ weak transitivity



as soon as one is directed
refine 1998

series parallel partial order N free
hasse diagram 60s

other Rewriting that preserves correctness

interchange law (90 Abramsk)
naturality (Godement sheaves)

instances of

$$(A * B) \circ (C * D)$$

$$\rightarrow (A \circ C) * (B \circ D)$$

one of them can be 1 the unit

$*$ weaker than $\circ$

$\otimes$ $<$

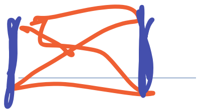
$\otimes$ $⅋$ (mix)

$<$ $⅋$

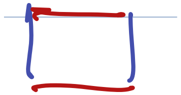
Retoré 1998

X Y U V can be 1

rule name	dicograph	$\rightsquigarrow$	dicograph'
☠ $\widehat{\otimes}\widehat{\parr}4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\parr3$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\parr}} Y$
$\otimes\parr2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$
$\otimes\triangleleft4$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft3l$	$(X \mathbin{\widehat{\triangleleft}} Y) \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\otimes}} U) \mathbin{\widehat{\triangleleft}} Y$
$\otimes\triangleleft3r$	$Y \mathbin{\widehat{\otimes}} (U \mathbin{\widehat{\triangleleft}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} (Y \mathbin{\widehat{\otimes}} V)$
$\otimes\triangleleft2$	$Y \mathbin{\widehat{\otimes}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\triangleleft}} Y$
$\triangleleft\parr4$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr3l$	$(X \mathbin{\widehat{\parr}} Y) \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$(X \mathbin{\widehat{\triangleleft}} U) \mathbin{\widehat{\parr}} Y$
$\triangleleft\parr3r$	$Y \mathbin{\widehat{\triangleleft}} (U \mathbin{\widehat{\parr}} V)$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} (Y \mathbin{\widehat{\triangleleft}} V)$
$\triangleleft\parr2$	$Y \mathbin{\widehat{\triangleleft}} U$	$\rightsquigarrow$	$U \mathbin{\widehat{\parr}} Y$



$(a \parr a^{\perp}) \otimes (b \parr b^{\perp})$

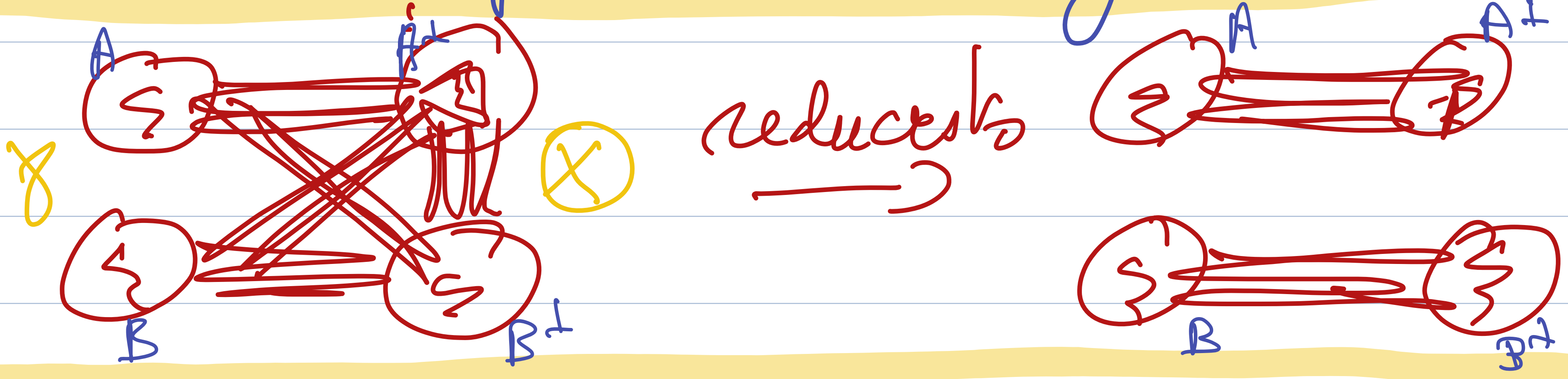


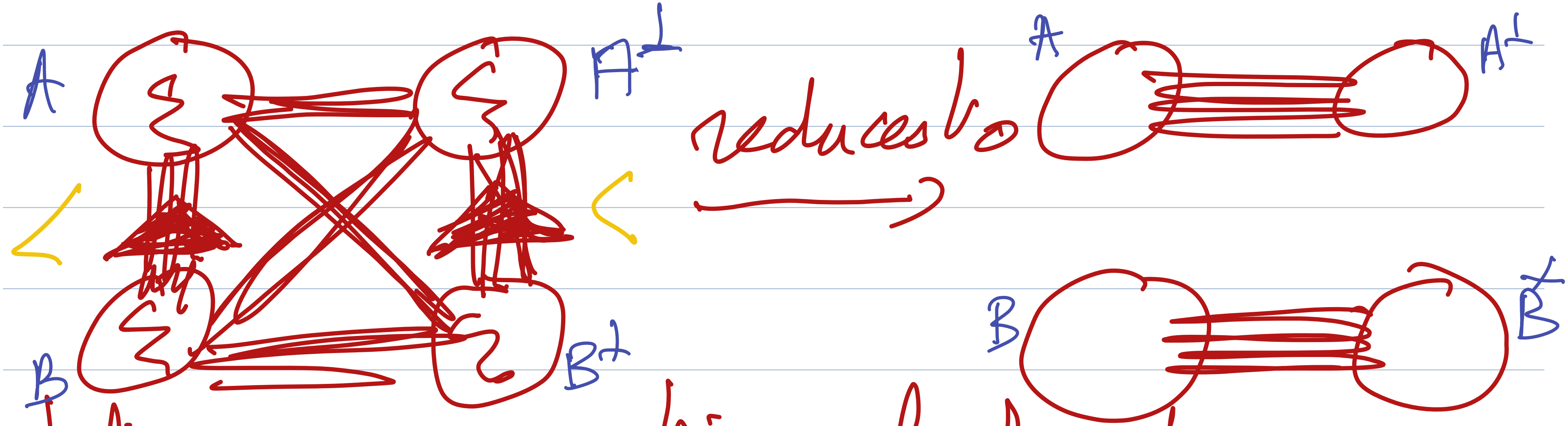
$a \otimes b$

$\parr(a^{\perp} \otimes b^{\perp})$

Cut elimination can be proved correct using this rewriting

$\text{cut} = (\exists X)\, X \otimes X^{\perp}$ Kruised





cut elim = preservation of chords.

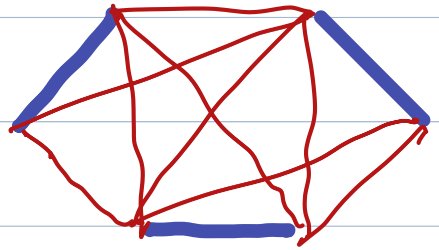
(S)BV Alessio Guglielmi

deeper view of rewriting

deep inference

We can define
a rewriting calculus
with handsome proofnets
starting with

Axiome



$(a_1 \parr a_1^\perp) \otimes (a_2 \parr a_2^\perp) \otimes (a_3 \parr a_3^\perp)$

I explored
this for
MLL in 1996
but not
for pomset logic
good that
Alessio did this!

Unit 1 : $(\varepsilon ⅋ \varepsilon^{\perp})$ ε special notation

rule by rule mapping

handsome proof net rewriting $\longrightarrow$ SBV derivation

cut in SBV i↑ replaces an internal $K \otimes K^{\perp}$ with 1

We can prove cut elimination with properties of (handsome proof net
graph rewriting

Pmn set as rewriting
is SBV

(outcome ant-diminution)
for SBV

$$\left(\begin{array}{c}\text{SBV}\\ \text{handsome PN}\\ \text{rewriting}\end{array}\right) \xrightarrow{?} \left(\begin{array}{c}\text{ALL?}\\ \text{handsome}\\ \text{proofnets}\end{array}\right)$$

NO

Lutz Straßburger
Tito N'guyen

$<$ is the collapse of $<^1$ $<^2$

dual of each other

on conclusion

a binary relation between $\leq n$ tuples.

A selfdual modality on $<$

(back to coherence spaces)

$$(?A)^{\perp} = !A^{\perp}$$

$$\begin{tikzcd} {!A} \arrow[r, -o] \arrow[d, -o] \arrow[dr, -o] & {!A \otimes !A} \\ 1 & {!A} \end{tikzcd} \qquad \begin{tikzcd} {?A} & {?A \parr ?A} \arrow[l, -o] \\ 1 \arrow[u, -o] & {?A} \arrow[ul, -o] \end{tikzcd}$$

$$(4A)^{\perp} = 4A^{\perp}$$

$$4A \overset{\text{linear iso}}{\circ\!\!-\!\!\!-\!\!\!-\!\!\circ} 4A < 4A$$

bigger

$$A \overset{\text{retract of}}{\underset{\circ\!\!-\!\!\!-\!\!\!-}{-\!\!\!-\!\!\!-\!\!\circ}} 4A$$

more elts

$$A \cancel{\underset{\circ\!\!-\!\!\!-\!\!\!-}{-\!\!\!-\!\!\!-\!\!\circ}} 1$$

idea $\mathbb{Q}$ / 2^{ω} copies of A

$< 4A \cong A < A < A \cdots \not< \cdots < A$

but we want a <u>countable</u> web $|4A|$

+ finite description

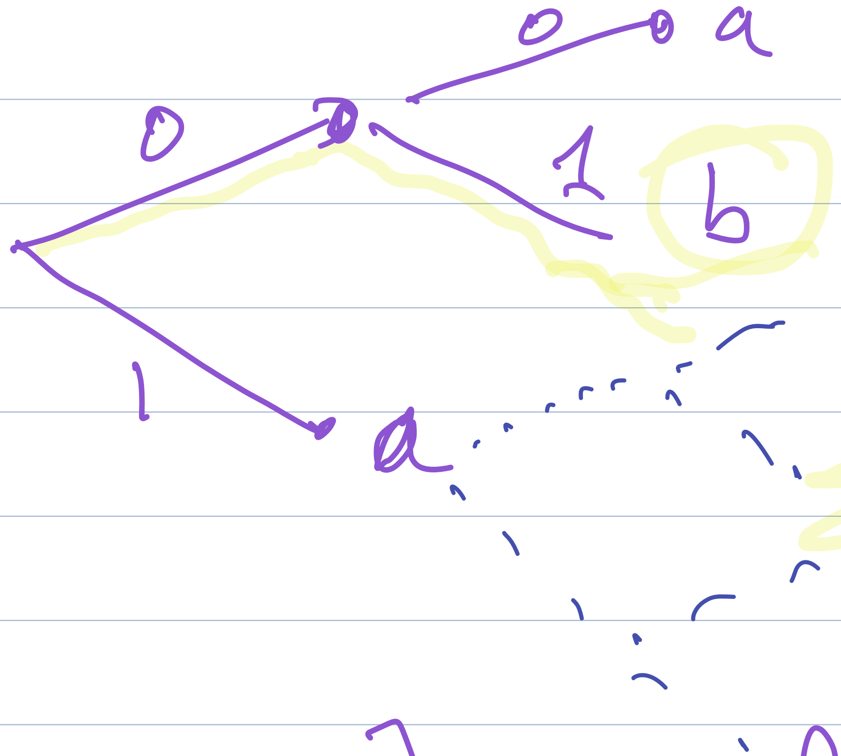
only finitely many different tokens

Tokens of $|4A|$

continuous maps

$2^{\omega} \longrightarrow |A|$ $\quad$ $|A|$ discrete topology

$f:$



$f(1 \cdots) = a$

$f(01 \cdots) = b$

$f(00 \cdots) = a$

$f \frown g$ $\exists w$ $f(w) \frown f(w') \{A\}$

$\& \ \forall w' > w \quad f(w) = f(w')$

This gives the required morphisms

$$\begin{tikzcd} 4A \arrow[r, o-o, "iso"] & 4A < 4A \end{tikzcd}$$

$(4A)^{\perp} = 4A^{\perp}$

$$\begin{tikzcd} 4A \arrow[r, shift left, -o] & A \arrow[l, shift left, -o] \end{tikzcd}$$ (the retract of $4A$)

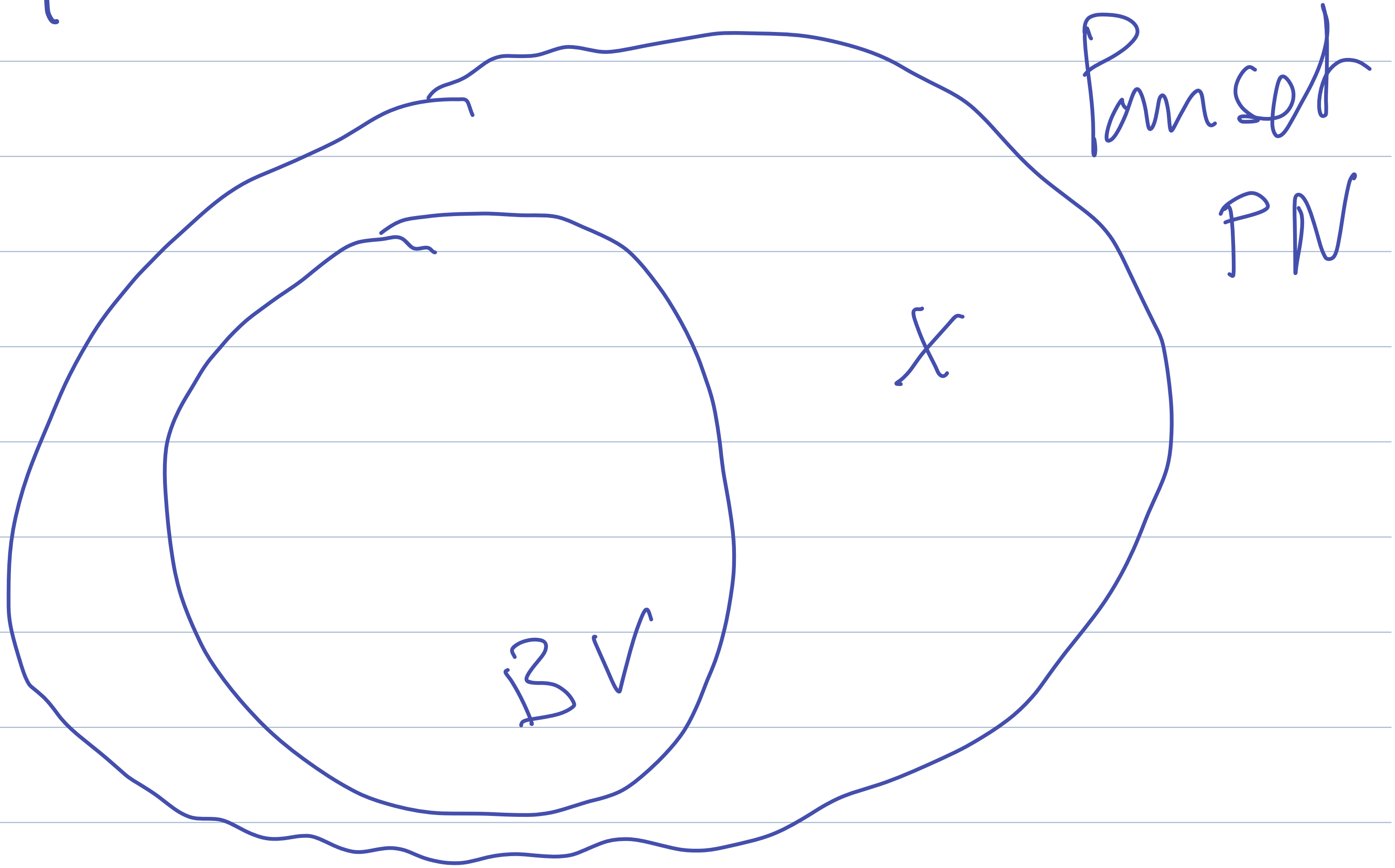
What could the syntax of $4A$ be?

Alessio has some ideas about that

$$\left(\frac{4A \,\&\, A}{4A} \right) ?$$

Thanks for your attention.

Any Questions?



$\boxed{2^{\omega}}$

A — $< \cancel{B}$

1. $A \multimap B$

$(4A) \multimap B$

$4A \multimap 4B$

$(<A^{\perp}) < B$

$< B$

$\cancel{A^{\perp} ⅋ B}$

$A^{\perp} < A$

$A^{\perp} < B$

$A \quad B$

$U^{\perp} < \cancel{B}W$

$\cancel{U^{\perp} < U}$

1011 — $\overset{R}{=}$ $11\,0\cancel{000}$

$\ne 2^{\omega}$