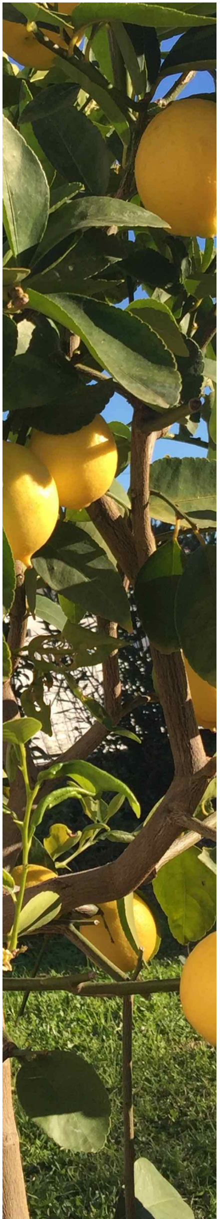

A self dual modality

for non-commutative contraction and duplication

in the category of coherence spaces

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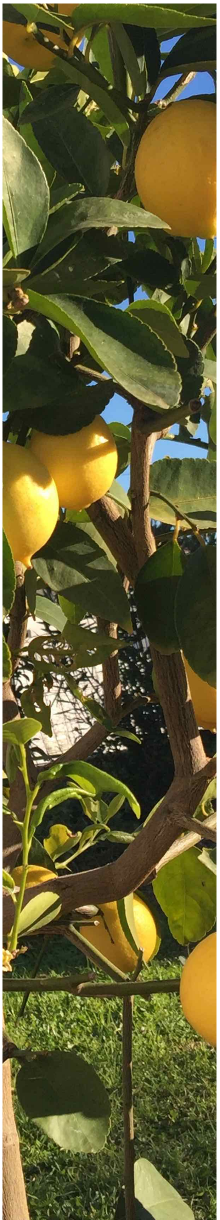
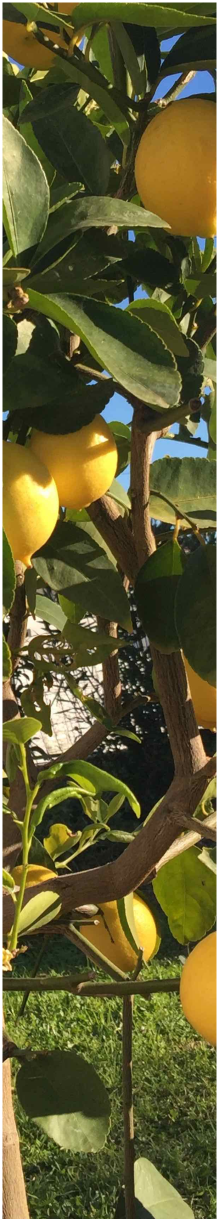
FSCD Linearity TLLA
Virtually in Paris, but almost anywhere, June 30, 2020



1. Initial Motivation

Girard (93): — Would you be able to find with ”your before connective” a self dual modality answering this?

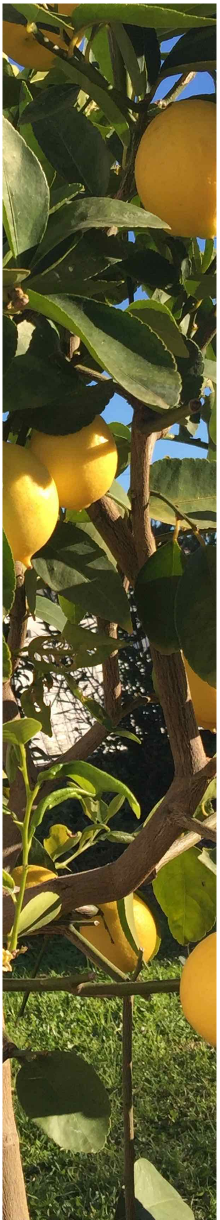
“The obvious candidate for a classical semantics was of course coherence spaces which had already given birth to linear logic; the main reason for choosing them was the presence of the involutive linear negation. However the difficulty with classical logic is to accommodate structural rules (weakening and contraction); in linear logic, this is possible by considering coherent spaces $?X$. But since classical logic allows contraction and weakening both on a formula and its negation, the solution seemed to require the linear negation of $?X$ to be of the form $?Y$, which is a nonsense (the negation of $?X$ is $!X^\perp$ which is by no means of isomorphic to a space$?Y$). Attempts to find a self-dual variant §Y of $?Y$ (enjoying $(\S Y)^\perp = \S Y^\perp$) systematically failed. The semantical study of classical logic stumbled on this problem of self-duality for years.” (J.-Y. Girard A new constructive logic classical logic, MSCS, 1991)



2. Today's Motivation

Renewed interest in Pomset logic and on the related developments by Guglielmi and Straßburger Calculus of Structure (SBV) and Deep Inference, a complete sequent calculus for pomset logic published by Slavnov.

Further more, e.g. for process calculi it makes sense to repeat a sequence of operations.



3. The category COH : the privileged categorical interpetation of linear logic

Categorical interpretation:

Formula/type : object

proof $\pi : A \vdash B$: morphism $[\![\pi]\!] : A \mapsto B$

whenever $\pi \rightsquigarrow \pi'$: $[\![\pi]\!] = [\![\pi']\!]$.

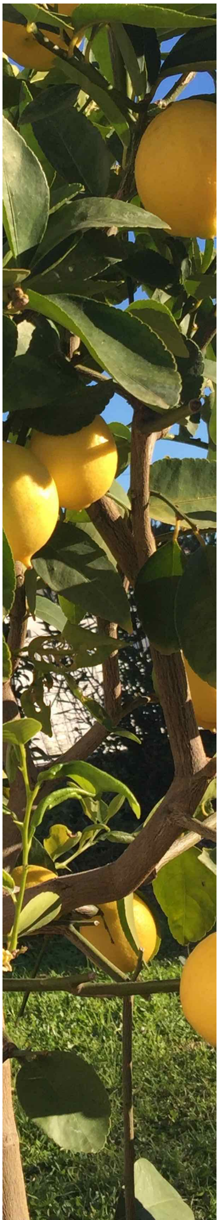
$Hom(A,B)$ corresponds to an *object* B^A.

CCC intuitionistic logic

COHerence spaces: intially introduced to interpret second order intutionistic logic because the endofunctor $X \longrightarrow T[X]$ can be represented as a coherence space.

Linear logic is issued from coherence spaces:

$$A \to B = (!A) \multimap B$$

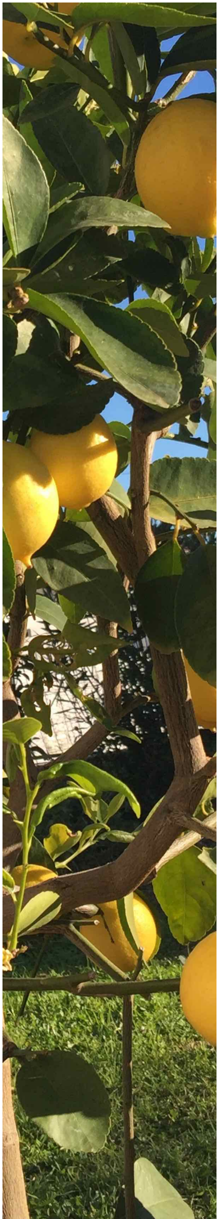


4. The category COH. Objects: coherence spaces

A coherence space $A = (|A|, \frown_A)$ is a simple graph undirected, without loops nor multiple edges

vertices are called tokens and they constitute the web $|A|$ endowed with a binary symmetric and irreflexive relation called strict coherence.

The objects under consideration are the cliques of this graph, i.e. the sets of pairwise related tokens. Cliques interpret proofs of A up to cut-elimination / normalisation.

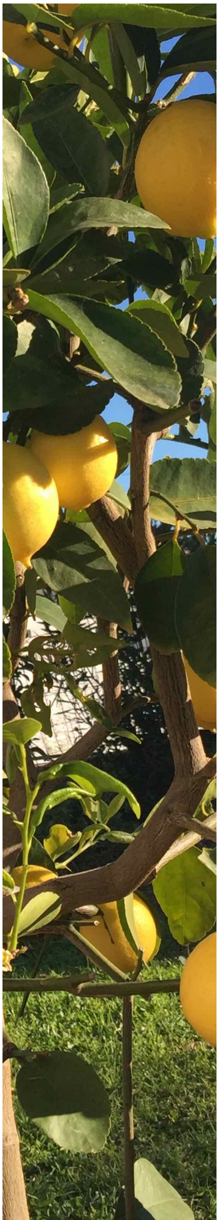


5. The category COH. Arrows: linear maps

A linear morphism F from A to B is a morphism mapping cliques of A to cliques of B such that:

- For all $x \in A$ if $(x' \subset x)$ then $F(x') \subset F(x)$
- For every family $(x_i)_{i \in I}$ of pairwise compatible cliques of A — that is to say $(x_i \cup x_j) \in A$ holds for all $i, j \in I$ — $F(\cup_{i \in I} x_i) = \cup_{i \in I} F(x_i)$.
- For all $x, x' \in A$ if $(x \cup x') \in A$ then $F(x \cap x') = F(x) \cap F(x')$.

Linear functions from A to B identify with cliques in $A^{\perp} \parr B = A \multimap B$.



6. Commutative Multiplicative Connectives

$$\alpha \smile \alpha'[A] \text{ iff } \alpha \not\frown \alpha'[A] \text{ and } \alpha \neq \alpha'$$

Given two tokens α, α' in $|A|$ exactly one of the following relation holds:

$\alpha \smile \alpha'[A]$ or $\alpha = \alpha'$ or $\alpha \frown \alpha'[A]$.

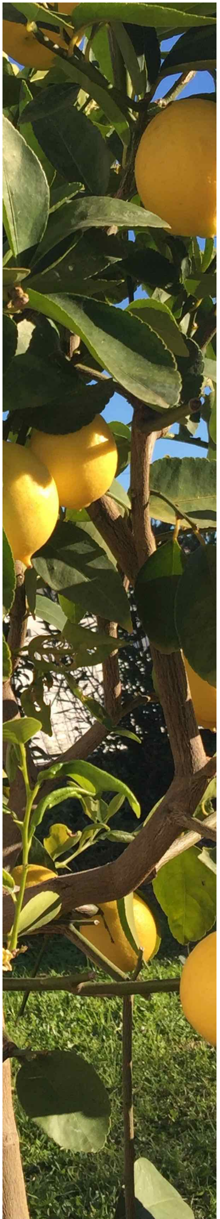
Multiplicative connectives $A * B$: $|A * B| = |A| \times |B|$. Unit $= \mathbb{1} = \{*\}$.

We may assume they are covriant in both their arguments.

Commutative multiplicative (binary) connectives, just two of them:

$A \parr B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$A \otimes B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$



7. The category COH. Arrows as cliques of the linear function space

A linear map F corresponds to

$$\{(\alpha, \beta) \mid \alpha \in |A| \beta \in |B| \; \beta \in F(\{\alpha\})\}$$

clique of $A^{\perp} \parr B = A \multimap B$.

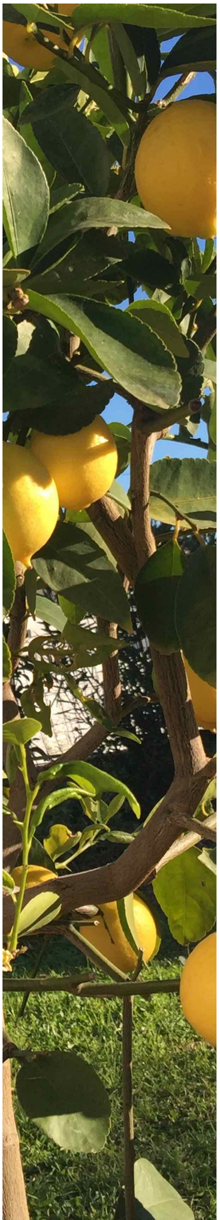
Linearity $\rightarrow$ for any clique x of A and any $\beta \in F(x)$ there is a unique $\alpha \in x$ such that $\beta \in F(\{\alpha\})$.

Conversely, given a clique f of $A^{\perp} \parr B$ a linear function can be defined by

$$F(x) = \{\beta \in |B| | \exists \alpha \in x (\alpha, \beta) \in f\}$$

Strict coherence in $A^{\perp} \parr B = A \multimap B$ is characterised as follows:

$(\alpha, \beta) \frown (\alpha', \beta')$ whenever $\alpha \frown \alpha' [A]$ entails $\beta \frown \beta' [B]$.



8. Before / Seq — pomset logic, calculus of structures, deep inference and other heresies

But, there is another (non commutative) multiplicative connective:

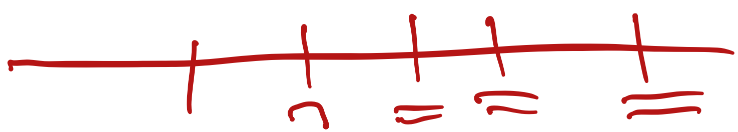
$A \lhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

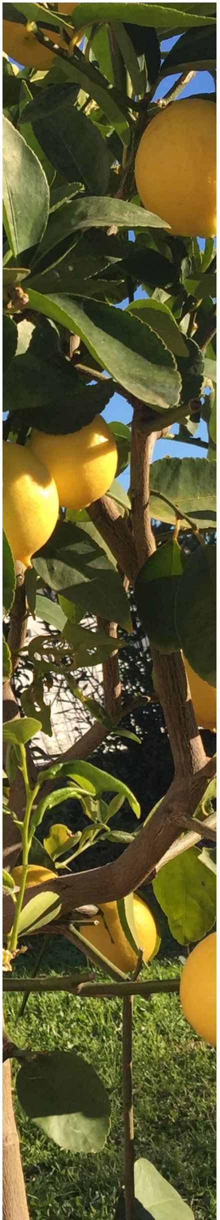
$A \rhd B$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$(\alpha,\beta) \frown (\alpha',\beta')[A \lhd B]$ whenever $\begin{cases} \alpha \frown \alpha'[A] \text{and} \beta = \beta' \\ \text{or} \\ \beta \frown \beta' \end{cases}$

Associative, self dual $(A \lhd B)^\perp = A^\perp \lhd B^\perp$ *no swap*

Generalisation: $\succ$ finite (partial) order over $\{1,\ldots,n\}$, web: $|A_1| \times \cdots \times |A_n|$ coherence: $(\alpha_1,\ldots,\alpha_n) \frown (\alpha'_1,\ldots,\alpha'_n)$ when there exists i s.t. $\alpha_i \frown \alpha'_i$ and $\alpha_j = \alpha'_j$ for all $j \succ j'$.





9. What we are looking for?

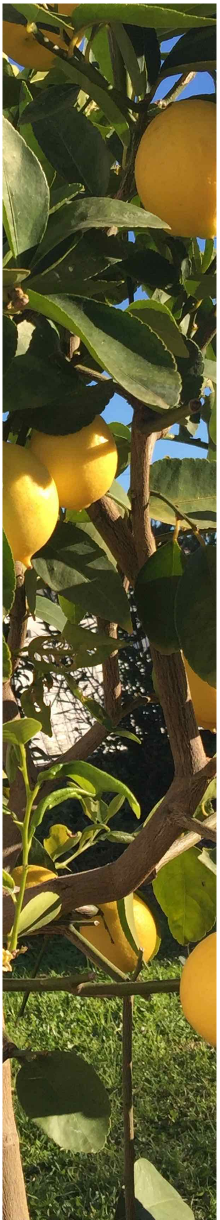
Usual modalities:

$$
\begin{array}{ccccccccc}
!A & \multimap & (!A \otimes !A) & \qquad & ?A & \multimapinv & ?A \parr ?A \\
\downarrow & \searrow & & & \uparrow & \nwarrow & \\
1 & & A & & 1 & & A
\end{array}
$$

Self dual contraction/duplication Flag:

$$\triangleleft\!\!|A \multimapinv \text{linear iso} \multimap (\triangleleft\!\!|A \triangleleft \triangleleft\!\!|A)$$

Of course there is no relation between $\triangleleft\!\!|A$ and 1, otherwise, with a self dual modality, the system would collapse.



10. Continuous functions from Cantor space to a discrete topological space

$\mathbb{2}^\omega$, infinite words on $\mathbb{2}$, with standard order and topology:

- usual total lexicographical order:

$$w_1 < w_2 \text{ iff } \exists m \in \mathbb{2}^* \ \exists w'_1, w'_2 \in \mathbb{2}^\omega \ w_1 = m0w'_1 \ \textit{and} \ w_2 = m1w'_2$$

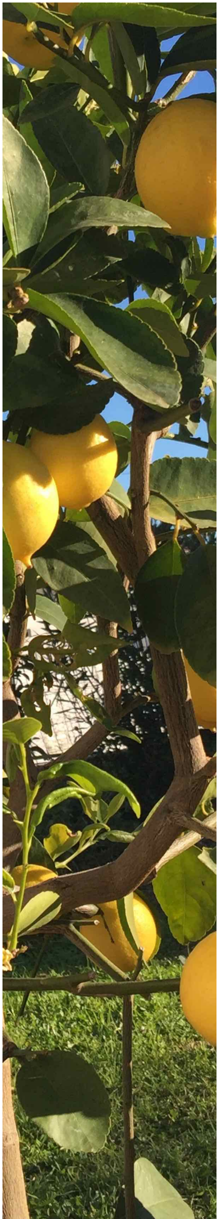
- usual product topology generated by clopen sets $(U_m)_{m \in \mathbb{2}^*}$

$$U_m = \{w \in \mathbb{2}^\omega | \exists w' \in \mathbb{2}^\omega \ w = mw'\}$$

Continuous function from $\mathbb{2}^\omega$ to a set M (discrete topology) = finite binary tree with M-labelled leaves without two sister leaves with the same M-label.

gt_M generic trees over M = binary tree representing continuous functions $\mathbb{2}^\omega \mapsto M$.

Let $f \in \mathrm{gt}_M$ for $w \in \mathbb{2}^\omega$ there is a unique prefix of w that is a root-to -leaf path of f. If the M-label is a then $f(w) = a$.



11. Justification

$2^\omega = \cup_{\alpha \in M} f^{-1}(\{\alpha\})$

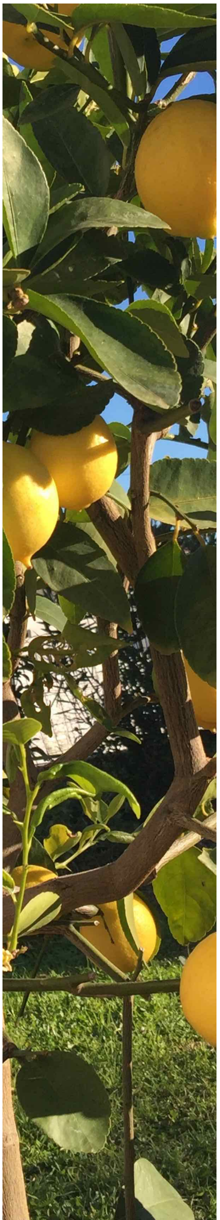
$\{\alpha\}$ are clopen sets and so is $f^{-1}(\{\alpha\})$.

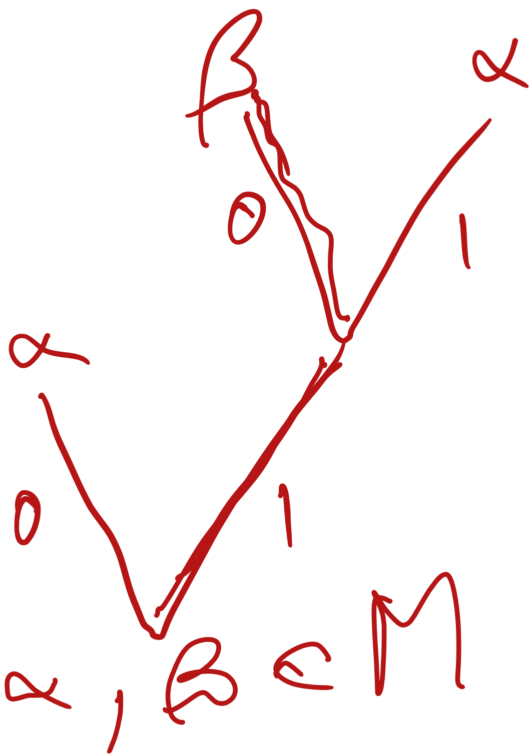
Hence one can extract a finite covering of 2^ω from the $f^{-1}(\{\alpha\})$ (compacity of 2^ω).

So the function has finitely many values.

Each of these $f^{-1}(\{\alpha\})$ can be written as a finite union of base clopen sets and a finite union of finite union is a finite union, and this gives the tree structures. Observe that two base clopen sets never have a non trivial intersection: their intersection is either empty of one of the two.

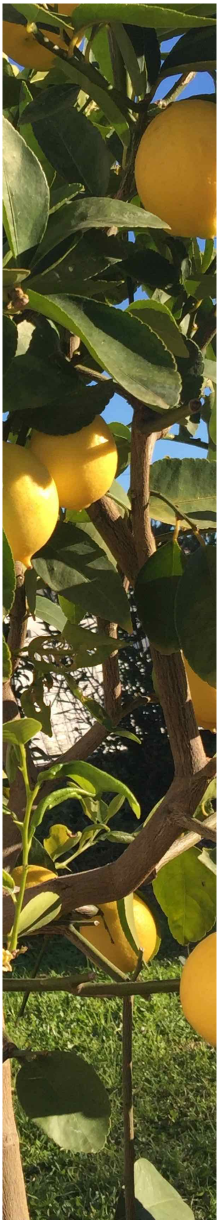
12. A generic tree, i.e. continuous function from 2^ω to a set M





$f(10\ldots)=\beta$

etc



13. A remark on continuous functions from the Cantor space to a discrete topological space

(RkCantor) Let $f, g \in \mathrm{gt}_M$. If $f \neq g$, then there exists $w \in \mathbb{2}^\omega$ such that

$$f(w) \neq g(w) \text{ and } \forall w' > w \quad f(w') = g(w')$$

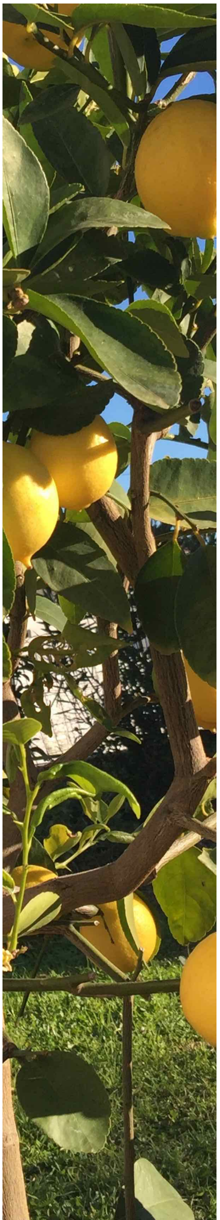
Consider the maps

$w \mapsto (f(w), g(w))$ from $\mathbb{2}^\omega$ to $M \times M$ endowed with the discrete topology which is the product of the discrete topology on M by itself.

$\Delta : M \times M \mapsto \mathbb{2}$ (discrete topology on $\mathbb{2}$) with $\Delta(x, y) = 1$ iff $x = y$.

The compound of those two continuous functions is continuous, so it is a finite tree of $\mathrm{gt}_{\mathbb{2}}$ and it has a maximum $w = u(1)^*$.

For w the property holds: $f(w) \neq g(w)$ and $f(w') = g(w')$ for all $w' > w$.



14. The flag modality

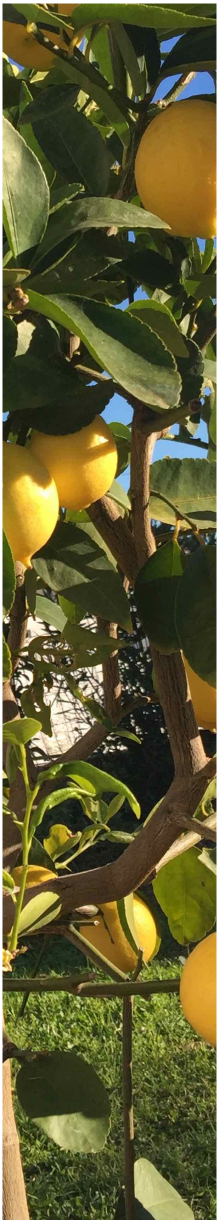
Web of $\triangleleft A$**:** $\mathrm{gt}_{|A|}$

the continuous functions from 2^{ω} to $|A|$ the web of A.

Observe that if $|A|$ is countable so is $\mathrm{gt}_{|A|}$.

Coherence $f \frown g[\triangleleft A]$ with $f, g \in |\triangleleft A = \mathrm{gt}_{|A|}$ whenever

$$\exists w \in 2^{\omega} \begin{cases} f(w) \frown g(w)[A] \\ \text{and} \\ \forall w' > w \quad f(w') = g(w') \end{cases}$$



15. Flag is self dual

The modality $\triangleleft\!|$ is self-dual, i.e. $(\triangleleft\!|A)^\perp \equiv \triangleleft\!|(A^\perp)$

Those two coherence spaces obviously have the same web.

Consider $f \neq g$ two distinct continuous functions from 2^ω to $|A|$.

(RkCantor) There exists $w \in 2^\omega$ such that $f(w) \neq g(w)$ and $\forall w' > w.\ f(w') = g(w')$.

Either $f(w) \frown g(w)[A]$ and $f \frown g[\triangleleft\!|A]$

or $f(w) \frown g(w)[A^\perp]$ and $f \frown g[\triangleleft\!|(A^\perp)]$

Hence for any distinct f, g either $f \frown g[\triangleleft\!|A]$ or $f \frown g[\triangleleft\!|(A^\perp)]$ so $\triangleleft\!|A^\perp = (\triangleleft\!|A)^\perp$.

16. A linear iso between Flag A and Flag A Before Flag A: definition

$$C = \{(h,(h_0,h_1)) \mid \forall w \in 2^\omega \; h(0w) = h_0(w) \text{ and } h(1w) = h_1(w)\}$$

defines a linear isomorphism between $\triangleleft A$ and $\triangleleft A < \triangleleft A$.

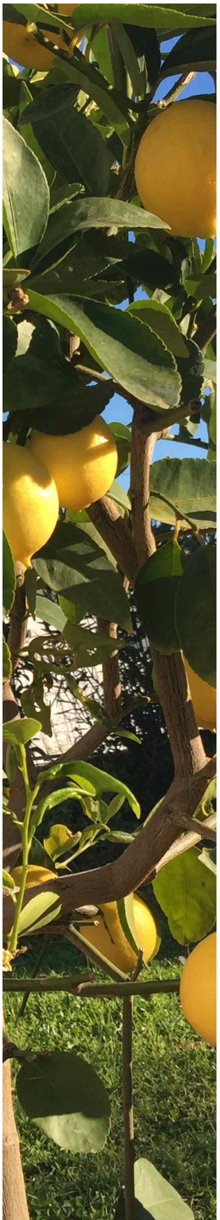
It is a bijection between the webs, between pairs of continuous functions from 2^ω to $|A|$ and continuous functions from 2^ω to $|A|$.

17. A linear iso from Flag A to Flag A Before Flag A

We have to check that given two pairs $(f,(f_0,f_1))$ and $(g,(g_0,g_1))$ in C whenever $f \frown g[\triangleleft\!|A]$ then $(f_0,f_1) \frown (g_0,g_1)[\triangleleft\!|A \lhd \triangleleft\!|A]$

If $f \frown g[\triangleleft\!|A]$ then there exists $w \in 2^{\omega}$ s.t. $f(w) \frown g(w)[A]$ and $h(w') = g(w')$ for all $w' \succ w$.

- if $w = 0m$ then $f_0 \frown g_0[\triangleleft\!|A]$ and $f_1 = g_1$, therefore $(f_0,f_1) \frown (g_0,g_1)[\triangleleft\!|A \lhd \triangleleft\!|A]$
- if $w = 1m'$ then $f_1 \frown g_1$, therefore $(f_0,f_1) \frown (g_0,g_1)[\triangleleft\!|A \lhd \triangleleft\!|A]$

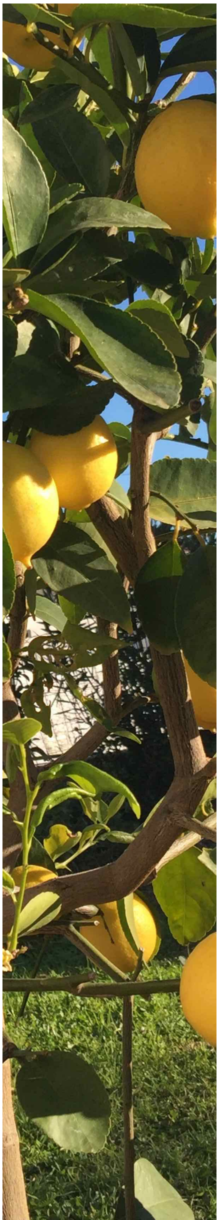


18. A linear iso from Flag A Before Flag Ato Flag A

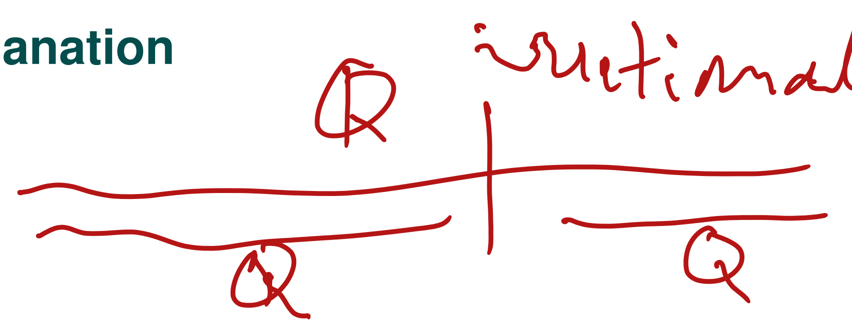
Conversely it is similar.

If $(f_0, f_1) \frown (g_0, g_1)[\triangleleft\!\!|A \lhd \triangleleft\!\!|A]$ either

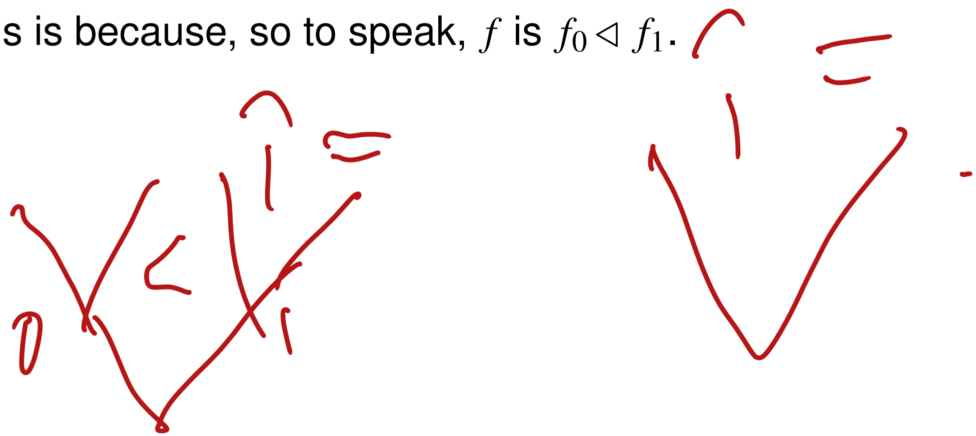
- $f_1 = g_1$ and $f_0 \frown g_0[\triangleleft\!\!|A]$ so there exists a w such that $f_0(w) \frown g_0(w)[A]$ and $f_0(w') \frown g_0(w')[A]$ for all $w' > w$. If $(f, (f_0, f_1)$ and $(g, (g_0, g_1)$ are in C then $f(0w) \frown g(0w)$ and for all $u > w$ if $u = 0v$ then $v > w$ and $f(u) = g(u)$ and if $u = 1m$ then $f(u) = g(u)$ as well.
- $f_1 \frown g_1[\triangleleft\!\!|A]$ so there exists a w such that $f_1(w) \frown g_1(w)[A]$ and $f_1(w') \frown g_1(w')[A]$ for all $w' > w$. COnsequently $f(1w) \frown g(1w)$ and for all $w' > 1w$, we have $f(w') = g(w')$ because $w' = 1m$ and $m > w/$



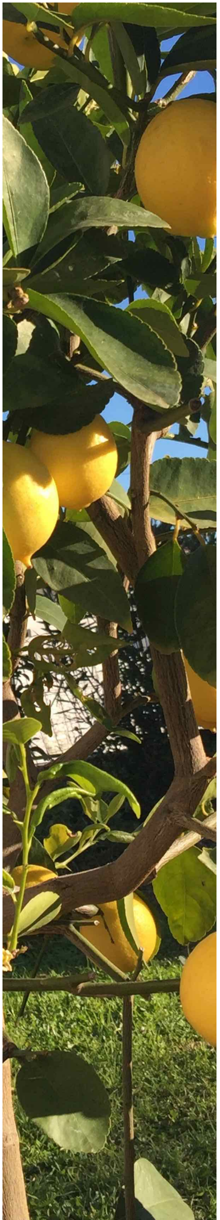
19. Explanation



This is because, so to speak, f is $f_0 \lhd f_1$.



I initially defined the web of $\triangleleft\!|A$ as $\lhd_{i \in Q} A$ (Q copies of $|A|$, a token was a function from Q to A) and Achim ~~young~~ suggested to use 2^ω to get a finite representation of the tokens.

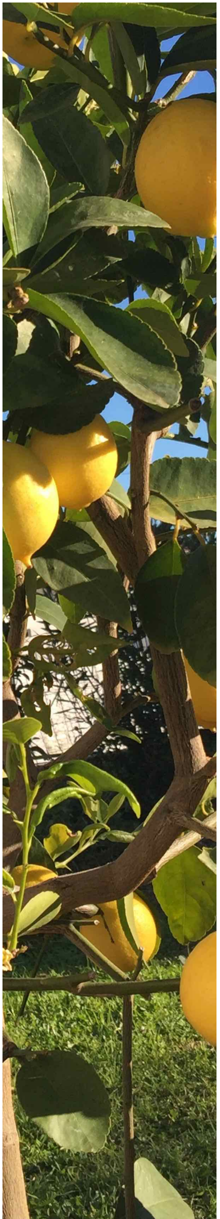


20. A is a retract of Flag A

Consider the $\{\alpha, \underline{\alpha})|\alpha \in |A|\}$ where α is the constant continuous function from 2^ω to $|A|$ mapping every word to α.

It is linear.

One compound is Id_A, while the other compound is identity but only for constant functions.



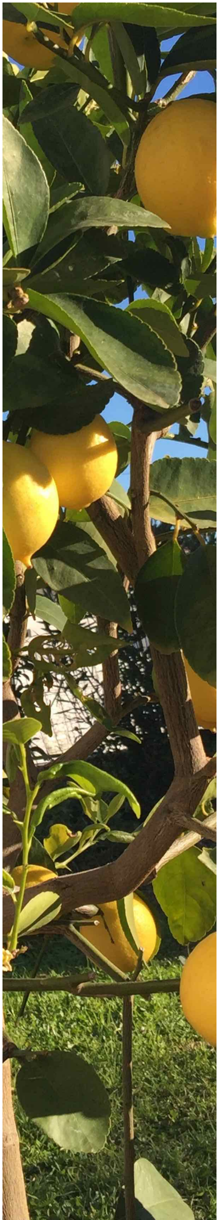
21. Flag is functorial

Given $\ell : A \to B$ defines $\triangleleft\!\!\mid\ell : \triangleleft\!\!\mid A \to \triangleleft\!\!\mid B$ by the following linear map:

$$\triangleleft\!\!\mid\ell = \{(f,g) / \forall w \in 2^{\omega} (f(w), g(w)) \in \ell\}$$

makes $\triangleleft\!\!\mid$ an endo-functor.

This is not difficult but a bit tedious to prove.



22. Concluding question: syntax?

Pomset logic is better defined with (handsome) proof nets,

or as a rewriting system like Deep Inference.

The design of a self dual modality should perhaps proceed with handsome proof nets

whose correction is equivalent to their interpretability in coherence spaces.

However modalities are complicated in the the proof net framework

an exception being the essential nets of Lamarche for intuitionistic logic.

Guglielmi proposed in the last years several versions of a self dual modality with deep inference coherence semantics should be a guideline to find the right one, if any.