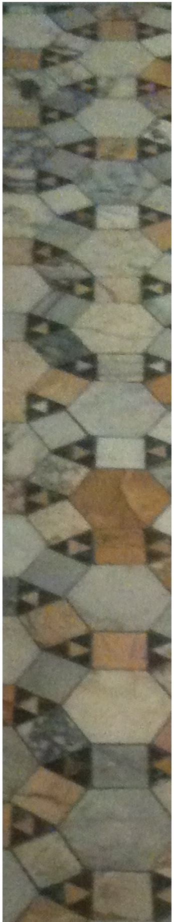

# Logic and topology
# some connections, old and new

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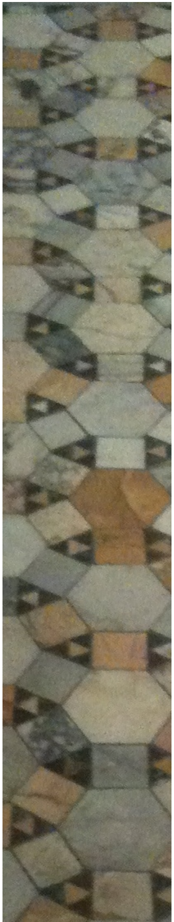
# A A list of connections from the early days

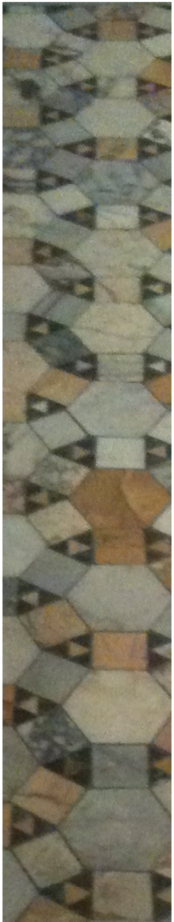


## Topological models for various logic:

- Opens as interpretations of propositional intuitionistic logic. — or modal logic.
- Models of first order intuitionistic logic as (pre)sheaves of classical models.

## Categorical interpretation of intuitionistic proofs

- Proofs as "Scott continuous" maps between object/structured sets/ types —– e.g. coherence spaces taht work for second order as well.
- Intuitionnistic types theory and its homotopic model.

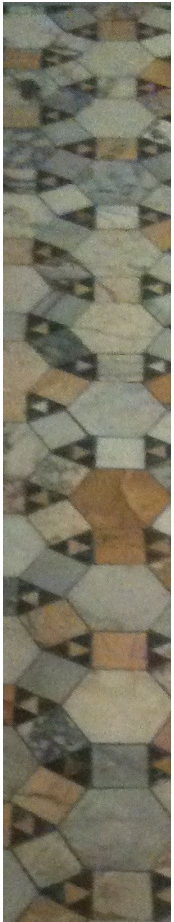


**Topological methods in logic:**

- Topological semantics for logic with a provability modality.
- Completion of theories endowed with a topology as a dynamical system. $T \mapsto T + Co(T)$.

**Logical methods for topology:**

- Logics for topological and spatial description.



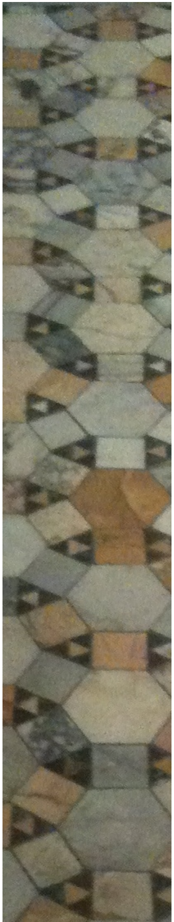
## A.1. Our selection

- Opens as interpretations of propositional intuitionistic logic.
- Models of first order intuitionistic logic as (pre)sheaves of classical models.
- Topological semantics for logic with a provability modality.

Some other choices are possible... ma al bar con uno bicchiere di Ramandolo.

# B Interpretations of (intuitionistic) propositions by open sets

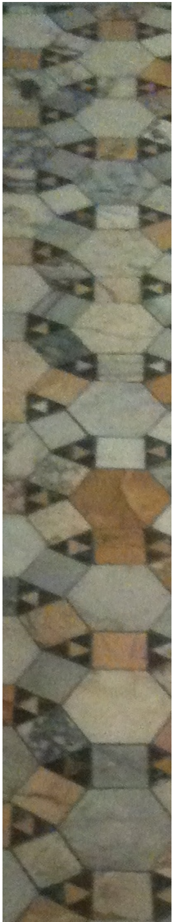
## B.1. Intuitionistic propositional calculus



0-ary connective / constant: $\perp$.

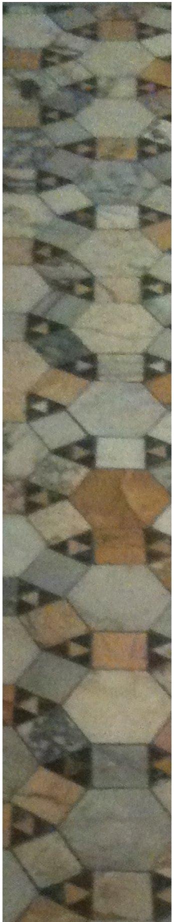
Binary connectives: $\wedge, \vee, \rightarrow$
— beware that they are all independent.

$\neg P$ is defined as $P \rightarrow \perp$.



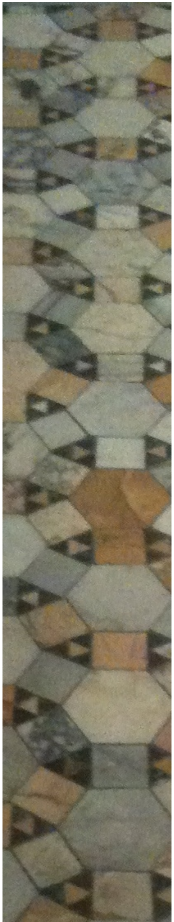
## B.2. Intuitionistic natural deduction

Deduction rules as expected, but a single conclusion! (so no contraction, no weakening on the conclusion side!)



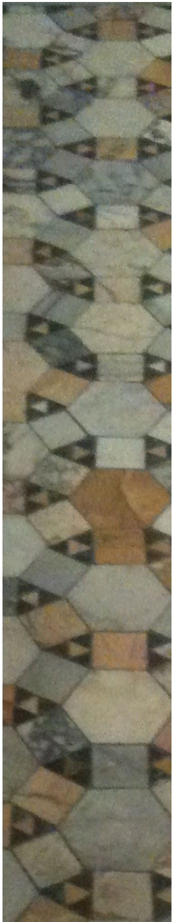
## B.3. Form formulas to open sets

Given a topology $\mathcal{T}$ on $X$ define $[\![p]\!] \in \mathcal{T}$ for any propositional variable $p$.
This extends to any propositional formula:
$[\![\bot]\!] = \emptyset$
$[\![p \wedge q]\!] = [\![p]\!] \cap [\![q]\!]$
$[\![p \vee q]\!] = [\![p]\!] \cup [\![q]\!]$
$[\![p \rightarrow q]\!] = \bigcup_{(x \cap [\![p]\!]) \subset [\![q]\!]} x$
A formula $F$ (resp. $F \vdash G$) is said to be true in some model whenever $[\![F]\!] = X$ the whole topological space itself (resp. whenever $[\![F]\!] \subset [\![G]\!]$).



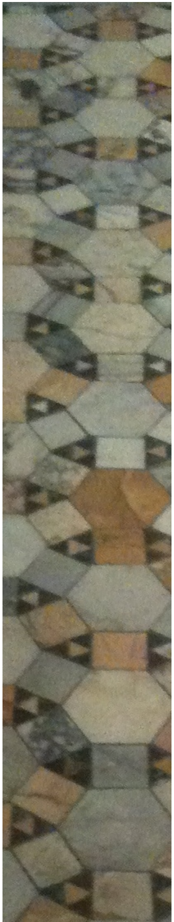
## B.4. Soundness and completeness

Soundness by induction on the (e.g. natural) deduction rules.
Completeness from the topological space induced by trees of classical models.
Standard $\mathbb{R}$ (or $\mathbb{R}^n$) models are enough to obtain completeness. This can be proved by defining a continuous map from $\mathbb{R}^n$ to any finite tree.
$[\![\neg P]\!]$ simply is the interior of $X \setminus [\![P]\!]$.
Example $\neg\neg P \vdash P$ is not provable:
define $[\![P]\!] = ]0;1[ \cup ]1;2[$. Then $[\![\neg\neg P]\!] = ]0;2[$

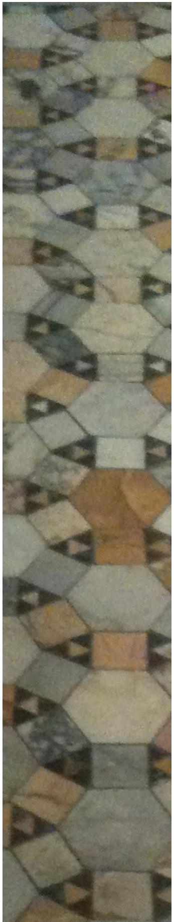


## B.5. Comments

Oldest connection between logic and topology?
Propositions up to equivalence: Heyting algebra lattice with a $\to$ relation satisfying some relation the order.
In a complete Heyting algebra (any subset as a sup) $\to$ can be defined as we did.
Opens of a topological space ordered with inclusion are a complete Heyting algebra.

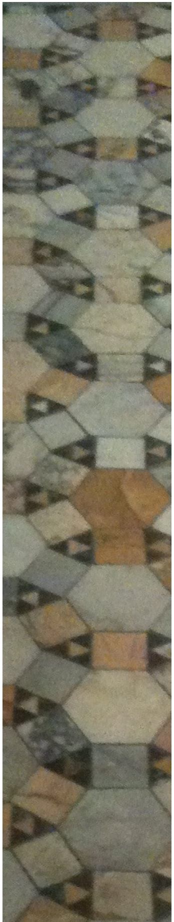


# C (Pre)sheaves and intuitionistic predicate calculus



## C.1. Deduction rules

Intuitionistic existential statements are stronger:
$\neg\forall x \neg P(x) \nvdash \exists x P(x)$
(rules as classical rules, limited by structural rules).



## C.2. (Pre)sheaves

Idea: continuous variation of an *algebraic structure.*
Topology or Grothendieck topology $\mathcal{T}$.
Pre sheaf: $u \in \mathcal{T} \mapsto M_u$ is a contravariant function from the topology with inclusion morphisms to the category of structures: when $v \subset u$ there is a morphism $\rho_{u,v} : M_u \to M_v$ (functoriality: $\rho_{u_3,u_2} \circ \rho_{u_1,u_2} = \rho_{u_1,u_3}$ whenever is makes sense, i.e. $u_3 \subset u_2 \subset u_1$).

Example of pre-sheafs on the topological space $R$:
$B : u \mapsto B(u, R)$ the ring of bounded functions from $u$ to $R$.
$C : u \mapsto C_u$ the ring of bounded functions from $u$ to $R$.

## C.3. Sheaves

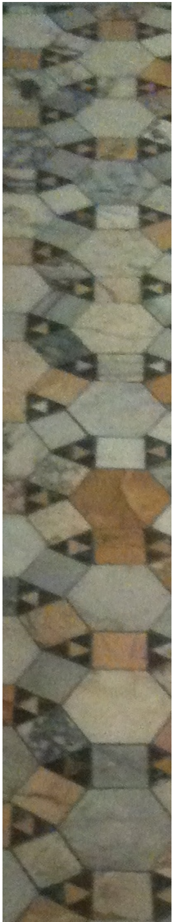
The presheaf is said to be a sheaf if every family of compatible elements has unique glueing:

> given a cover $U_i$ of an open set $U$,
>
> with for every $i$ an element $c_i \in M_{u_i}$ such that
>
> > for every pair $i, j$ $\rho_{u_i,u_j}(c_i) = \rho_{u_j,u_i}(c_j)$ there is a unique $c$ in $M_u$ such that $c_i = \rho_{u,u_i}(c)$.

Example of sheaf on the topological space $R$:
$u \mapsto C(u, R)$ the ring of continuous functions from $u$ to $R$.
Presheaf, but not a sheaf $u \mapsto B(u, R)$ the ring of bounded functions from $u$ to $R$.

## C.4. Sheaf of $\mathcal{L}$-structures

Classical model of "group" : a group.
Intuitionistic model of "group" : a sheaf of groups.
Additional property: For any $n-$ary relation symbol $R$, for any tuple $(a_1, \ldots, a_n)$ from $M_u$
if for a cover $u_i$ of $u$ such that $(a_1^i, \ldots, a_n^i) \in R_{u_i}$ for all $i \in \mathcal{I}$ then $(a_1, \ldots, a_n) \in R_u$.



## C.5. Presheaf semantics: Kripke-Joyal forcing — 1/3 atoms and conjunction

Formulas of $\mathcal{L}$ can be inductively interpreted on an object $u$ of a given presheaf model $M$ ($\nu$: assignment into $M_u$):

- $u \Vdash_\nu R(t_1, \ldots, t_n)$ iff $([t_1]_\nu, \ldots, [t_n]_\nu) \in R_u$.
- $u \Vdash_\nu t_1 = t_2$ iff $\quad [t_1]_\nu = [t_2]_\nu$.
- $u \Vdash_\nu \bot$ iff $u \lhd \emptyset$ $(M_\emptyset \Vdash \bot)$
- $u \Vdash_\nu \phi \wedge \psi$ iff $u \Vdash_\nu \phi$ and $u \Vdash_\nu \psi$.

## C.6. Presheaf semantics: Kripke-Joyal forcing — 2/3 disjunction and existential

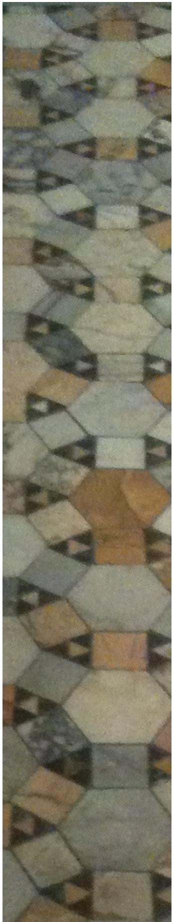
Formulas of $\mathcal{L}$ can be inductively interpreted on an object $u$ of a given presheaf model $M$ ($\nu$: assignment into $M_u$):

- $u \Vdash_\nu \phi \vee \psi$ iff there is a covering $u_i$ of $u$ such that for any $i \in \mathcal{I}$ we have $u_i \Vdash_\nu \phi$ or $u_i \Vdash_\nu \psi$.
- $u \Vdash_\nu \exists x \phi$ iff iff there is a covering $u_i$ of $u$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathcal{I}$ such that $u_i \Vdash_{\nu[x \mapsto a_i]} \phi$ for any index $i$.

## C.7. Presheaf semantics: Kripke-Joyal forcing — 3/3 implication and universal

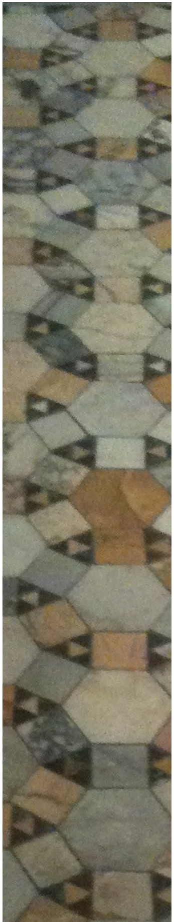
Formulas of $\mathcal{L}$ can be inductively interpreted on an object $u$ of a given presheaf model $M$ ($\nu$: assignment into $M_u$):

- $u \Vdash \phi \to \psi$ iff for all $f : v \to u$, if $v \Vdash \phi$ then $v \Vdash \psi$.
- $u \Vdash \neg\phi$ iff for all $f : v \to u$, with $v \neq \emptyset$, $v \nVdash \phi$.
- $u \Vdash_\nu \forall x\phi$ iff for all $f : v \to u$ and all $a \in M_v$, $v \Vdash_{\nu[x \mapsto a]} \phi$.



## C.8. Properties of Kripke-Joyal forcing

Functoriality of $\Vdash$:
if $u_i \subset u_j$ and $u_j \Vdash F(t_1, \ldots, t_n)$ then $u_i \Vdash F(t_1^i, \ldots, t_n^i)$
where $t_k^j$ is simply the restriction of $t_k$ to $u_i$.
Locality of validity:
we asked for the validity of atoms to be local, but Krike-Joyal forcing propagates this property to all formulae:
If there exist a covering $u_i$ of $u$ and if for all $i$ one has $u_i \Vdash F(t_1^i, \ldots, t_n^i)$ then $u \Vdash F(t_1, \ldots, t_n)$

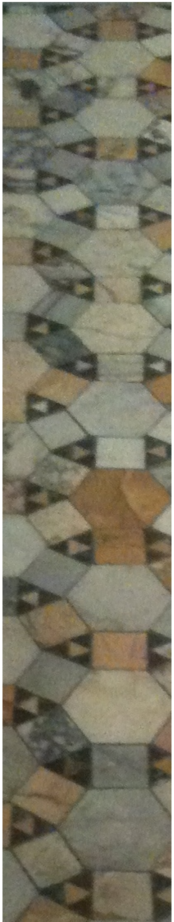


## C.9. Soundness

Whenever $\vdash F$ in IQC then any presheaf semantics satisfies $F$.
Whenever $\Gamma \vdash F$ in IQC then any presheaf semantics that satisfies $\Gamma$ satisfies $F$ as well.

## C.10. Soundness and Completeness

IQC proves $F$ iff $X \Vdash \top$

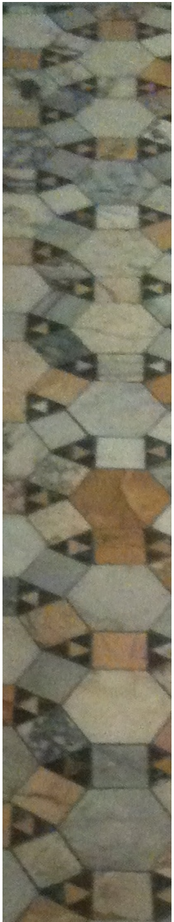


## C.11. A remark on $C_{\mathbb{R}}$

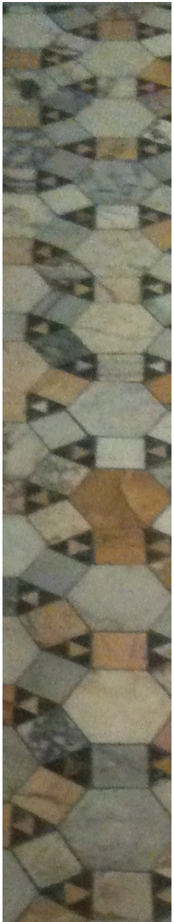
$]a, b[\not\Vdash (\ell = 0 \vee \neg(\ell = 0))$
with $\ell(x) = 0$ for $x \in ]a, (a + b)/2[$,
and $\ell(x) = x - (a + b)/2$ for $x \in ](a + b)/2, b[$.
Indeed, otherwise there would exist $u_1$, $u_2$ with $u_1 \cup u_2 = ]a, (a + b)/2[$, such that $\ell(x_1) = 0$ for all $x_1$ in $u_1$ and $\ell(x_2) \neq 0$ for all $x_2$ in $u_2$.
This is impossible because $(a + b)/2$ ought to be either in $u_1$ or in $u_2$.
If $(a+b)/2$ is in $u_1$ then $\ell$ should be constantly 0 around $(a + b)/2$.
If $(a + b)/2$ is in $u_2$ then $\ell$ should not be constantly 0 around $(a + b)/2$.

## C.12. Example

Let $A[f] = (f = 0 \vee \neg(f = 0))$ $C_{\mathbb{R}}$ the sheaf of rings of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ validates both:
(1) $\neg\forall f \ A[f]$
(2) $\forall f \ \neg\neg A[f]$ (provable)
Remark: $\neg\neg(p \vee \neg p)$ is provable in IQC hence (2) is provable and true in any particular sheaf model.
Let us see that $\neg\forall f \ A[f]$ is true in $C_{\mathbb{R}}$
$\neg\forall f \ A[f]$ is true in $C_{\mathbb{R}}$ according to Kripke Joyal forcing means that for any $u \neq \emptyset$ $u \not\Vdash \forall f \ A[f]$ there exists $v \subset u$ and $f$ a continuous function on $v$ such that $v \not\Vdash A[f]$.
Since $u$ is a non empty open of R $u$ contains $v = ]a, b[$, and $f = \ell$ defined above shows that $\neg\forall f \ A[f]$ is true in $C_{\mathbb{R}}$.
This shows that $(1) \to \neg(2)$ is not true intuitionistic logic (while $\neg(2)$ and (1) are classically equivalent.



# D Topological models of logic with a provability modality



# D.1. Gödel-Löb logic

$\Box\varphi$ : T (including PA) proves $\varphi$.
$\Diamond\varphi$ : $\neg\Box\neg\varphi$

## Language:

$$p \qquad \neg\varphi \qquad \varphi \land \psi \qquad \Box\varphi$$

## Axioms:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

## Second incompleteness theorem:

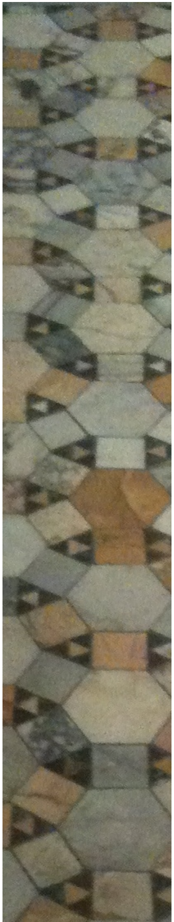
$$\Box\Diamond\top \to \Box\bot$$

## D.2. Arithmetical interpretation

An arithmetical interpretation assigns a formula $p^*$ in the language of arithmetic to each propositional variable $p$.

- $p \mapsto p^*$
- $\Box\varphi \mapsto \exists x\, \mathtt{Proof}_{\mathsf{PA}}(x, \ulcorner\varphi^*\urcorner)$

**Theorem 1** (Solovay). *If* GL $\vdash \varphi$ *if and only if, for every arithmetical interpretation* $*$, PA $\vdash \varphi^*$.



## D.3. Topological semantics:

- GL-spaces: scattered topological spaces $\langle X, \mathcal{T} \rangle$

  Scattered: Every non-empty subset contains an isolated point.

- Valuations: $dA$ is the set of limit (or accumulation) points of $A$.

$$[\![ \Diamond \varphi ]\!] = d[\![ \varphi ]\!]$$

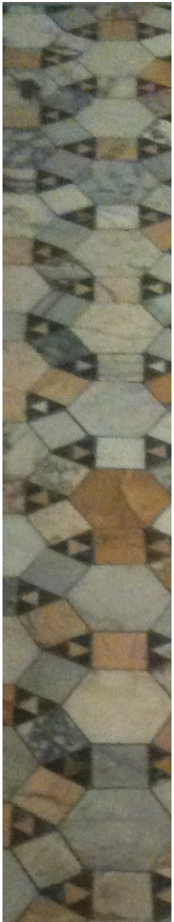
GL is also sound and complete for this interpretation.

## D.4. Some scattered spaces

- A finite partial order $\langle W, < \rangle$ with the downset topology
- An ordinal $\xi$ with the initial segment topology
- An ordinal $\xi$ with the order topology

Non-scattered:

- The real line
- The rational numbers
- The Cantor set



## D.5. Ordinal numbers

Ordinals serve as canonical representatives of well-orders.

Well-order: Structure $\langle A, \preccurlyeq \rangle$ such that

- $A$ is any set,
- $\preccurlyeq$ is a linear order on $A$, and
- if $B \subseteq A$ is non-empty, then it has a $\preccurlyeq$-minimal element.

The class Ord of ordinals is itself well-ordered:

$$\xi \leq \zeta \Leftrightarrow \xi \subseteq \zeta.$$

Examples:

- Every interval $[0, n)$ is an ordinal for $n \in \mathbb{N}$.
- The set of natural numbers can itself be seen as the first infinite ordinal, and is denoted $\omega$.

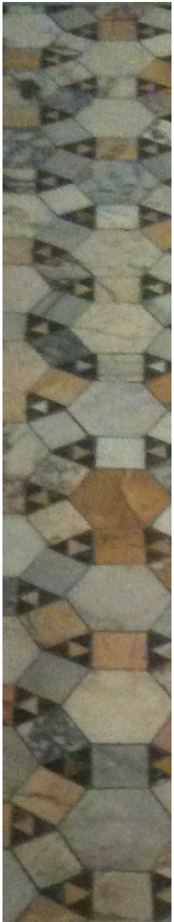
## D.6. Ordinal topologies

Intervals on ordinals are defined in the usual way, e.g.

$$[\alpha, \beta) = \{\xi : \alpha \leq \xi < \beta\}.$$

- Initial topologies: Topology $\mathcal{I}_0$ on an ordinal $\Theta$ generated by sets of the form $[0, \alpha)$.

- Interval topologies: Topology $\mathcal{I}_1$ on an ordinal $\Theta$ generated by sets of the form $[0, \alpha)$ and $(\alpha, \beta)$.



## D.7. Iterated derived sets

Recall that if $\langle X, \mathcal{T} \rangle$ is any topological space and $A \subseteq X$, $dA$ denotes the set of limit points of $A$.

If $\xi$ is an ordinal, define $d^{\xi}A$ recursively by:

1. $d^0 A = A$
2. $d^{\zeta+1}A = dd^{\zeta}A$
3. $d^{\lambda}A = \bigcap_{\zeta<\lambda} d^{\zeta}A$ ($\lambda$ a limit).

**Theorem 2.** *The following are equivalent:*

- $\langle X, \mathcal{T} \rangle$ *is scattered*
- *there exists an ordinal* $\Lambda$ *such that* $d^{\Lambda}X = \varnothing$*.*

## D.8. Ranks on a scattered space

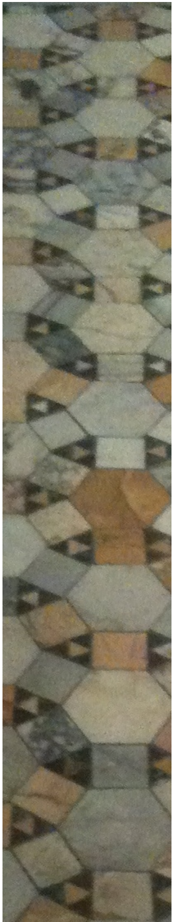
Let $\mathfrak{X} = \langle X, \mathcal{T} \rangle$ be a scattered space.

- Define $\rho(x)$ to be the least ordinal such that $x \notin d^{\rho(x)+1}X$.
- Define $\rho(\mathfrak{X})$ to be the least ordinal such that $d^{\rho(\mathfrak{X})}X = \varnothing$.

Fact: The rank on $\langle \Theta, \mathcal{I}_0 \rangle$ is the identity.

Henceforth:

- $\rho_0$ is the rank with respect to $\mathcal{I}_0$
- $\rho_1$ is the rank with respect to $\mathcal{I}_1$.



## D.9. Completeness

Observation:

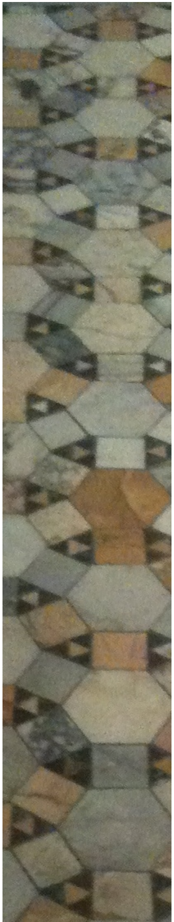
- The initial topology validates

  $$\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p).$$

- Any space of rank $n < \omega$ validates $\Box^{n+1}\bot$.

- The first ordinal with infinite $\rho_1$ is $\omega^\omega$.

**Theorem 3** (Abashidze, Blass). *If* $\Theta \geq \omega^\omega$, *then* GL *is complete for* $\langle \Theta, \mathcal{I}_1 \rangle$.



Conclusion

It's quite difficult to give an overview on the connection between logic and topology in 15'.

We hope the talk will suggest some discussions during the logic colloquium.

Some of those connections are quite active, and that's the most important.