Proof nets without links

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1. Multiplicative linear logic

\[ \mathcal{G} ::= P \mid \mathcal{G} \& \mathcal{G} \mid \mathcal{G} \otimes \mathcal{G} \mid \mathcal{G} \rightarrow \mathcal{G} \]

De Morgan laws:

\[
\begin{align*}
(A^\bot)^\bot & \equiv A \\
(A \& B)^\bot & \equiv (B^\bot \otimes A^\bot) \\
(A \otimes B)^\bot & \equiv (B^\bot \& A^\bot)
\end{align*}
\]

Deductive system:
\[ \Theta, A, B, \Gamma \vdash \Delta \]
\[ \Theta, B, A, \Gamma \vdash \Delta \]

\[ XT_l \]

\[ axiom \]
\[ A \vdash A \]

\[ \Gamma \vdash \Delta, A \]
\[ A^\perp, \Gamma \vdash \Delta \]

\[ \perp_l \]

\[ \Gamma \vdash \Delta, A \]
\[ A^\perp, \Gamma \vdash \Delta \]

\[ \perp_r \]

\[ \Gamma \vdash \Delta, A \]
\[ \Gamma \vdash \Delta, A^\perp \]

\[ \text{exchange} \]
\[ \Gamma \vdash \Delta, A, B, \Psi \]
\[ ET_r \]

\[ \Gamma \vdash \Delta, X, \Gamma \vdash \Delta' \]

\[ \text{cut} \]
\[ \Gamma, \Gamma' \vdash \Delta, \Delta' \]

\[ \Gamma \vdash \Delta, A \]
\[ \Gamma \vdash \Delta, B \]
\[ \Gamma' \vdash \Delta', \Psi \]
\[ \text{exchange} \]
\[ \Gamma \vdash \Delta, B, A, \Psi \]

\[ \text{conjunction} \]
\[ A, B, \Gamma \vdash \Delta \]
\[ A \otimes B, \Gamma \vdash \Delta \]

\[ \otimes_l \]

\[ \Gamma \vdash \Delta, A, B \]
\[ \Gamma \vdash \Delta, A \circ B \]

\[ \circ_r \]

\[ \Gamma \vdash \Delta, A \circ B, \Gamma' \vdash \Delta', A \circ B \]
\[ \circ_l \]

\[ \Gamma \vdash \Delta, B, \Gamma' \vdash \Delta', B \]
\[ \Gamma \vdash \Delta, A \circ B \]

\[ \neg \circ_l \]

\[ \Gamma \vdash \Delta, A \circ B \]
\[ \Gamma' \vdash \Delta, A \circ B \]

\[ \neg \circ_r \]
2. **An equivalent but simpler calculus**

For the following reasons:

- de Morgan laws

- implication is definable:
  \[ A \circ B \equiv A^\perp \varnothing B \]

- complex axioms are derivable
The language can be restricted to:

\[ \mathcal{F} ::= P \mid P^\perp \mid \mathcal{F} \otimes \mathcal{F} \]

with the following rules:

\[
\begin{align*}
\vdash \Gamma, A, B & \quad XT \\
\vdash \Gamma, B, A & \quad \text{axiom} \\
\vdash p, p^\perp & \quad \text{axiom} \\
\vdash \Gamma, A & \quad \text{XC} \\
\vdash \Gamma, A, \Gamma & \quad \text{cut} \\
\vdash \Gamma, A \otimes B, \Gamma' & \quad \otimes \\
\vdash \Gamma, A \otimes B, \Gamma' & \quad \phi \\
\end{align*}
\]
3. Proof nets with links

- abstract mathematical structure independent from the names of the formulae
- two equivalent definitions:
  - derivational (generative, inductive, existential)
  - representational (model theoretic, global, universal)
- contexts are not copied
- quotient proof trees by permuting rules
- more efficient cut-elimination procedure (local, sharing)
- allows new proof search (parsing?) techniques
Formulae: blue edges (perfect matching)

Connectives: red edges

Criterion: no $\mathcal{A}\mathcal{E}$ cycle

$\mathcal{A}\mathcal{E}$ path/cycle alternate elementary path/cycle:
a blue edge, a red edge, a blue edge, a red edge, .... not crossing itself

Liens

<table>
<thead>
<tr>
<th>Name</th>
<th>axiom link</th>
<th>par link</th>
<th>tensor link</th>
</tr>
</thead>
<tbody>
<tr>
<td>Premisses</td>
<td>none</td>
<td>$A \text{ et } B$</td>
<td>$A \text{ et } B$</td>
</tr>
<tr>
<td>Conclusions</td>
<td>$a \text{ et } a^\perp$</td>
<td>$A \text{ et } B$</td>
<td>$A \text{ et } B$</td>
</tr>
</tbody>
</table>

edge bicolored graph

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3.1. From proofs to graphs

Proof $\rightarrow$ graphs endowed with a perfect matching (by induction)

Grosso modo: sub-formula tree + axioms.

Notice that

- no $\mathcal{AE}$ cycle

- two vertices are always connected via an $\mathcal{AE}$ path

one does not have MIX : $\not\vdash \Gamma, \Delta \vdash MIX(A \otimes B) - \circ(A \varnothing B)$
Example: \(((a \varnothing a^\perp) \otimes (b \varnothing b^\perp)) \otimes (c \varnothing c^\perp)\):
3.2. From graphs to proofs

3.2.1. Lemma The following properties are equivalent:

1. no $\emptyset$ cycle
2. the perfect matching is unique
3. the graph recursively contains a blue bridge
• $\neg 2 \Rightarrow \neg 1$ easy

• $\neg 1 \Rightarrow \neg 2$ easy
3 ⇒ 1 obvious

1 ⇒ 3

extend as much as possible an $\mathcal{AE}$ path

- pending bridge
- loop

- $\mathcal{AE}$ path in the reduced graph ⇒ $\mathcal{AE}$ path in the initial graph
- bridge in the reduced graph ⇒ bridge in the initial graph
- by induction on the size of the graph
3.2.2. Proof net $\rightarrow$ proof

- One can assume w.l.o.g. that all conclusions of the proofs are conjunctions: (final disjunctions are easy to handle).

- One takes off the final conjunctions, except the red edges between the two premisses.
  - there exists a blue bridge (lemma)
    - If the bridge is one of the premisses of a conclusion conjunction the conjunction rule applies
    - Otherwise the blue bridge is like this:
      By induction hypothesis we have:
■ a proof $\delta$ of $\vdash A^\perp, U, V, W$ one axiom of which is $\vdash A, A^\perp$

■ a proof $\gamma$ of $\vdash X, Y, A$.

\[ \vdash X, Y, A \]

\[ \vdash \gamma \]

\[ \vdash [A^\perp := X, Y], A \]

\[ \vdash \cdots \]

\[ \vdash \cdots \]

\[ \vdash [A^\perp := X, Y], U, V, W \]
4. Proof nets without links

Idea: identifying more proofs, but not all of them.

Algebraic properties of the connectives:

- **Associativity**

  - $A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C$
  
  - $A \varnothing (B \varnothing C) \equiv (A \varnothing B) \varnothing C$

- **Commutativity**

  - $A \otimes B \equiv B \otimes A$
  
  - $A \varnothing B \equiv B \varnothing A$
Sequent or disjunction of the formulae in a sequent \( \rightarrow \) cograph

- \( p \rightarrow \) vertex (no edges)
  vertices = propositional variables

- \( A \varnothing B \rightarrow \) disjoint union

- \( A \otimes B \rightarrow \) series composition

Example: \((a \varnothing a\bot) \otimes (b \varnothing b\bot) \otimes (c \varnothing c\bot)\)
4.1. From proofs to graphs

Inductive definition:

- axiom $\rightarrow$ blue edge

- $\emptyset$ rule $\rightarrow$ no effect
\( \otimes \) rule \( \rightarrow \) we select a family of connected components in each cograph and we compose them in the series mode

\[
A = p \lor (a \land (b \lor c)) \\
B = (x \land (y \lor z)) \lor t
\]
Proof net = cograph + perfect matching

- every $\mathcal{AE}$ cycle contains a chord

- between any two vertices there exists a chordless $\mathcal{AE}$ path (no MIX)

Easy to prove by induction on the number of rules.
4.2. From graphs to proofs

One way to prove this is to go back to the previous case, little by little, by considering intermediate structures

- axioms

- subformulae trees of the conclusions

- cographs whose vertices are conclusions

\[(a \triangleleft a^\perp) \otimes (b \triangleleft b^\perp) \otimes (c \triangleleft c^\perp) \tag{correct}\]
The transformation and its inverse preserve the criterion:

- every $\mathcal{AE}$ cycle contains a chord

- between any two vertices, there exists a chordless $\mathcal{AE}$ path

Since for a proof net with links there cannot be any chord in an $\mathcal{AE}$ path, we have:

every graph satisfying the criterion corresponds to (at least) one proof.
4.3. A definition of proofs by proof net rewriting

Every proof nets is obtained from the complete bicolored graph

\[ \land_{i \in I} (a_i \varnothing a_i^\perp) \quad I \text{ multiset} \]

by the following rewrite rule (modulo commutativity and associativity):

\[ X \otimes (Y \varnothing Z) \rightarrow (X \otimes Y) \varnothing Z \]

If MIX is allowed, add the following rule:

\[ X \otimes Y \rightarrow X \varnothing Y \]
5. Proof Nets for Lambek Calculus

5.1. Reminder: Proof Nets with Links
(Abrusci & Maringelli, 1998, JoLLI)

Red edges of links are **directed**.
Criterion:

- the underlying undirected graph is a correct MLL proof net (every $\&\exists$ path contains a chord)

- there always exist an $\&\exists$ path from the right premise of a disjunction to its left premise

- there is a black cyclic order between the conclusions, and for every black arc $x \rightarrow y$ there is a directed $\&\exists$ path from $y \longrightarrow x$

Idea: similar black arcs for $\emptyset$ links, that we unfold.
5.2. Non commutative proof nets without links

5.2.1. Directed cographs

- Disjoint union

- Directed series composition between \((V, A)\) and \((V', A')\)

  - vertices \(V \uplus V'\)

  - arcs\(= A \cup A' \cup (V \times V')\) (and not \(A \cup A' \cup (V \times V') \cup (V' \times V)\))

  - \(V\) is called the first component and \(V'\) the second component of the directed series composition.
5.2.2. **Cyclic orders**  A total cyclic order is a ternary relation $C(x, y, z)$ (moving from $x$ to $z$ in the right direction, one meets $y$):

- $C(x, y, z) \Rightarrow C(y, z, x)$

- $C(x, y, z) \land C(y, u, z) \Rightarrow C(x, y, u) \land C(x, u, z)$

- $C(x, y, z) \lor C(z, y, x)$

The interval $[a, b]$ is $\{ z \mid C(a, z, b) \}$.
5.2.3. K-graph of a formula

\(K(a) \quad a \in A:\)

- **vertices:** \(\{a\}\)
- **\(N\)-arcs** \(\emptyset\)
- **\(R\)-arcs** \(\emptyset\)

\(K(X \otimes Y)\)

- **vertices:** \(V(F) = V(X) \cup V(Y)\)
- **\(N\)-arcs** \(N(K(F)) = N(K(X)) \cup N(K(Y))\)
- **\(R\)-arcs** \(R(K(F)) = R(K(X)) \cup R(K(Y)) \cup V(X) \times V(Y)\)

\(K(X \varnothing Y)\)

- **vertices:** \(V(F) = V(X) \cup V(Y)\)
- **\(R\)-arcs** \(R(K(F)) = R(K(X)) \cup R(K(Y))\)
- **\(N\)-arcs** \(N(K(F)) = N(K(X)) \cup N(K(Y)) \cup V(X) \times V(Y)\)

Total order on the atoms.
Underlying graph: Komplet graph.
5.2.4. **Sequent structure** \( \vdash A_1, \ldots, A_n \)

Cyclic order on the family of \( n \geq 1 \) \( K \) graphs \( K(A_1), \ldots, K(A_n) \), materialised by a black arc from the last vertex in \( K_i \) to the first vertex in \( K_{i+1}[n] \) **whose main connective is an** \( R \) **series-composition**. \( (\otimes) \).

In particular, if there is a single \( K_1 \) then there is an edge from the last vertex in \( K_1 \) to the first vertex in \( K_1 \).

Notice that one has a unique Hamiltonian circuit (total order + cyclic order).

The graph is invariant with respect to associativity, but makes a difference between comma and \( \emptyset \). (Otherwise the criterion is slightly more complicated).
5.3. Cyclic proof nets without links

Proof structure:

- a sequent structure (set of cyclically ordered K graphs)
- a set of B-edges which are a perfect matching.

Correctness criterion:

**MLL proof-net** the underlying undirected $BR$-graph is an MLL proof-net, that is to say:

- **acyclicity** Every alternate elementary cycle contains a chord.
- **connectedness** There exists a chordless alternate elementary path between any two vertices.

**Hamiltonian adequacy** $B$-edges are adequate to the Hamiltonian circuit: that is whenever $ab'a', bb'b'$ and $H(a, b, a')$ one has $H(a, b', a')$ as well.
5.3.1. **Alternative definition**  One black arc per \( \varnothing \) composition (from the last atom of the first component to the first atom of the second component) \( L(A) \) Leight.

The Hamiltonian circuit makes use of all black arcs.

Advantage: less arcs, and there is no difference between commas and \( \varnothing \).

Disadvantage: in this case, the criterion has a supplementary condition: each R series composition is an interval of the cyclic order.
6. Conclusion

- **Pomset logic** Similar notion of proof net for an extension with a non commutative and autodual connective (further studied by Alessio Guglielmi as a term calculus). This connective corresponds to directed series composition.

- **Redundancy?** Can the criterion and/or structure can be simplified?

- **What about cuts?** Undirected even in the non commutative case.

- **Mixed system** What about partially commutative linear logic (de Groote, 96) and its extension, Non-Commutative logic (Ruet 1999, Abrusci-Ruet 2000)

- **Algorithms** Can proof nets without links be used for optimised proof net construction as in Moot’s proof search?