Conjugacy and episturmian morphisms *

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Abstract

Episturmian morphisms generalize Sturmian morphisms. Here, we study some intrinsic properties of these morphisms: invertibility, presentation, cancellativity, unitarity, characterization by conjugacy. Most of them are generalizations of known properties of Sturmian morphisms. But we present also some results on episturmian morphisms that have not already been stated in the particular case of Sturmian morphisms: characterization of the episturmian morphisms that preserve palindromes, new algorithms to compute conjugates.

We also study the conjugation of morphisms in the general case and show that the monoid of invertible morphisms on an alphabet containing at least three letters is not finitely generated.

Keywords. Combinatorics on words, Sturmian morphisms, conjugacy, presentation, invertible morphisms.

Résumé

Les morphismes épisturmiens généralisent les morphismes sturmiens. Ici, nous étudions des propriétés intrinsèques de ces morphismes : inversibilité, unitarité, régularité, caractérisation par conjugaison. La plupart de ces résultats sont des généralisations de résultats connus sur les morphismes sturmiens. Mais nous présentons aussi des résultats non encore établis pour les morphismes sturmiens : caractérisations des morphismes épisturmiens qui préservent les palindromes, nouveaux algorithmes pour calculer des conjugués.

Nous donnons également des résultats généraux sur la conjugaison et montrons que le monoïde des morphismes inversibles sur un alphabet d'au moins trois lettres n'est pas finiment engendré.

Mots Clés. Combinatoires des mots, morphismes sturmiens, conjugaison, présentation, morphismes inversibles.

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Contents

1 Introduction

Since the works of Morse and Hedlund [13], Sturmian words have been widely studied (see [2] for a recent survey). These infinite words, that are defined over a two-letter alphabet, have a lot of equivalent definitions. When larger alphabets are considered, these definitions give different generalizations of Sturmian words (see for instance [1, 3, 4, 7, 8, 10, 11, 14]). *Episturmian words* is one of these generalizations [7, 9], and it partially coincides with previous generalizations [1, 4].

Sturmian (endo)morphisms are defined on two-letter alphabets. They were initially introduced as the morphisms which preserve Sturmian words. In [15], Séébold prove that the monoid of Sturmian morphisms is generated by the exchange (of the two letters) morphism and two other morphisms (L and R). In [7, 9, 10], working on alphabet of arbitrary size, Justin et al. call episturmian the (endo)morphisms generated by the permutations and a family of morphisms that generalizes L and R. One can note that these morphisms already appear (even not explicitly) in some works around generalization of Sturmian words [1, 4, 14]. In [10], Justin and Pirillo show that episturmian morphisms are the morphisms that preserve episturmian words. The reader will find a recent survey on Sturmian morphisms in [2]. In this survey, a lot of properties of Sturmian morphisms concern relations with Sturmian words. Our aim is to present a study of episturmian morphisms. *We state only intrinsic properties of episturmian morphisms (without any reference to episturmian words).* We generalize results given in [2] and give new results. Note that we state all our results using only two morphisms (that we call also L and R) from the family of morphisms introduced in [7, 9, 10]. These results can be easily adapted to use all the family.

The paper is constituted of nine sections (except this introduction). In Section 2 we recall basic notions and results on words and morphisms. We present definitions of episturmian morphisms and of the special case of permutations. We recall a presentation of the monoid of permutations with the exchange morphisms as generators.

Most of the following sections deal with conjugacy. The notion of conjugation of Sturmian morphisms was introduced by Séébold [16]. On two-letter alphabets, the definition of conjugation is a bit different than the notion of conjugacy given in [2] but the ideas are the same, and we obtain similar results. Conjugacy can be easierly generalized to arbitrary alphabets than conjugation. In Section 3, we present general results on conjugacy of arbitrary morphisms. In particular we study the number of conjugates of a morphism stating a linear algorithm to compute it. We show also that the conjugates of a morphism can be naturally ordered.

Epistandard morphisms are particular episturmian morphisms (they are defined in Section 2) that generalizes the notion of standard morphisms which play a central role in the study of characteristic Sturmian words (see [2] for instance). It is a basic property (stated in Section 2) that an epistandard morphism f defined on an alphabet A verifies: each letter of A is the final letter of the image of a word by f. Among all the morphisms that verify this property, we show in Section 4 that epistandard morphisms are characterized by their number of (right) conjugates. Moreover this number is maximal in the family.

In Section 5 we state algorithms to compute the first and the previous conjugate of a given episturmian morphism (first and previous in accordance with the order on conjugates). As a consequence we generalize a result from Séébold [16]: *a morphism is episturmian if and only if it is the conjugate of an epistandard morphism.*

In Section 6, we study the number of conjugates of an episturmian morphism f (not necessarily epistandard). We relate this number to any decomposition of f over exchange morphisms, L and R.

The algorithms studied in Section 5 show that an episturmian morphism can have different decompositions over exchange morphisms, L and R. In Section 7, generalizing a result from Séébold [15], we state a presentation of the monoid of episturmian morphisms. As an intermediary result, we show that this monoid is both left and right cancellative.

In Section 8, we study relations between palindromes and episturmian morphisms. In particular, we characterize the episturmian morphisms that preserve palindromes. These relations explain in some way the presentation given in Section 7.

In Section 9, as done in [2] for Sturmian morphisms, we show that the monoid of episturmian morphisms is both right and left cancellative. As in [2], we deduce an algorithm to determine if a morphism is episturmian. and, if it is, to compute a decomposition of this morphism over exchange morphisms, L and R.

We end this paper by investigating relations between episturmian morphisms and invertible morphisms. In the case of two-letter alphabets, Wen and Wen [17] show that a morphism is invertible if and only if it is Sturmian. In Section 10, we show that episturmian morphisms are invertible, but on three-letter (or larger) alphabets the converse is not true. We give a deep reason for this. The monoid of invertible morphisms is not finitely generated (for alphabets containing at least three letters). In [17], Wen and Wen use inner automorphisms. We show a relation between these group morphisms and conjugacy.

2 Words and morphisms

The aim of this section is to introduce episturmian morphisms (see Section 2.3). Before this, after generalities on words and morphisms given in the next section, we recall in Section 2.2 properties on permutations which are particular episturmian morphisms.

2.1 Basic definitions

Given a finite set X, we will denote by $\#X$ its cardinal, that is, the number of its elements. A *monoid* M is a set equipped with an associative internal operation and a neutral element e for this operation. We denote the internal operation by juxtaposition. Given a subset X of M, the iteration X^* of X is the subset of M constituted of e and of all elements $x_1 \dots x_n$ with $n \geq 1$, $x_1, \dots, x_n \in X$. Sometimes, we will compute elements $x_k \dots x_l$ with $l < k$; in this case $x_k \dots x_l = e$.

A set of *generators* of M is a subset G of M such that any element of M can be decomposed in elements of G. The monoid M is said *finitely generated* if it has a finite set of generators. A *presentation* of the monoid M (based on a set of generators G) is a set S of equalities between elements of G such that given any elements $f_1, \dots, f_n, g_1, \dots, g_p$ in G, the equality $f_1 \dots f_n = g_1 \dots g_p$ holds if and only if it can be stated using the equalities of S (and the trivial relations $em = me = m$ for any element m of M). When there is no relation between generators, the monoid is said *free*.

An *alphabet* A is a set of symbols called *letters*. Here we consider only finite alphabets. A *word* over A is a finite sequence of letters from A. The *empty word* ε is the empty sequence of letters. Equipped with the concatenation operation, the set A^* of words over A is a free monoid with neutral element ε and set of generators A. Given a non-empty word $u = a_1 \dots a_n$ with $a_i \in A$, the *length* $|u|$ of u is the integer n. One has $|\varepsilon| = 0$. For a word u and a letter a, $|u|_a$ is the number of occurrences of a in u. If for some words u, v, p, s (possibly empty), $u = pvs$, then v is a *subword* of u, p is a *prefix* of u and s is a *suffix* of u. Given a non empty word $u = a_1 \dots a_n$ with a_i in A, we will denote the *last* letter a_n of u by $\mathrm{last}(u)$. Two words u and v are said *conjugate* if there exists a word w such that $uw = wv$. Powers of a word are defined inductively by $u^0 = \varepsilon$, and for any integer $n \geq 1$, $u^n = uu^{n-1}$. Observe that the set of powers of u is $\{u\}^*$. A word u is said *primitive* if for any word w, the equality $w = u^n$ implies $n = 1$. Any non-empty word u can be uniquely written z^n with z primitive: z is called the *primitive root* of u. We recall

Lemma 2.1 [12, Proposition 1.3.2] *For two words x and y, the two following assertions are equivalent.*

- $xy = yx$.
- *There exists a primitive word z with $x, y \in \{z\}^*$.*

A *morphism* f on A is an application from A^* to A^* such that for all words u, v, $f(uv) = f(u)f(v)$. A morphism is entirely known by the images of the letters of A. The *empty morphism* ϵ is the morphism such that for each letter x in A, $\epsilon(x) = \varepsilon$. We will denote $||f|| = \sum_{x \in A} |f(x)|$. Given two morphisms f and g, we will denote fg their composition. For a subset X of A^*, $f(X)$ is the set $\{f(x) \ / \ x \in X\}$.

A morphism f is called *periodic* (this name is given in [5]; in [12] such morphisms are called *cyclic*) if there exists a word z such that $f(A) \subseteq \{z\}^*$. The empty morphism is periodic. For a non-empty periodic morphism there exists a unique shorter word z with $f(A) \subseteq \{z\}^*$. This word is primitive and it is called the *period* of f.

2.2 Permutations

In this part, we recall notions about permutations. Given an alphabet A, a *permutation* P on A is a morphism on A such that $A = \{P(x) \mid x \in A\}$. This means in particular that for each letter x in A, $P(x)$ is a letter. We denote by Perm(A) the set of permutations on A. Equipped with the composition of functions, it forms a monoid.

Lemma 2.2 below states a well known fact: any permutation P can be decomposed over the permutations that exchange only two letters. Given two letters x, y, the *exchange morphism* of x and y is the morphism defined on A by

$$E_{xy} : \begin{cases} x \to y \\ y \to x \\ z \to z, \quad \forall z \notin \{x, y\} \end{cases}$$

We observe that $E_{xy} = E_{yx}$. Moreover, for any $x \in A$, E_{xx} is the identity morphism (also denoted Id). We denote by Exch(A) the set of exchange morphisms defined on A (including the identity).

Given a letter x and a permutation P, the permutation $Q = E_{xP(x)}P$ verifies $Q(x) = x$ and $Q(y) = y$ for each letter y such that $P(y) = y$. If $P(x) \neq x$, we have $\#\{z/Q(z) \neq z\} < \#\{z/P(z) \neq z\}$. Thus by induction, we can prove

Lemma 2.2 *Each permutation is a composition of at most $\#A - 1$ exchange morphisms.*

Of course, such a decomposition is not unique. The following well-known result precises a presentation of the monoid Perm(A).

Lemma 2.3 *The set* Perm(A) *with set of generators* Exch(A) *has the following presentation.*
If x, y, z, t are pairwise different letters

$$E_{xy}E_{xy} = Id \tag{1}$$

$$E_{xy}E_{yz} = E_{yz}E_{zx}, \tag{2}$$

$$E_{xy}E_{zt} = E_{zt}E_{xy}, \tag{3}$$

Let us observe that, when $x = y$, Relations (1) and (3) are still valid and are trivial relations. One can also remark that if, in Relation (2), we replace x by z, y by x and z by y, we get the equivalent relation

$$E_{xy}E_{yz} = E_{zx}E_{xy}, \tag{4}$$

For the sake of completeness, we give a proof of Lemma 2.3.

Proof. By a direct computation, we can verify that Equalities (1), (2) and (3) hold.

Let $f_1, \dots, f_m, g_1, \dots, g_n$ be some exchange morphisms. Assuming that $f_1 \dots f_m = g_1 \dots g_n$, we have to show that the equality can be stated using Relations (1), (2) and (3).

If we apply n times Formula (1), we get $g_n g_{n-1} \dots g_1 f_1 \dots f_m = Id$.

Thus we have to prove that if $h_1 \dots h_p = Id$ holds for $p \geq 1$ exchange morphisms, then this equality can be stated using Relations (1), (2) and (3) (and trivial relations). First, if $p = 1$, then necessarily $h_1 = Id$. If $p \geq 2$, we can assume that each exchange morphism h_i is not the identity. Let α, β be two different letters, and let i be an integer such that $h_i = E_{\alpha\beta}$, and such that for each integer j with $\forall 1 \leq j < i$, $h_j(\alpha) = \alpha$. Observe $i < p$ else $h_1 \dots h_p(\alpha) \neq \alpha$. Only four cases are possible.

a) If $h_i = h_{i+1}$, then by Relation (1), $h_1 \dots h_p = h_1 \dots h_{i-1}h_{i+2} \dots h_p$.

b) If $h_{i+1} = E_{\alpha\delta}$ with $\delta \in A \setminus \{\alpha, \beta\}$, then $h_i h_{i+1} = E_{\alpha\beta}E_{\alpha\delta} = E_{\beta\alpha}E_{\alpha\delta}$. It follows by Relation (4) that $h_1 \dots h_p = h_1 \dots h_{i-1}E_{\delta\beta}E_{\beta\alpha}h_{i+2} \dots h_p$.

c) If $h_{i+1} = E_{\beta\delta}$ with $\delta \in A \setminus \{\alpha, \beta\}$, then $h_i h_{i+1} = E_{\alpha\beta}E_{\beta\delta}$. It follows by Relation (2) that $h_1 \dots h_p = h_1 \dots h_{i-1}E_{\beta\delta}E_{\delta\alpha}h_{i+2} \dots h_p$.

d) if $h_{i+1} = E_{\delta\gamma}$ for some letters δ, γ such that $\{\delta, \gamma\} \cap \{\alpha, \beta\} = \emptyset$, by Relation (3), $h_1 \dots h_p = h_1 \dots h_{i-1}E_{\delta\gamma}E_{\alpha\beta}h_{i+2} \dots h_p$.

Consequently one of three following situations holds.

1) We have stated $h_1 \dots h_p = Id$ (possible only in case a).

2) We have found an integer $q \geq 1$ and morphisms $h'_1, \dots h'_q$ such that $h_1 \dots h_p = h'_1 \dots h'_q$ and $\{x \in A / \exists k \in \{1, \dots, q\}, h'_k(x) \neq x\} \subseteq \{x \in A / \exists k \in \{1, \dots, p\}, h_k(x) \neq x\} \setminus \{\alpha\}$ (possible only in case a).

3) We have found an integer $r \geq 1$ and morphisms $h''_1, \dots h''_q$ such that $h_1 \dots h_p = h''_1 \dots h''_r$, $\{x \in A / \exists k \in \{1, \dots, r\}, h''_k(x) \neq x\} = \{x \in A / \exists k \in \{1, \dots, p\}, h_k(x) \neq x\}$ and $r - i' < p - i$ where i' is the integer such that $h''_{i'}(\alpha) \neq \alpha$ and $\forall 1 \leq j < i'$, $h''_j(\alpha) = \alpha$.

By induction on $p - i$, we can get only situation 1 or 2.

Finally, we can state $h_1 \dots h_p = Id$ by induction on $\#\{x \in A / \exists k \in \{1, \dots, k\}, h_k(x) \neq x\}$. □

2.3 Episturmian morphisms

Let A be an alphabet. In [7, 9, 10], Droubay, Justin and Pirillo have introduced for each letter α, the morphisms Ψ_α and $\overline{\Psi}_\alpha$

$$\Psi_\alpha\colon \begin{cases} \alpha \to \alpha \\ x \to \alpha x, & \forall x \neq \alpha \end{cases} \qquad \overline{\Psi}_\alpha\colon \begin{cases} \alpha \to \alpha \\ x \to x\alpha, & \forall x \neq \alpha \end{cases}$$

They call *epistandard morphism* any morphism obtained by composition of exchange morphisms and morphisms Ψ_α with $\alpha \in A$. In other words, an epistandard morphism is a morphism in

$$\text{Epistand(A)} = (\text{Exch(A)} \cup \{\Psi_\alpha / \alpha \in A\})^*.$$

Similarly ([9, 10]), an *episturmian morphism* is an element of

$$\text{Episturm(A)} = (\text{Exch(A)} \cup \{\Psi_\alpha, \overline{\Psi}_\alpha / \alpha \in A\})^*.$$

When $\#A = 2$, epistandard (resp. episturmian) morphisms are exactly the standard (resp. sturmian) morphisms (see [2]).

In the rest of the paper, we will always consider a finite alphabet A containing at least two letters. We will also distinguish a letter a in A. Following the original notation of Séébold [15], we denote $L = \Psi_a$ ($\Psi(a) = a$, $\Psi(x) = ax$ for $x \neq a$) and $R = \overline{\Psi}_a$ ($\overline{\Psi}(a) = a$, $\overline{\Psi}(x) = ax$ for $x \neq a$). For any letter α, we have

$$\Psi_\alpha = E_{a\alpha} L E_{a\alpha} \tag{5}$$

$$\overline{\Psi}_\alpha = E_{a\alpha} R E_{a\alpha} \tag{6}$$

Thus Epistand(A) $= (\text{Exch(A)} \cup \{L\})^*$ and Episturm(A) $= (\text{Exch(A)} \cup \{L, R\})^*$. Of course each result that implies a decomposition of an episturmian morphism over Exch(A) $\cup \{L, R\}$ can be reformulated using a decomposition over Exch(A) $\cup \{\Psi_\alpha, \overline{\Psi}_\alpha / \alpha \in A\}$.

Note that, in the particular case where $\#A = 2$, L and R are the morphisms G and $\tilde{G}$ in [2].

To end this section we observe that episturmian morphisms are injective and that we have

Lemma 2.4
A morphism f on A is epistandard if and only if f is episturmian and $A = \{\text{last}(f(x)) / x \in A\}$.

Proof. Let g, h be two non-erasing morphisms and let $B = \{\text{last}(h(x)) / x \in A\}$. We have

$$\{\text{last}(gh(x)) / x \in A\} = \{\text{last}(g(x)) / x \in B\}.$$

Moreover if $g \in \text{Exch(A)} \cup \{L\}$, $\{\text{last}(g(x)) / x \in A\} = A$. Thus by induction on n, we can show that $f_1, \ldots, f_n \in \text{Exch(A)} \cup \{L\}$ implies $\{\text{last}(f_1 \ldots f_n(x)) / x \in A\} = A$.

Conversely if f is an episturmian morphism which is not epistandard, we have $f = gRh$ with g, h two episturmian morphisms. It follows $\{\text{last}(f(x)) / x \in A\} = \{\text{last}(g(a))\} \neq A$ (because $\#A = 2$). $\square$

3 Conjugacy

In [2], one can find the following definition: a morphism g is a *right conjugate* of a morphism f defined on A, in symbols $f \lhd g$, if there exists a word w such that $f(x)w = wg(x)$ for all words x in A^*. Here, we will also say that f is a *left conjugate* of g, that w right-conjugates f (into g), and we will sometimes write $f \lhd_w g$. For instance, $L \lhd_a R$.

This section is devoted to a general study of conjugacy. After some first properties (Section 3.1), we study in Section 3.2 the number of conjugates of a morphism; we also show that the right conjugates of a morphism can be naturally ordered. We end the study with some algorithmic aspects. In Section 3.3, we show how to compute the number of right conjugates.

3.1 First properties

In this section, we show basic properties on conjugacy of morphisms.

Lemma 3.1 *Let f, f', g, g', h be some morphisms and let w_1, w_2 be some words.*

1. *If $f \lhd_{w_1} g$ and $g \lhd_{w_2} h$ then $f \lhd_{w_1w_2} h$.*
2. *If $g \neq \epsilon$, $f \lhd_{w_1} g$, $f' \lhd_{w_2} g$ and $|w_1| \leq |w_2|$, then there exists a word w_3 such that $w_2 = w_3w_1$ and $f' \lhd_{w_3} f$.*
3. *If $f \neq \epsilon$, $f \lhd_{w_1} g$, $f \lhd_{w_2} g'$ and $|w_1| \leq |w_2|$, then there exists a word w_3 such that $w_2 = w_1w_3$ and $g \lhd_{w_3} g'$.*
4. *If $f \lhd_{w_1} g$ and $f' \lhd_{w_2} g'$ then $ff' \lhd_{f(w_2)w_1} gg'$.*

Lemma 3.1 was already given (without the precision on words) in [2]. For the sake of completeness, we give its proof.

Proof.

1. For all words x in A^*, we have $f(x)w_1 = w_1g(x)$, and $g(x)w_2 = w_2h(x)$. Thus $f(x)w_1w_2 = w_1g(x)w_2 = w_1w_2h(x)$.
2. Since $g \neq \epsilon$, there exists a word w such that $|g(w)| \geq |w_2|$. From $f(w)w_1 = w_1g(w)$ and $f'(w)w_2 = w_2g(w)$, we get that w_1 and w_2 are both suffixes of $g(w)$. Since $|w_1| \leq |w_2|$, w_1 is also a suffix of w_2. So there exists a word w_3 such that $w_2 = w_3w_1$. Now for any word x in A^*, $f'(x)w_3w_1 = f'(x)w_2 = w_2g(x) = w_3w_1g(x) = w_3f(x)w_1$. Consequently $f' \lhd_{w_3} f$.
3. Proof similar to that of the previous point.
4. For any word x, we have $(ff')(x)[f(w_2)w_1] = f(f'(x))w_1g(w_2) = w_1g(f'(x))g(w_2) = w_1g(f'(x)w_2) = w_1g(w_2g'(x)) = w_1g(w_2)g(g'(x)) = [f(w_2)w_1](gg')(x)$, i.e., $ff' \lhd_{f(w_2)w_1} gg'$.

□

A consequence of Lemma 3.1(3) is that if two words right-conjugate a same non-empty morphism, one is prefix of the other. We also have

Property 3.2 *If a word right-conjugates a morphism f, all its prefixes also right-conjugate f.*

Proof. The result is immediate for the empty word. Let f, g be two morphisms, y a letter and w a word such that $f \lhd_{wy} g$. We now define a morphism h as follows. Let $x \in A$. If $f(x) = \varepsilon$ (so $g(x) = \varepsilon$), then let $h(x) = \varepsilon$. Otherwise there exists a word u such that $g(x) = uy$. In this case, let $h(x) = yu$. It is straightforward that $f \lhd_w h$. The result follows by induction. □

3.2 Numbers of conjugates

In this section, we give general results about the number of conjugates of a morphism. We also show that these conjugates can be ordered.

We denote by $\mathrm{NbR}(f)$ the number of right conjugates of a morphism f. For instance, since the right conjugates of L are L and R, and since the unique right conjugate of R is R itself, $\mathrm{NbR}(L) = 2$ and $\mathrm{NbR}(R) = 1$. For any morphism f, we have $\mathrm{NbR}(f) \geq 1$ since f is its own right conjugate ($f \triangleleft_{\varepsilon} f$). Let us also note that $\mathrm{NbR}(f)$ is always finite. Indeed if $f \triangleleft\ g$ then for each letter x the words $f(x)$ and $g(x)$ are conjugate. But $f(x)$ has at most $|f(x)|$ conjugates if $f(x) \neq \varepsilon$ and one conjugate if $f(x) = \varepsilon$. It follows that $\mathrm{NbR}(f) \leq \Pi_{x \in A} \max(|f(x)|, 1)$. In Lemma 3.5 below, we will show that this bound is quite large. First we study a special case. Observe that given a morphism f and a word w that right-conjugates f, there is only one morphism g such that $f \triangleleft_w g$. The following lemma shows that periodic morphisms are exactly those morphisms which are right-conjugated by at least two words. For these morphisms, we compute the number of right conjugates.

Lemma 3.3 *Let f be a morphism. The four following assertions are equivalent.*

1. *There exists a word u such that u right-conjugates f and $|u| \geq \mathrm{NbR}(f)$.*
2. *There exist (at least) two different words that right-conjugate f into the same morphism.*
3. *The morphism f is periodic.*
4. *Given a right conjugate g of f, there exist infinitely many words that right-conjugate f into g.*

Moreover, if f is a non-empty periodic morphism with period z, we have

a. $\mathrm{NbR}(f) = |z|$.

b. if w and w' right-conjugate f into the same morphism, then $|z| \leq ||w| - |w'||$.

Proof. The results are true for the empty morphism ($\mathrm{NbR}(\epsilon) = 1$ and $\forall w$, $\epsilon \triangleleft_w \epsilon$). Let f be a non-empty morphism.

$1 \Rightarrow 2$. Assume that there exists a word u such that u right-conjugates f and $|u| \geq \mathrm{NbR}(f)$. By Property 3.2, all the prefixes of u also right-conjugates f. The number of prefixes of u is $|u| + 1$ and $|u| + 1 \geq \mathrm{NbR}(f) + 1$. Thus (at least) two different prefixes of u right-conjugate f into the same morphism h.

$2 \Rightarrow 3$ (and b.). Assume there exist two different words w, w' that right-conjugate f into a same morphism g: $f \triangleleft_w g$, $f \triangleleft_{w'} g$. Without loss of generality, we can assume $|w| \leq |w'|$. Since $f \neq \epsilon$, we have $g \neq \epsilon$. By Lemma 3.1(2), there exists a word t such that $w' = tw$ and $f \triangleleft_t f$. Since $w \neq w'$, we have $t \neq \varepsilon$. For each letter y, $f(y)t = tf(y)$. By Lemma 2.1, $f(y)$ is a power of a primitive word z. Thus the morphism f is periodic with period z. Note that $|z| \leq |t| \leq ||w'| - |w||$ which proves Part b.

$3 \Rightarrow 4$. Assume f is a periodic morphism with period z ($z \neq \varepsilon$). For any integer $n \geq 0$, $f \triangleleft_{z^n} f$. Let g be a right conjugate of f and w a word such that $f \triangleleft_w g$. By Lemma 3.1(1), $f \triangleleft_{z^n w} g$ for any $n \geq 0$.

$4 \Rightarrow 1$ is immediate since $\mathrm{NbR}(f)$ is finite.

Now assume f is periodic (and still non-empty) with period z. Part b has already been stated. We prove $\mathrm{NbR}(f) = |z|$. We have $f \triangleleft_z f$. By Property 3.2, the prefixes of z right-conjugate f.

By Lemma 3.1(1), it follows that the words $z^n z_1$, with $n \geq 0$ and z_1 prefix of z, right-conjugate f. By Lemma 3.1(3), there is no other word that right-conjugates f (we can see that any word that right-conjugates f is a prefix of z^n for an integer n). Moreover, given a prefix z_1 of z with $z_1 \neq z$, and given an integer n, by Lemma 3.1(1), z_1 and $z^n z_1$ right-conjugate f into the same morphism. It follows $\mathrm{NbR}(f) \leq |z|$. Now let z_1, z_2 be two prefixes of z different from z. We have $|z| > ||z_1| - |z_2||$. Thus by Part b, z_1 and z_2 right-conjugate f into two different morphisms. Thus $\mathrm{NbR}(f) = |z|$.

□

In the previous proof, we have also stated that for a non-empty periodic morphism with period z, there exists a one-to-one correspondance between the strict prefixes of z and the right-conjugates of f. More generally,

Corollary 3.4 *A morphism g is a right conjugate of a morphism f if and only if there exists a unique word w such that $0 \leq |w| \leq \mathrm{NbR}(f) - 1$ and $f \triangleleft_w g$.*

Proof. The result was already proved for a non-empty periodic morphism f. If $f = \epsilon$, then the result is also true since ϵ has exactly one right congugate (itself) and $f \triangleleft_\varepsilon f$ Note that if $f \triangleleft\ g$, f is periodic if and only g is periodic. Assume f, g are two non-periodic morphisms. Let w be a word such that $f \triangleleft_w g$. By Lemma 3.3, $|w| \leq \mathrm{NbR}(f) - 1$ and w is the unique word that right-conjugates f into g (thus there is exactly $\mathrm{NbR}(f)$ words that right-conjugate f).

□

According to the previous corollary and following [16], we will say that g is the $|w|^{\text{th}}$ right conjugate of f if, for a word w, $f \triangleleft_w g$ and $0 \leq |w| \leq \mathrm{NbR}(f) - 1$. Of course, f is the 0^{th} conjugate of f. If $|w| = 1$, g will be called the *first* (right) conjugate of f, and f will be called the *previous* conjugate of g. If $|w| = \mathrm{NbR}(f) - 1$, g will be called the *last* (right) conjugate of f. Note that if g is a non-periodic morphism and is the p^{th} conjugate of a morphism f, then there exists a unique word w such that $f \triangleleft_w g$. Of course, $|w| = p$.

Now we state a better bound for $\mathrm{NbR}(f)$ than the one given in the introduction of this section.

Lemma 3.5 *Let f be a non-empty morphism.*

1. $\mathrm{NbR}(f) \leq ||f||$.
2. *If f is not periodic,* $\mathrm{NbR}(f) \leq \max\{|f(xy)| \ / \ x, y \in A, x \neq y\} - 1$

Proof. We first prove the second part. Let f be a non-periodic morphism. Assume there exists a word w that right-conjugates f with $|w| > \max\{|f(xy)| \ / \ x, y \in A, x \neq y\} - 1$. Since f is not periodic, at least two different letters x and y exist such that $f(x) \neq \varepsilon$ and $f(y) \neq \varepsilon$. For two such letters, let $f(x) = X\alpha$ and $f(y) = Y\beta$ where α, β are letters and X, Y are words. Since $f(xy)w = wg(xy)$, $f(yx)w = wg(yx)$ and $|X\alpha Y| = |Y\beta X| < |w|$, one has $X\alpha Y = Y\beta X$ prefix of w. From $|X\alpha Y|_\alpha = |Y\beta X|_\alpha$, it follows $\alpha = \beta$. So $f(xy) = f(yx)$. By Lemma 2.1, $f(x)$ and $f(y)$ are powers of a same word. Consequently a word z exists such that $f(x)$ is a power of z for each x in A. A contradiction with f non-periodic.

The first part of Lemma 3.5 is a direct consequence of Part 2 and Lemma 3.3(a). Indeed, if f is not periodic, $\max\{|f(xy)| \ / \ x, y \in A, x \neq y\} - 1 < ||f||$, and if f is periodic with period z, $|z| \leq ||f||$.

□

The bounds in the previous lemma are optimal. Let z be a primitive word. The periodic morphism f defined by $f(a) = z$ and $f(x) = \varepsilon$ for $x \in A \setminus \{a\}$ verifies $\mathrm{NbR}(f) = |z| = ||f||$. Now if b is a letter different from a, and if n is an integer, let us consider the morphism g defined by $g(a) = a$, $g(b) = a^n b$, and $g(x) = \varepsilon$ for $x \in A \setminus \{a, b\}$. This morphism is not periodic. Moreover the right conjugates of f are the morphisms h with $h(a) = a$, $h(\varepsilon) = \varepsilon$ for $x \in A \setminus \{a, b\}$, and $h(b) = a^i b a^j$ for integers $i, j \geq 0$ such that $i + j = n$. Thus $\mathrm{NbR}(g) = n + 1 = |g(ab)| - 1$.

As for the number of right conjugates, given a morphism f, we define $\mathrm{NbL}(f)$ the number of left conjugates. As previously done, we can see:

Lemma 3.6 *Let f, g be two morphisms.*

1. *If $f \neq \epsilon$, $\mathrm{NbL}(f) \leq ||f||$.*
2. *If f is not periodic, $\mathrm{NbL}(f) \leq \max\{|f(xy)| \ / \ x, y \in A, x \neq y\} - 1$*
3. *If $f \neq \epsilon$ is a periodic morphism with period z, $\mathrm{NbL}(f) = |z|$.*
4. *The morphism g is a left conjugate of f if and only if there exists a unique word w such that $0 \leq |w| \leq \mathrm{NbL}(f) - 1$ and $g \triangleleft_w f$.*
5. *The following assertions are equivalent:*
   - *there exists a word u such that $g \triangleleft_u f$ and $|u| \geq \mathrm{NbL}(f)$,*
   - *there exist two word that right-conjugate f into the same morphism,*
   - *f is periodic,*
   - *there is infinitely many words that right-conjugate g into f.*

Finally, we define $\mathrm{NbC}(f)$ as the total number of left or right conjugates of a morphism f, that is, $\mathrm{NbC}(f) = \#\{g \ / \ g \triangleleft \ f \text{ or } f \triangleleft \ g\}$. For instance $\mathrm{NbC}(\epsilon) = 1$, $\mathrm{NbC}(L) = \mathrm{NbC}(R) = 2$. We have

Lemma 3.7 *Let f be a non-empty morphism.*

1. *If f is periodic, $\mathrm{NbC}(f) = \mathrm{NbR}(f) = \mathrm{NbL}(f)$.*
2. *If f is not periodic, $\mathrm{NbC}(f) = \mathrm{NbR}(f) + \mathrm{NbL}(f) - 1$.*
3. *For any right conjugate g of f, $\mathrm{NbC}(f) = \mathrm{NbC}(g)$.*

Proof.

1. Let f be a non-empty periodic morphism with period z. By Lemma 3.3(a), $\mathrm{NbR}(f) = |z|$. Let g be a right conjugate of f. By Corollary 3.4 there exists a word z_1 such that $f \triangleleft_{z_1} g$ and $|z_1| < |z|$. Since $f \triangleleft_z f$, by Lemma 3.1(3), there exists a word z_2 such that ($z = z_1 z_2$ and) $g \triangleleft_{z_2} f$. Thus any right conjugate of f is also a left conjugate of f. Similary we can prove that any left conjugate of f is also a right conjugate of f. Thus $\mathrm{NbC}(f) = \mathrm{NbR}(f) = \mathrm{NbL}(f)$.
2. Now assume that f is not periodic. Let g be a morphism which is both right and left conjugate of f (f is such a morphism). Let w_1, w_2 be two words such that $f \triangleleft_{w_1} g$ and $g \triangleleft_{w_2} f$. By lemma 3.1(1), $f \triangleleft_{w_1 w_2} f$. Thus for all $x \in A$, $f(x) w_1 w_2 = w_1 w_2 f(x)$, and since f is not periodic, $w_1 w_2 = \varepsilon$. So $f = g$. Thus the only morphism which is both a left and right conjugate of f is f itself. Consequently $\mathrm{NbC}(f) = \mathrm{NbR}(f) + \mathrm{NbL}(f) - 1$.

3. Let g be a morphism such that $f \lhd g$ and let h be a morphism.
   If $g \lhd h$ then $f \lhd h$ by Lemma 3.1(1).
   If $h \lhd g$ then $f \lhd h$ or $h \lhd f$ by Lemma 3.1(2).
   If $f \lhd h$ then $g \lhd h$ or $h \lhd g$ by Lemma 3.1(3).
   If $h \lhd f$ then $h \lhd g$ by Lemma 3.1(1).
   In other words any conjugate of g is a conjugate of f and conversely.

□

3.3 Computation of the numbers of conjugates

The aim of this section is to prove the next proposition. We also give consequences of the intermediary Lemma 3.9.

Proposition 3.8 *Given a morphism f, the values* NbC(f), NbR(f) *and* NbL(f) *can be computed in time $O(||f||)$.*

This proposition is a consequence of

Lemma 3.9 *Let f be a non-periodic morphism and let $n \geq 0$ be an integer.*
The morphism f has at least $(n+1)$ right conjugates (resp. left conjugates) if and only if it has at least n right conjugates (resp. left conjugates) and there exists a letter y such that for each letter x with $f(x) \neq \varepsilon$, y is the n^{th} (modulo $|f(x)|$) letter of $f(x)$ (resp. the n^{th} (modulo $|f(x)|$) letter from the end of $f(x)$).

(Recall that f has at least $n+1$ right conjugates if and only if f has a n^{th} right conjugate.)
Proof. We prove this lemma only for right conjugacy. The proof for left conjugacy is symmetric. First assume that f has at least $(n+1)$ right conjugates. Let g be the n^{th} right conjugate of f, and let w be the word of length n such that $f \lhd_w g$. For each $x \in A$, we have $f(x)w = wg(x)$ and thus $f(x)^{|w|}w = wg(x)^{|w|}$. It follows that the n^{th} letter of w is also the n^{th} (modulo $|f(x)|$) of $f(x)$ when $f(x) \neq \varepsilon$.

Conversely assume that f has at least n right conjugates and that there exists a letter y such that for each letter x with $f(x) \neq \varepsilon$, y is the n^{th} (modulo $|f(x)|$) letter of $f(x)$. Let g be the $(n-1)^{\text{th}}$ right conjugate of f and let w be the word of length $n-1$ such that $f \lhd_w g$. For $x \in A$, we have $f(x)w = wg(x)$. Thus $f(x)^{|w|+1}w = wg(x)^{|w|+1}$. When $f(x) \neq \varepsilon$, y appears to be the first letter of $g(x)$. So, for each letter x such that $f(x) \neq \varepsilon$, let u_x be the word such that $g(x) = yu_x$. We then define a morphism h on A by $h(x) = \varepsilon$ if $f(x) = \varepsilon$ and $h(x) = u_x y$ otherwise. For all letters x, $f(x)wy = wyh(x)$ (because $f(x)w = wg(x) = wyu_x$ if $f(x) \neq \varepsilon$). Thus $f \lhd_{wy} h$ which means h is the n^{th} conjugate of f ($|wy| = n$). In other words, f has at least $(n+1)$ right conjugate. □

Proof of Proposition 3.8. First note that $\text{NbC}(\epsilon) = \text{NbR}(\epsilon) = \text{NbL}(\epsilon) = 1$.

From now, we assume that $f \neq \epsilon$. The morphism f is periodic if and only if for all different letters x, y, we have $f(x)f(y) = f(y)f(x)$. This can be checked in time $O(||f||)$ since for two different letters x, y, $|f(xy)| \leq ||f||$, and since there is a constant $(\frac{\#A(\#A-1)}{2})$ number of equations to verify.

If f is periodic, by Lemmas 3.3(a) and 3.7, it is enough to compute its period z. Since $f \neq \epsilon$, there is a letter x such that $f(x) \neq \varepsilon$. The word z is the least word such that $f(x) = z^n$ for an integer n. Observe that if $z \neq f(x)$, z is the least non-empty border (that is which is both suffix and prefix) of $f(x)$ such that $f(x) = zt$ with t another border of $f(x)$ ($f(x) = zt = tz$). By classical string-matching algorithms, it is well-known that the borders of the word $f(x)$ can be computed in time $O(|f(x)|)$ (see for instance [6]). Moreover since $f(x)$ has at most $|f(x)|+1$ borders, one can compute in time $O(||f||)$ the value of $|z|$.

Now assume that f is non-periodic. To compute the number of right conjugates of f, we have just to apply:

```
NbR← 1
While f has a NbR^th right conjugate do
      NbR← NbR+1
end while
```

The validity of this algorithm is given by Corollary 3.4 and by the definition of the n^{th} conjugate of f (with n an integer). The test "f has a NbR$^{\text{th}}$ right conjugate" can be done in time $O(1)$ by Lemma 3.9 (note that we have the hypothesis "f has a (NbR-1)$^{\text{th}}$ right conjugate"). Finally, the algorithms acts in time $O(||f||)$ since $\text{NbR}(f) \leq ||f||$ by Lemma 3.5(1).

The number of left conjugates can be obtained similary (defining the p^{th} left conjugate of a morphism f for $0 \leq p \leq \text{NbL}(f)$).

The total number of conjugates is obtained by Lemma 3.7. □

We end this section with some consequences of Lemma 3.9. First when it is considered with $n = 1$, we get

Corollary 3.10 *Let f be a morphism Assume that f is not a periodic morphism.*

1. *A morphism g exists such that $f \triangleleft g$ and $g \neq f$ if and only if there exists a unique letter x such that for each letter y, $f(y) = \varepsilon$ or x is the first letter of $f(y)$.*
2. *A morphism g exists such that $g \triangleleft f$ and $g \neq f$ if and only if there exists a unique letter x such that for each letter y, $f(y) = \varepsilon$ or x is the last letter of $f(y)$.*

Note that the first part of Corollary 3.10 was already (partially) mentionned in [2, beginning of Section 2.3.4]. Note also that Corollary 3.10 can also be stated for periodic morphisms with period a word of length at least two.

When Corollary 3.10 is applied to episturmian morphisms, using also Lemma 2.4, we obtain

Corollary 3.11 *Let f be an episturmian morphism.*

1. *The morphism f is epistandard (i.e. belongs to $(\{L\} \cup \text{Exch(A)})^*$) if and only if it is not the right conjugate of another morphism.*
2. *The morphism f belongs to $(\{R\} \cup \text{Exch(A)})^*$ if and only if it has no right conjugate except itself.*

Finally, Corollary 3.10 can be used also to get informations on the last right conjugate of a morphism. The following result generalizes [16, Lemma 5].

Lemma 3.12 *For the last right conjugate g of a non-periodic morphism f, there exist (at least) two letters x and y such that $g(x)$ and $g(y)$ start with different letters (g has no right conjugate except itself).*

Proof. Let g be a morphism which is the last right conjugate of a non-periodic morphism f. The morphism g is non-periodic, and by definition there exists a word w such that $f \triangleleft_w g$ and $|w| = \mathrm{NbR}(f) - 1$. Let h be a right conjugate of g, and let w' be a word such that $g \triangleleft_{w'} h$. By Lemma 3.1(1), $f \triangleleft_{ww'} h$. By Lemma 3.3, ww' is the unique word that right-conjugates f into h. Thus by Corollary 3.4, $|ww'| \leq \mathrm{NbR}(f) - 1$, that is, $|w'| = 0$ and $g = h$. By Corollary 3.10(1), there exist two letters x and y such that $g(x)$ and $g(y)$ start with different letters. □

4 A characterization of epistandard morphisms

By Lemma 2.4, we know that an epistandard morphism f verifies $A = \{\mathrm{last}(f(x))/x \in A\}$. Among all such morphisms, we show that epistandard morphisms are those with a maximal number of right conjugates (Proposition 4.1).

Proposition 4.1 *Let f be a non-erasing morphism such that $A = \{\mathrm{last}(f(x))/x \in A\}$.*

1. $\mathrm{NbR}(f) \leq \dfrac{||f|| - 1}{\#A - 1}$
2. $\mathrm{NbR}(f) = \dfrac{||f|| - 1}{\#A - 1}$ *if and only if f is epistandard.*

We first give the proof of this proposition. After it, we show consequences in particular in the case of standard morphisms. In order to prove Proposition 4.1, we need the following lemma.

Lemma 4.2 *Given a morphism f,*

1. $||fL|| = ||f|| + (\#A - 1)|f(a)|$
2. *If f is not periodic,* $\mathrm{NbR}(fL) = \mathrm{NbR}(f) + |f(a)|$.
3. *If f is not periodic,* $\mathrm{NbL}(fR) = \mathrm{NbL}(f) + |f(a)|$.

Proof.

1. $||fL|| = |fL(a)| + \sum_{x \in A \setminus \{a\}} |fL(x)| = |f(a)| + \sum_{x \in A \setminus \{a\}} |f(a)f(x)| = ||f|| + (\#A - 1)|f(a)|$.

2. Let g be the last conjugate of f, and let w be the word of length $\mathrm{NbR}(f) - 1$ such that $f \triangleleft_w g$. Recall $L \triangleleft_a R$. From Lemma 3.1(4), $fL \triangleleft_{f(a)w} gR$. Since f is non-periodic, fL is also non-periodic. Thus by Lemma 3.3, $|f(a)w| \leq \mathrm{NbR}(fL) - 1$. It follows $\mathrm{NbR}(fL) \geq |f(a)| + \mathrm{NbR}(f)$.

   Let G be the last conjugate of fL and let W be the word of length $\mathrm{NbR}(fL) - 1$ such that $fL \triangleleft_W G$. Note that $|f(a)| \leq \mathrm{NbR}(fL) - \mathrm{NbR}(f) = |W| + 1 - \mathrm{NbR}(f) \leq |W|$. From $f(a)W = WG(a)$, it follows $|G(a)| = |f(a)|$ and thus $G(a)$ is a suffix of W. For a letter x different from a, $|G(x)| = |fL(x)| = |f(a)| + |f(x)| \geq |f(a)| = |G(a)|$ and $f(x)W = WG(x)$. It follows that $G(a)$ is a suffix of $G(x)$. Thus we can define a morphism h as follows. Let $h(a) = G(a)$. For $x \in A \setminus \{a\}$, there exists a word u such that $G(x) = uG(a)$. In this case, let $h(x) = u$. One has $G = hR$

   From $f(a)W = Wg(a)$, one gets also that $f(a)$ is a prefix of W. Let v be the word such that $W = f(a)v$. We have $f(a)[f(a)v] = [f(a)v]g(a)$, that is, $f(a)v = vg(a) = vh(a)$. Moreover for

$x \in A \setminus \{a\}$, $f(a)f(x)vh(a) = f(a)f(x)f(a)v = (fL)(x)W = Wg(x) = f(a)vh(x)h(a)$, that is, $f(x)v = vh(x)$. Thus $f \triangleleft_w h$. Since f is non-periodic, from Lemma 3.3, $|v| \leq \mathrm{NbR}(f) - 1$. Thus $|W| = |f(a)| + |v| \leq |f(a)| + \mathrm{NbR}(f) - 1$. We get $|W| + 1 = |f(a)| + \mathrm{NbR}(f)$, that is, $\mathrm{NbR}(fL) = \mathrm{NbR}(f) + |f(a)|$.

3. The proof is similar to the proof of Part 2.

$\square$

In addition to the previous lemma, observe also that for an exchange morphism E and a morphism f, $\mathrm{NbR}(fE) = \mathrm{NbR}(f)$ and $||fE|| = ||f||$. Now, we can state the following lemma that proves both the first point of Proposition 4.1 and the only if part of the second point of the same proposition.

Lemma 4.3 *Let f be a non-erasing morphism such that $A = \{\mathrm{last}(f(x))/x \in A\}$.*
If $\mathrm{NbR}(f) \geq \dfrac{||f|| - 1}{\#A - 1}$, *then f is epistandard and* $\mathrm{NbR}(f) = \dfrac{||f|| - 1}{\#A - 1}$.

Proof. We act by induction on $||f||$. Since f is non-erasing $||f|| \geq \#A$. If $||f|| = \#A$, then since $A = \{\mathrm{last}(f(x))/x \in A\}$, f is a permutation (so is an epistandard morphism) and $\mathrm{NbR}(f) = \dfrac{||f|| - 1}{\#A - 1}$.

Let f be a morphism such that $||f|| > \#A$. Assume that the property is verified for all morphisms f' with $||f'|| < ||f||$.

Let $m := \min\{|f(x)| \ / \ x \in A\}$. We have $m \leq \mathrm{NbR}(f) - 1$. Indeed if $m \geq \mathrm{NbR}(f)$, then $||f|| \geq \#A.m \geq \#A.\mathrm{NbR}(f) \geq \#A\dfrac{||f|| - 1}{\#A - 1}$. This implies $\#A \geq ||f||$: a contradiction.

Let y be a letter such that $|f(y)| = m$. Let g be the last right conjugate of f, and let w be the word of length $\mathrm{NbR}(f) - 1$ such that for each $x \in A$, $f(x)w = wg(x)$. Since $|w| \geq |f(y)|$, $f(y)$ is a prefix of w, and consequently a prefix of $f(x)$ for each letter x in A. If x is a letter different from y, $f(x)$ and $f(y)$ end with different letter. Thus $f(y)$ is a prefix of $f(x)$ with $f(x) \neq f(y)$.

We define a morphism f' as followed. Let $f'(a) = f(y)$. For $x \in A \setminus \{a\}$, $E_{ay}(x) \in A \setminus \{y\}$. There exists a non-empty word u such that $f(E_{ay}(x)) = f(y)u$. Let $f'(x) = u$. We have $f = f'LE_{ay}$. By Lemma 4.2 (and the observation just before Lemma 4.3), $\mathrm{NbR}(f) = \mathrm{NbR}(f'L) = \mathrm{NbR}(f') + |f'(a)|$ and $||f|| = ||f'|| + (\#A - 1)|f'(a)|$. Since $\mathrm{NbR}(f) \geq \dfrac{||f|| - 1}{\#A - 1}$, we get $\mathrm{NbR}(f') \geq \dfrac{||f'|| - 1}{\#A - 1}$. Moreover since $|f'(a)| \neq 0$ and $\#A \geq 2$, $||f|| > ||f'||$. By construction, f' is not erasing, and moreover

$$\{\mathrm{last}(f'(x))/x \in A\} = \{\mathrm{last}(f(E_{ay}(x))/x \in A\}.$$

Thus by induction, f' is epistandard and $\mathrm{NbR}(f') = \dfrac{||f'|| - 1}{\#A - 1}$. It follows $\mathrm{NbR}(f) = \dfrac{||f|| - 1}{\#A - 1}$ and f is epistandard. $\square$

Proof of Proposition 4.1. In addition to Lemma 4.3, we have just to prove that for an epistandard morphism f, $\mathrm{NbR}(f) = \dfrac{||f|| - 1}{\#A - 1}$. This is true if f is an exchange morphism. Indeed in this case, $||f|| = \#A$ and $\mathrm{NbR}(f) = 1$.

The result then holds by induction on the number of elements of a decomposition of f over $\{L\} \cup \mathrm{Exch(A)}$. To verify it, it is sufficient to prove that given an epistandard morphism f with $\mathrm{NbR}(f) = \frac{||f||-1}{\#A-1}$, and given an exchange morphism E, one has $\mathrm{NbR}(fE) = \frac{||fE||-1}{\#A-1}$ and $\mathrm{NbR}(fL) =$

$\frac{||fL||-1}{\#A-1}$. The result for fE is immediate since $||fE|| = ||f||$ and $\mathrm{NbR}(fE) = \mathrm{NbR}(f)$. For fL, using Lemma 4.2 (note that f is epistandard and thus non-periodic), one has $\mathrm{NbR}(fL) = \mathrm{NbR}(f) + |f(a)| = \frac{||f||-1}{\#A-1} + |f(a)| = \frac{||f||+(\#A-1)|f(a)|-1}{\#A-1} = \frac{||fL||-1}{\#A-1}$. □

Now, let us make some remarks about Proposition 4.1.

First, given a morphism f, Proposition 4.1 allows to determine in time $O(||f||)$ whether f is epistandard. Indeed in Section 3.3, we have seen how to compute $\mathrm{NbR}(f)$ in time $O(||f||)$.

Second, observe that, when $\#A > 2$ and the condition $A = \{\mathrm{last}(f(x))/x \in A\}$ is not verified, there exist some morphisms f such that $\mathrm{NbR}(f) > \dfrac{||f||-1}{\#A-1}$. For an integer $n \geq 1$, this is the case for instance with the morphism f, defined by $f(a) = a^n b$ and $f(x) = a$ for all $x \neq a$, for which

$$\frac{||f||-1}{\#A-1} = \frac{(n+1)+(\#A-1)-1}{\#A-1} = 1 + \frac{n}{\#A-1} \leq 1 + \frac{n}{2} < n+1 = \mathrm{NbR}(f).$$

When $\#A = 2$, standard morphisms are the morphisms with a maximum number of right conjugates among non-periodic morphisms. Indeed Proposition 4.1 has the following corollary (Note that Séébold has already proved [16] the if part of the second point of this corollary).

Corollary 4.4 *Let f be a non-periodic morphism on $\{a,b\}$.*

1. $\mathrm{NbR}(f) \leq ||f|| - 1$
2. $\mathrm{NbR}(f) = ||f|| - 1$ *if and only if f is standard.*

Proof. Let f be a non-periodic morphism on $A = \{a,b\}$. By Lemma 3.6(1 and 4), there exist a morphism g and a word w of length $\mathrm{NbL}(f) - 1$ such that $g \triangleleft_w f$. The morphism g has no left conjugate except itself (otherwise if $h \triangleleft_u g$ with $u \neq \varepsilon$, by Lemma 3.1(1), $h \triangleleft_{uw} f$: a contradiction with Lemma 3.6(1) since $|uw| > |w| = \mathrm{NbL}(f) - 1$). By Corollary 3.10(2), $g(a)$ and $g(b)$ end with different letters. By Lemma 3.1(1), any right conjugate morphism of f is a right conjugate of g. Thus $\mathrm{NbR}(f) \leq \mathrm{NbR}(g)$. Since $A = \{\mathrm{last}(g(x))/x \in A\}$, by Proposition 4.1, $\mathrm{NbR}(g) \leq \dfrac{||g||-1}{\#A-1} = ||g|| - 1$. Since f and g are conjugates, $||f|| = ||g||$. It follows $\mathrm{NbR}(f) \leq ||f|| - 1$.

Now assume $\mathrm{NbR}(f) = ||f|| - 1$. It follows $f = g$ (else $\mathrm{NbR}(g) > ||g|| - 1$). Consequently $A = \{\mathrm{last}(f(x))/x \in A\}$ and by Proposition 4.1, f is standard. Conversely if f is standard, by Proposition 4.1, $\mathrm{NbR}(f) = \dfrac{||f||-1}{\#A-1} = ||f|| - 1$. □

To end this section, let us observe that for a periodic morphism f on $\{a,b\}$, $\mathrm{NbR}(f) > ||f|| - 1$ if and only if there exists a primitive word z such that $f(a) = z$, $f(b) = \varepsilon$ or $f(a) = \varepsilon$, $f(b) = z$.

5 A characterization of episturmian morphisms

The aim of this section is to generalize a result given in [2] and first proved by Séébold [16]. We state

Theorem 5.1 *A morphism f is episturmian if and only if it is a right conjugate of a unique epistandard morphism. This epistandard morphism is obtained from any decomposition of f in elements of* Exch(A) $\cup \{L, R\}$ *by replacing all the occurrences of R by L.*

To prove this theorem, we first give two propositions. In Proposition 5.2 (resp. Proposition 5.4), we show how to compute a decomposition on Exch(A) $\cup \{L, R\}$ of the first right conjugate (resp. the previous conjugate) of a morphism.

Proposition 5.2 *Let f be an episturmian morphism on A. Let $f_1, \dots, f_n$ be some elements of* Exch(A) $\cup \{L, R\}$ *such that $f = f_1 \dots f_n$.*

The morphism f has a right conjugate different from f if and only if there exists an integer k between 1 and n such that $f_k = L$.

When it is the case, let k be the least integer between 1 and n such that $f_k = L$. For each i between 1 and $k-1$, let g_i be the morphism defined by:

- *$g_i = L$ if $f_i = R$ and the first letter of $f_{i+1} \dots f_k(a)$ is different from a.*
- *$g_i = f_i$ otherwise.*

Then the first right conjugate of f is the morphism $g_1 g_2 \dots g_{k-1} R f_{k+1} \dots f_n$ ($R f_2 \dots f_n$ when $k = 1$).

Before proving this proposition, we give an example on the alphabet $\{a, b, c\}$. We consider the morphism

$$f = E_{bc} R E_{bc} E_{ac} R E_{ac} R E_{bc} R E_{ab} L E_{ab} R L L E_{bc} R$$

that is $n = 17$ and $f_1 = f_3 = f_8 = f_{16} = E_{bc}$, $f_2 = f_5 = f_7 = f_9 = f_{13} = f_{17} = R$, $f_4 = f_6 = E_{ac}$, $f_{10} = f_{12} = E_{ab}$, $f_{11} = f_{14} = f_{15} = L$.

Here, $k = 11$. Let us now define the g_i's ($1 \le i \le 10$). When f_i is an exchange, $g_i = f_i$. This is the case for $i \in \{1, 3, 4, 6, 8, 10\}$. Now we have to consider the case where $f_i = R$. For this, let x_i be the first letter of $f_{i+1} \dots f_{11}(a)$ ($1 \le i \le 10$). Note that $x_{10} = a$ and x_i is the first letter of $f_{i+1}(x_{i+1})$ (for $1 \le i \le 9$). We have $x_9 = b$, $x_8 = b$, $x_7 = c$, $x_6 = c$, $x_5 = a$, $x_4 = a$, $x_3 = c$, $x_2 = b$ and $x_1 = b$. Thus we set $g_9 = g_7 = g_2 = L$ and $g_5 = R$. Proposition 5.2 claims that the first conjugate of f is

$$f = E_{bc} L E_{bc} E_{ac} R E_{ac} L E_{bc} L E_{ab} R E_{ab} R L L E_{bc} R.$$

To verify it, it is sufficient from Lemma 3.1(4) to prove that the first conjugate of $f_1 \dots f_{10} L$ is $g_1 \dots g_{10} R$. This is true since

$f_1 \dots f_{10} L(a) = c(aac)^2 a$,	$g_1 \dots g_{10} R(a) = (aac)^2 ac$,
$f_1 \dots f_{10} L(b) = c(aac)^3 a$,	$g_1 \dots g_{10} R(b) = (aac)^3 ac$,
$f_1 \dots f_{10} L(c) = c(aac)^2 abac(aac)^2 a$,	$g_1 \dots g_{10} R(c) = (aac)^2 abac(aac)^2 ac$.

Proposition 5.2 is a corollary of the following result.

Lemma 5.3 *Let f be a morphism.*

1. *The first right conjugate of Lf is Rf.*

   Moreover if f has at least one right conjugate different from itself and if g is the first right conjugate of f, then the three following assertions hold.

2. *For any exchange E, the first right conjugate of Ef is Eg.*
3. *If the first letter of $f(a)$ is a, then the first right conjugate of Rf is Rg.*
4. *If the first letter of $f(a)$ is different from a, then the first right conjugate of Rf is Lg.*

Note that this lemma can be applied to obtain a recursive version of the method given in Proposition 5.2.

Proof of Lemma 5.3.

Part 1 is due to the fact that R is the first conjugate of L: $\forall w \in A^*$, $L(f(w))a = aR(f(w))$.

Parts 2, 3 and 4 are direct uses of Lemma 3.1(4). We have $E \triangleleft_\varepsilon E$ and $R \triangleleft_\varepsilon R$. Let x be the first letter of $f(a)$. We have $f \triangleleft_x g$ since g is the first conjugate of f. By Lemma 3.1(4), it follows $Ef \triangleleft_{E(x)} Eg$ (that is Eg is the first conjugate of Ef) and $Rf \triangleleft_{R(x)} Rg$. When $x = a$, $Rf \triangleleft_a Rg$, that is Rg is the first conjugate of Rf. When $x \neq a$, we have $\forall w \in A^*$, $[Rf](w)xa = xa[Rg](w)$. But from $L \triangleleft_a R$, we get $aRg(w) = Lg(w)a$. It follows $Rf \triangleleft_x Lg$, that is Lg is the first conjugate of Rf.

□

Proof of Proposition 5.2. (See also Section 10.2) If f has a right conjugate different from itself then, by Corollary 3.11(2), $f \notin (\mathrm{Exch(A)} \cup R)^*$. Thus there exists an integer k such that $f_k = L$. Conversely if such an integer exists, since $L \triangleleft R$, by Lemma 3.1(4) $f_1 \dots f_{k-1} R f_{k+1} \dots f_n$ is a right conjugate of f.

Now let k be the least integer between 1 and n with $f_k = L$, and let (for $1 \leq i \leq k-1$) g_i be the morphism defined in the hypotheses of Proposition 5.2. By Lemma 5.3(1), the first conjugate of $Lf_{k+1} \dots f_n$ is $Rf_{k+1} \dots f_n$. For $1 \leq i \leq k-1$, by induction using Parts 2, 3 and 4 of Lemma 5.3, we get that the first conjugate of $f_i \dots f_{k-1} L f_{k+1} \dots f_n$ is $g_i \dots g_{k-1} R f_{k+1} \dots f_n$. □

When considering left conjugates in place of right conjugates, by similar methods, we get the following result:

Proposition 5.4 *Let f be an episturmian morphism on A. Let $f_1, \dots, f_n$ be some elements of* $\mathrm{Exch(A)} \cup \{L, R\}$ *such that $f = f_1 \dots f_n$.*

The morphism f is a right conjugate of another morphism if and only if there exists an integer k between 1 and n such that $f_k = R$.

When it is the case, let k be the least integer between 1 and n such that $f_k = R$. For each i between 1 and $k-1$, let g_i be the morphism defined by:

- *$g_i = R$ if $f_i = L$ and the last letter of $f_{i+1} \dots f_k(a)$ is different from a.*
- *$g_i = f_i$ otherwise.*

Then the previous right conjugate of f is the morphism $g_1 g_2 \dots g_{k-1} L f_{k+1} \dots f_n$ ($Lf_2 \dots f_n$ when $k = 1$).

The methods given in Propositions 5.2 and 5.4 are very similar. One can think that they are inverse transformations that is, given a decomposition $f_1 \dots f_n$ (with $f_i \in \mathrm{Exch(A)} \cup \{L, R\}$) of an episturmian morphism, applying first Proposition 5.2 and then Proposition 5.4 (or first Proposition 5.4 and then Proposition 5.2) we get the initial decomposition. This is false. For instance, applying Proposition 5.2 on RL we get RR, and, applying Proposition 5.4 on RR we get LR. In a similar way applying Proposition 5.4 on LR we get LL, and, applying Proposition 5.2 on LL we get RL. Of course $RL = LR$ (Note that in Section 7, we study relations between morphisms L, R and exchanges).

The algorithms described in Propositions 5.2 and 5.4 can be implemented in time $O(n)$. Indeed, a first phase analyses one by one each possible k, a second phase determines the g_i's by decreasing integer from $k-1$ to 1 (note that the first letter of $f_i \dots f_k(a)$ is also the first letter of the image by f_i of the first letter of $f_{i+1} \dots f_k(a)$), and finally a third phase copies the f_i's for $i \geq k+1$. The three phases are in time $O(n)$.

Now let us come back to the proof of Theorem 5.1. First, we mention two corollaries of Propositions 5.2 and 5.4, respectively.

Corollary 5.5 *Any right conjugate of an episturmian morphism is episturmian.*

Corollary 5.6 *If a morphism f has a right conjugate which is episturmian then f is also episturmian.*

Proof of Theorem 5.1. By Corollary 5.5, if a morphism f is a right conjugate of an epistandard morphism, then f is episturmian. Conversely, assume that a morphism f is episturmian. By Lemma 3.6, the left conjugates of f can be ordered by the length of the words that right-conjugate them into f. Let us denote $f_0 = f$, f_1, $\dots$, $f_{\mathrm{NbL}(f)-1}$ the left conjugates of f. By Corollary 5.6, all these conjugates are episturmian. By Lemmas 3.1(1) and 3.6(1 and 4), since f is not periodic, $f_{\mathrm{NbL}(f)-1}$ has no left conjugates except itself, and for i, $0 \leq i < \mathrm{NbL}(f) - 1$, f_{i+1} is the previous conjugate of f_i. By Corollary 3.11, $f_{\mathrm{NbL}(f)-1}$ is epistandard, and for each integer i, $0 \leq i < \mathrm{NbL}(f) - 1$, f_i is not epistandard. In other words, f is the left conjugate of a unique epistandard morphism $g = f_{\mathrm{NbL}(f)-1}$. The decomposition of g can be obtain from the decomposition of f by using several times Proposition 5.4 until there is no "R" in the decomposition. Thus g is obtained from any decomposition of f in elements of $\mathrm{Exch(A)} \cup \{L, R\}$ by replacing all the occurrences of R by L. □

6 Number of conjugates of an episturmian morphism

In this section, we study the number of conjugates of an episturmian morphism f, that is the values $\mathrm{NbC}(f)$, $\mathrm{NbR}(f)$, $\mathrm{NbL}(f)$. We start with

Proposition 6.1 *For any episturmian morphism f,* $\mathrm{NbC}(f) = \dfrac{||f|| - 1}{\#A - 1}$

Proof. Let f be an episturmian morphism. By Theorem 5.1, there exists an epistandard morphism g such that $g \triangleleft f$. By definition of conjugacy, $||g|| = ||f||$. It follows by Proposition 4.1(2), that $\mathrm{NbR}(g) = \dfrac{||f|| - 1}{\#A - 1}$. By Corollary 3.11, $\mathrm{NbL}(g) = 1$. It follows by Lemma 3.7(2), $\mathrm{NbC}(g) = \mathrm{NbR}(g)$. Lemma 3.7(3) shows that $\mathrm{NbC}(f) = \frac{||f||-1}{\#A-1}$. □

Now for an episturmian morphism f, we show relations between $\mathrm{NbC}(f)$, $\mathrm{NbR}(f)$, $\mathrm{NbL}(f)$ and any decomposition of f over $\mathrm{Exch(A)} \cup \{L, R\}$

Proposition 6.2 *If $f = f_1 \dots f_n$ is an episturmian morphism with $f_i \in \mathrm{Exch(A)} \cup \{L, R\}$,*

a) $$\mathrm{NbL}(f) = 1 + \sum_{1 \leq i \leq n | f_i = R} |f_1 \dots f_{i-1}(a)|.$$

b) $$\mathrm{NbR}(f) = 1 + \sum_{1 \leq i \leq n | f_i = L} |f_1 \dots f_{i-1}(a)|.$$

c) $$\mathrm{NbC}(f) = 1 + \sum_{1 \leq i \leq n | f_i \in \{L, R\}} |f_1 \dots f_{i-1}(a)|.$$

(Let recall that if $i = 1$, $f_1 \dots f_{i-1} = Id$).

The proof of this proposition uses the following result. We denote by Stand(f) the unique epistandard morphism which is a left conjugate of an episturmian morphism f (see Theorem 5.1).

Lemma 6.3 *For any episturmian morphisms f and g,*

$$\mathrm{NbL}(fg) = \mathrm{NbL}(f) + \mathrm{NbL}(\mathrm{Stand}(f)g) - 1.$$

Consequently, for any exchange morphism E,

$$\mathrm{NbL}(fL) = \mathrm{NbL}(fE) = \mathrm{NbL}(f).$$

Proof. Let f, g be episturmian morphisms. By definition of the previous conjugate of a morphism (see also Lemma 3.6(4) and Lemma 3.1(1)), all the left conjugates of f can be obtained by successive applications of Proposition 5.4. In particular, Stand(f) (the "last" left conjugate of f) is obtained from f using $\mathrm{NbL}(f)-1$ applications of this proposition. It follows that Stand$(f)g$ is obtained from fg using $\mathrm{NbL}(f) - 1$ applications of Proposition 5.4. From Stand$(f)g$, successive applications of Proposition 5.4 give both the remaining left conjugates of fg and the left conjugates of Stand$(f)g$. We need $\mathrm{NbL}(\mathrm{Stand}(f)g)-1$ applications of the proposition. Moreover the total number of applications to obtain left conjugates of fg is $\mathrm{NbL}(fg)-1$. Thus $\mathrm{NbL}(fg) = \mathrm{NbL}(\mathrm{Stand}(f)g)+\mathrm{NbL}(f)-1$.

When g is epistandard (this is the case in particular when $g = L$ or when g is an exchange morphism), Stand$(f)g$ is epistandard. In this case $\mathrm{NbL}(\mathrm{Stand}(f)g) = 1$ and $\mathrm{NbL}(fg) = \mathrm{NbL}(f)$. $\square$

Proof of Proposition 6.2. First we prove the first part of the proposition by induction on n. If $n = 1$ and $f_1 \in \mathrm{Exch(A)} \cup \{L\}$, f_1 is epistandard and $\mathrm{NbL}(f_1) = 1$. If $n = 1$ and $f_1 = R$, $\mathrm{NbL}(f_1) = 2$. In both cases, the formulas of the proposition are verified.
Now assume that $n \geq 2$. Let $f' = f_1 \dots f_{n-1}$ and assume that

$$\mathrm{NbL}(f') = 1 + \sum_{1 \leq i \leq n-1 | f_i = R} |f_1 \dots f_{i-1}(a)|.$$

If $f_n \in \mathrm{Exch(A)} \cup \{L\}$, by Lemma 6.3, $\mathrm{NbL}(f) = \mathrm{NbL}(f'f_n) = \mathrm{NbL}(f')$. The formula of the proposition is verified.
If $f_n = R$, by Lemma 6.3, $\mathrm{NbL}(f) = \mathrm{NbL}(fR) = \mathrm{NbL}(f')+\mathrm{NbL}(\mathrm{Stand}(f')R)-1$. By Lemma 4.2(3), $\mathrm{NbL}(\mathrm{Stand}(f')R) = \mathrm{NbL}(\mathrm{Stand}(f'))+|\mathrm{Stand}(f')(a)| = 1+|\mathrm{Stand}(f')(a)|$. Now since $\mathrm{Stand}(f') \triangleleft f'$, one has $|\mathrm{Stand}(f')(a)| = |f'(a)| = |f_1 \dots f_{n-1}(a)|$. Once again, the formula of the proposition is verified.

Note that the second part of the proposition can be deduced from the first and the third part and from the formula $\mathrm{NbR}(f) = \mathrm{NbC}(f) - \mathrm{NbL}(f) + 1$ (see Lemma 3.7(2)).

Thus it remains to prove the third part of the proposition. Let g be the last conjugate of f, and let $g_1, \dots, g_n \in \mathrm{Exch(A)} \cup \{R\}$ be the morphisms such that $g_1 \dots g_n$ is the decomposition of g obtained by successive applications of Proposition 5.2. We have $f_i \in \{L, R\}$ if and only if $g_i = R$ for $1 \leq i \leq n$. For each integer i, $1 \leq i \leq n$, $f_1 \dots f_{i-1} \triangleleft g_1 \dots g_{i-1}$. It follows that $|f_1 \dots f_{i-1}(a)| = |g_1 \dots g_{i-1}(a)|$. Since $g \in (\mathrm{Exch(A)} \cup R)^*$, by Corollary 3.11(2), $\mathrm{NbR}(g) = 1$. By Lemma 3.7(3), $\mathrm{NbC}(f) = \mathrm{NbC}(g)$. Also by Lemma 3.7(2) $\mathrm{NbC}(f) = \mathrm{NbR}(g) + \mathrm{NbL}(g) - 1 = \mathrm{NbL}(g)$. By the first part (already proved) of this proposition :

$$\mathrm{NbC}(f) = 1 + \sum_{1 \leq i \leq n | g_i = R} |g_1 \dots g_{i-1}(a)|.$$

It follows $\mathrm{NbC}(f) = 1 + \sum_{1 \le i \le n | f_i \in \{L,R\}} |f_1 \dots f_{i-1}(a)|$. □

As a final remark in this section, observe that as a consequence of Propositions 6.1 and 6.2, for any episturmian morphism $f = f_1 \dots f_n$ with $f_i \in \mathrm{Exch(A)} \cup \{L, R\}$, we have

$$\frac{||f|| - 1}{\#A - 1} = 1 + \sum_{1 \le i \le n | f_i \in \{L,R\}} |f_1 \dots f_{i-1}(a)|.$$

7 A presentation of the monoid Episturm(A)

In the previous sections, we have noticed that there exist some relations between morphisms L, R and the exchange morphisms. In this section, we study all the existing relations. We prove

Theorem 7.1 *The monoid* Episturm(A) *with set of generators* $\mathrm{Exch(A)} \cup \{L, R\}$ *has the following presentation (x, y, z, t are pairwise different letters):*

$$\begin{array}{rcll} E_{xy}E_{xy} & = & Id, & (1) \\ E_{xy}E_{yz} & = & E_{yz}E_{zx}, & (2) \\ E_{xy}E_{zt} & = & E_{zt}E_{xy}, & (3) \\ E_{xy}L & = & LE_{xy} \quad \text{when } a \notin \{x, y\}, & (7) \\ E_{xy}R & = & RE_{xy} \quad \text{when } a \notin \{x, y\}, & (8) \\ LE_1LE_2 \dots LE_kR & = & RE_1RE_2 \dots RE_kL & (9) \end{array}$$

where $k \ge 1$ is an integer and $E_1, \dots, E_k$ are exchange morphisms such that $E_1 \dots E_k(a) = a$, and for each integer i, $2 \le i \le k$, $E_i \dots E_k(a) \neq a$.

Theorem 7.1 generalizes a result of Séébold [15]. More precisely, Séébold gives a presentation of the monoid of Sturmian morphisms. In the case of a two-letter alphabet, his presentation coincides with the presentation given in Theorem 7.1.

Relations (1), (2) and (3) have already been established in Section 2. We let the reader verify Relations (7) and (8).

Now let $k \ge 1$ be an integer and let $E_1, \dots, E_k$ be exchange morphisms such that $E_1 \dots E_k(a) = a$, and for each integer i, $2 \le i \le k$, $E_i \dots E_k(a) \neq a$. By Proposition 5.4, the previous conjugate of $LE_1LE_2 \dots LE_kR$ is $LE_1RE_2 \dots RE_kL$. By Proposition 5.2, the first conjugate of this morphism is $RE_1RE_2 \dots RE_kL$. It follows that Relation (9) holds (see also Section 8). Note that the same proof shows the same relation with permutations in place of the exchange morphisms.

To prove Theorem 7.1, we use three intermediate lemmas. The first lemma states that the monoid of episturmian morphisms is both left and right cancellative. The second one states a kind of normalization of the decomposition of an episturmian morphism. It will also be used in Section 8. The third lemma is an inductive step for the proof of Theorem 7.1. It will also be used in Section 9.

Lemma 7.2 *Given an alphabet A, the monoid* Episturm(A) *is both left and right cancellative, i.e., for any three episturmian morphisms*

- *If $hf = hg$ then $f = g$.*
- *If $fh = gh$ then $f = g$.*

Proof. (see also Fact 10.3) Left cancellativity is an immediate consequence of the fact that episturmian morphisms are injective.

For right cancellativity, it is sufficient to prove the lemma when $h \in \mathrm{Exch(A)} \cup \{L, R\}$. Indeed the general result follows by induction.

If h is an exchange morphism then $fh = gh$ implies $f = fhh = ghh = g$.

Assume now $fL = gL$. We have $L(a) = a$. Thus $f(a) = g(a)$. Let $x \in A \backslash \{a\}$. Since $L(x) = ax$, we get $f(a)f(x) = g(a)g(x)$, that is $f(x) = g(x)$. So $f = g$.

In a similar way, we can see that $fR = gR$ implies $f = g$. $\square$

Lemma 7.3 *Let $f_1, \ldots, f_n$ be some morphisms in* $\mathrm{Exch(A)} \cup \{L, R\}$. *If $f_1 \ldots f_n$ is not a permutation, there exist some integers $k \geq 1$, $p \geq 2$, and some morphisms $g_1, \ldots, g_p$ such that*

a) $g_{2i} \in \{L, R\}$ for $1 \leq i \leq k$,

b) $g_{2i-1} \in \{E_{ax} \mid x \in A\}$ for $1 \leq i \leq k$,

c) $g_i \in \mathrm{Exch(A)}$, for $2k + 1 \leq i \leq p$,

d) $g_1 \ldots g_p = f_1 \ldots f_n$, and the equality can be stated using Relations (1), (2), (3), (7), (8).

Note that a corollary of this lemma is mentionned without proofs in [10]: any episturmian morphisms can be decomposed into gh with $g \in \{\Psi_\alpha, \overline{\Psi}_\alpha / \alpha \in A\}^*$, $h \in \mathrm{Perm(A)}$.

Proof. We prove this lemma by induction on the number N of indices i such that $f_i \in \{L, R\}$. Since $f_1 \ldots f_n$ is not a permutation, $N \geq 1$.

Let j be the least integer such that $f_j \in \{L, R\}$. One can assume $j \geq 2$, since $E_{aa} f_1 \ldots f_n = f_1 \ldots f_n$.

Using Relations (1), (2), and (3), one can find some exchange morphisms $f'_1, \ldots, f'_m$ ($m \geq 1$) such that $f'_1 \ldots f'_m = f_1 \ldots f_{j-1}$, $f'_1 = E_{ax}$ for a letter x in A (possibly $x = a$), and for i, $2 \leq i \leq m$, $f'_i(a) = a$.

Using Relation (7) or (8), one obtains $f_1 \ldots f_n = f'_1 f_j (f'_2 \ldots f'_m) f_{j+1} \ldots f_n$.

In the sequence $f'_2, \ldots, f'_m, f_{j+1}, \ldots, f_n$, there are $N - 1$ occurrences of elements in $\{L, R\}$.

If $N = 1$, the result holds with $k = N = 1$ ($p = m + n - 1$).

If $N > 1$, by induction, there exist $p - 2$ morphisms $g_3, \ldots, g_p$ and an integer k such that $g_{2i} \in \{L, R\}$ ($2 \leq i \leq k$), $g_{2i-1} \in \{E_{ax} \mid x \in A\}$ ($2 \leq i \leq k$), $g_i \in \mathrm{Exch(A)}$ ($2k + 1 \leq i \leq p$), $g_3 \ldots g_p = f'_2 \ldots f'_m f_{j+1} \ldots f_n$ and the equality can be stated using Relations (1), (2), (3), (7), (8). Taking $g_1 = f'_1$, $g_2 = f_j$, one gets the result. $\square$

Lemma 7.4 *Let $f, g, h_1, \ldots, h_n$ be $n+2$ morphisms ($n \geq 1$) in* $\mathrm{Exch(A)} \cup \{L, R\}$. *If $fg = h_1 \ldots h_n$, then there exists $p \geq 1$ morphisms $h'_1, \ldots, h'_p$ in* $\mathrm{Exch(A)} \cup \{L, R\}$, *such that $g = h'_1 \ldots h'_p$. Moreover the equality $fh'_1 \ldots h'_p = h_1 \ldots h_n$ can be stated using Relations (1), (2), (3), (7), (8) and (9).*

Proof. If f is an exchange morphism, then by Formula (1), $g = fh_1 \dots h_n$.

Assume $f = L$. If $n = 1$ then necessarily, $h_1 = L$ and $g = Id$. Now assume $n \geq 2$. We also assume that the decomposition $h_1 \dots h_n$ is normalized following Lemma 7.3. This means that there exists an integer $k \geq 1$ such that

1. $h_{2i} \in \{L, R\}$ for $1 \leq i \leq k$,
2. $h_{2i-1} \in \{E_{ax} \mid x \in A\}$ for $1 \leq i \leq k$,
3. $h_i \in \mathrm{Exch(A)}$, for $2k+1 \leq i \leq n$,

Let x be a letter such that $h_1 = E_{ax}$. We first prove that $x = a$, that is, $h_1 = Id$. For this, consider a word w such that each letter of A occurs in w. One can easily see that for any episturmian morphism h, each letter of A also occurs in $h(w)$. In particular, $|h_3 \dots h_n(w)|_a \geq 1$ (if $n = 2$, take $h_3 \dots h_n = Id$). The letter a is an internal subword of $h_3 \dots h_n(awa)$. Thus, since $h_2 \in \{L, R\}$, $h_2 \dots h_n(awa)$ contains aa, and xx is a subword of $h_1 \dots h_n(awa)$. Note that for any word u, if for a letter y, yy is a subword of $L(u)$ then $y = a$. From $h_1 \dots h_n(awa) = Lg(awa)$, it follows $x = a$.

Now observe that the first letter of $Lg(x) = h_2 \dots h_n(x)$ is the letter a for each letter x in A. For any morphism h in $(\{R\} \cup \mathrm{Exch(A)})^*$, the images by h of two different letters do not start with the same letter a. Consequently an integer $l \geq 1$ exists such that $h_{2l} = L$ (an even index because of the normalization). Assume $h_{2i} = R$ for i, $1 \leq i \leq l-1$.

Assume $l \geq 2$. Since $h_{2l} = L$, the first letter of $h_{2l} \dots h_n(a)$ is a. Moreover the first letter of $Lg(a) = h_2 \dots h_n(a)$ is also a. Thus the first letter of $h_2 \dots h_{2l-1}(a)$ is a. For $1 \leq i \leq l-1$, and for any letter x, $h_{2i}h_{2i+1}(x)$ starts with the letter $h_{2i+1}(x)$. It follows that $h_2h_3 \dots h_{2l-2}h_{2l-1}(a)$ starts with the letter $h_3h_5 \dots h_{2l-1}(a)$. So $h_3h_5 \dots h_{2l-1}(a) = a$.

Let i be the greater integer such that $1 \leq i \leq l-1$, and $h_{2i+1}h_{2i+3} \dots h_{2l-1}(a) = a$. Since $\forall j$, $i \leq j \leq l-1$, $h_{2i} = R$. One can apply Relation (9), and replace l by i. Thus by induction, one can assume $l = 1$.

If $l = 1$, one has $fg = h_2 \dots h_n$, i.e., $Lg = Lh_3 \dots h_n$. By Lemma 7.2, $g = h_3 \dots h_n$. The result holds.

The case $f = R$ is similar. □

Proof of Theorem 7.1. Let $f_1, \dots, f_n, g_1, \dots, g_p \in \mathrm{Exch(A)} \cup \{L, R\}$. Assuming $f_1 \dots f_n = g_1 \dots g_p$, we have to prove that the equality can be stated using Relations (1), (2), (3), (7), (8) and (9).

We act by induction on n.

Case $n = 1$

If $f_1 = g_1 \dots g_p$ is a permutation, then for any integer i, $g_i \notin \{L, R\}$. The result holds by Lemma 2.3.

Assume $f_1 = L$. By Lemma 7.3, one can assume $p \geq 2$, $g_1 = E_{ax}$ for some letter x, $g_2 \in \{L, R\}$. If $g_2 = R$, then given a letter b in $A \setminus \{a\}$, one gets a contradiction: $L(a)$ and $L(b)$ end with different letters, whereas $g_1 \dots g_p(a)$ and $g_1 \dots g_p(b)$ end by the same letter $g_1(a)$. Thus $g_2 = L$. The first letter of $f_1(a)$ is both a and $x = g_1(a)$. This implies $x = a$ and $g_1 = Id$. It follows $Id = g_3 \dots g_p$ by Lemma 7.2. We are back in the previous case.

The case $f_1 = R$ is similar.

Case $n \geq 2$

In this case, one can apply Lemma 7.4. There exists an integer $q \geq 1$ and morphisms $h_1, \dots, h_q$ in $\mathrm{Exch(A)} \cup \{L, R\}$ such that $f_2 \dots f_n = h_1 \dots h_q$. Moreover the equality $f_1h_1 \dots h_q = g_1 \dots g_p$ can be stated using Relations (1), (2), (3), (7), (8) and (9). By induction, the equality $f_2 \dots f_n = h_1 \dots h_q$ can be stated using these relations. Thus the equality $f_1 \dots f_n = g_1 \dots g_p$ can be stated using these relations. $\square$

We end this section considering the case of epistandard morphisms.

Corollary 7.5 *The monoid of epistandard morphisms with set of generators* $\mathrm{Exch(A)} \cup \{L\}$ *has the presentation given by Relations (1), (2), (3) and (7).*

Proof. The proof of this corollary is based on a simple remark. Let $f_1, \dots, f_n, g_1, \dots, g_p$ be morphisms in $\mathrm{Exch(A)} \cup \{L, R\}$. Assume $f_1 \dots f_n = g_1 \dots g_p$. Using Relations (1), (2), (3), (7), (8) and (9), we can see that $R \in \{f_1, \dots, f_n\}$ if and only if $R \in \{g_1, \dots, g_p\}$. $\square$

8 Episturmian morphisms and palindromes

In this section, we study connections between palindroms and morphisms that can be written $LE_1LE_2 \dots LE_kR$ as in Relation (9) in Theorem 7.1.

Let u be a word. The *mirror image* of u, denoted $\tilde{u}$ or $\tilde{\ }u$, is the empty word if $u = \varepsilon$, else it is the word $a_n \dots a_1$ where $a_1, \dots, a_n$ are the letters such that $u = a_1 \dots a_n$. The word u is a *palindrome* if it is equal to its mirror image, that is if $u = \tilde{u}$. Given any permutation P, it is quite immediate that

$$P(\tilde{u}) = \widetilde{P(u)} \tag{10}$$

In particular, if u is a palindrome, $P(u)$ is also a palindrome. The relations between palindromes and the morphisms L and R are based on the following equality

$$R(\tilde{u}) = \widetilde{L(u)} \tag{11}$$

Indeed if $u \neq \varepsilon$, $u = a_1 \dots a_n$ with $a_i \in A$, then we have $R(\tilde{u}) = a'_n a a'_{n-1} a \dots a'_1 a$ and $L(u) = a a'_1 \dots a a'_{n-1} a a'_n$ where $a'_i = \varepsilon$ if $a_i = a$ and $a'_i = a_i$ if $a_i \neq a$. Note that the previous equality is equivalent to

$$L(\tilde{u}) = \widetilde{R(u)} \tag{12}$$

In the same manner, we can see the following fact (already mentioned in [7, Lemma 3]).

Fact 8.1 *If u is a palindrome, then $L(u)a$ is also a palindrome.*

The first connection we show is

Lemma 8.2 *Let $k \geq 1$ be an integer. If $P_1, \dots, P_k$ are permutations such that $P_1 \dots P_k(a) = a$, and for each integer i, $2 \leq i \leq k$, $P_i \dots P_k(a) \neq a$ then for each letter x in A, $LP_1LP_2 \dots LP_kR(x)$ is a palindrome.*

Proof. Let x be a letter. Let $v_x = \varepsilon$ if $x = a$ and $v_x = x$ if $x \neq a$. The word v_x is a palindrome, and $R(x) = v_x a$. Let u_k be the palindrome $P_k(v_x)$. We have $P_kR(x) = u_kP_k(a)$. If $k = 1$, $P_k(a) = a$. In this case $LP_kR(x) = L(u_k)a$ is a palindrome by Fact 8.1.

Now assume $k \geq 2$ and that for an integer i, $2 \leq i \leq k$, there exists a palindrome u_i such that

$$P_iLP_{i+1} \dots LP_kR(x) = u_i(P_iP_{i+1} \dots P_k(a))$$

By Fact 8.1, $L(u_i)a$ is a palindrome. Let $u_{i-1} = P_{i-1}(L(u_i)a)$. This word is a palindrome. Since the letter $P_iP_{i+1}\dots P_k(a)$ is different from a, we have

$$L(u_i(P_i\dots P_k(a))) = L(u_i)a(P_i\dots P_k(a))$$

It follows

$$P_{i-1}LP_i\dots LP_kR(x) = u_{i-1}(P_{i-1}P_i\dots P_k(a))$$

Thus by induction, there exists a palindrome u_1 such that

$$P_1LP_2\dots LP_kR(x) = u_1(P_1P_2\dots P_k(a))$$

Since $P_1\dots P_k(a) = a$, again by Fact 8.1, $LP_1LP_2\dots LP_kR(x)$ is a palindrome. □

Lemma 8.2 can be used to give

Another proof of Relation 9.

Let $E_1,\dots,E_k$ be permutations such that $E_1\dots E_k(a) = a$, and for each integer i, $2 \le i \le k$, $E_i\dots E_k(a) \neq a$.

Let x be a letter. Since x is a palindrome, one has

$$LE_1LE_2\dots LE_kR(x) = LE_1LE_2\dots LE_kR(\tilde{x})$$

Then using Relation (10) and (11), one gets

$$LE_1LE_2\dots LE_kR(x) = \tilde{}(RE_1RE_2\dots RE_kL(x))$$

which implies

$$\tilde{}(LE_1LE_2\dots LE_kR(x)) = RE_1RE_2\dots RE_kL(x)$$

But by Lemma 8.2, $LE_1LE_2\dots LE_kR(x)$ is a palindrome, that is,

$$\tilde{}(LE_1LE_2\dots LE_kR(x)) = LE_1LE_2\dots LE_kR(x)$$

Consequently $LE_1LE_2\dots LE_kR = RE_1RE_2\dots RE_kL$. □

We end this section by Proposition 8.3 that characterizes the episturmian morphisms that preserve palindromes. We say that a morphism f *preserves palindromes* if for any palindrome u, $f(u)$ is also a palindrome. We observe that a morphism f on A preserves palindromes if and only if $f(x)$ is a palindrome for each letter x in A.

Proposition 8.3 *An episturmian morphism preserves palindromes if and only if it is a composition of permutations and morphisms $LE_1LE_2\dots LE_kR = RE_1RE_2\dots E_kL$ where $k \ge 0$, $E_1,\dots,E_k \in$ Exch(A), $E_1\dots E_k(a) = a$, and for any i, $2 \le i \le k$, $E_i\dots E_k(a) \neq a$.*

Proof. It is immediate that a permutation preserves palindromes, and if f, g are morphisms that preserve palindromes then fg also preserves palindromes. Thus the if part of Proposition 8.3 is a an immediate consequence of Lemma 8.2.

Let us now prove the only if part. Let f be an episturmian morphism that preserve palindromes. Assume that f is not a permutation. By Lemma 7.3, there exist an integer $k \ge 1$, some exchange morphisms $E_1,\dots,E_k$, some morphisms $X_1,\dots,X_k$ in $\{L, R\}$, and a permutation P such that $f = E_1X_1\dots E_kX_kP$. We have $E_1fP^{-1} = X_1E_2\dots E_kX_k$ where P^{-1} is the permutation such that $PP^{-1} = Id$.

We denote by $\tilde{X_i}$ the unique morphism in $\{L, R\} \setminus \{X_i\}$. Relations (11) and (12) imply that $X_i(u) = \tilde{}(\tilde{X_i}(\tilde{u}))$ for any word u. Using Relation (10), for each word u, and each integer i $(1 \leq i \leq k)$, we can see by induction that

$$X_1E_2X_2 \dots E_iX_i(u) = \tilde{}(\tilde{X_1}E_2\tilde{X_2} \dots E_i\tilde{X_i}(\tilde{u})) \tag{13}$$

Since for any letter x, $f(x) = \tilde{}(f(x))$, we have $E_1fP^{-1}(x) = \tilde{}(E_1fP^{-1}(x))$. Consequently,

$$X_1E_2X_2 \dots E_kX_k(x) = \tilde{X_1}E_2\tilde{X_2} \dots E_k\tilde{X_k}(x)$$

It follows that

$$X_1E_2X_2 \dots E_kX_k = \tilde{X_1}E_2\tilde{X_2} \dots E_k\tilde{X_k}$$

By Theorem 7.1, the previous equality can be deduced by Relations (1), (2), (3), (7), (8), (9). Since $X_1 \in \{L, R\}$, and $\tilde{X_1} \neq X_1$, the relation (9) has to be used to deduce the equality. This means that there exists three morphisms g_1, g_2, g_3 such that $X_1E_2X_2 \dots E_kX_k = g_1g_2g_3$, g_2 is a permutation, $g_1 = LE'_1LE'_2 \dots LE'_lR$ with $E'_1E'_2 \dots E'_l(a) = a$, $E'_i \dots E'_l(a) \neq a$ for $2 \leq i \leq l$.

By Lemma 8.2 and Formula (13), $g_1 = X_1E_2X_2 \dots X_lE_{l+1}X_{l+1} = \tilde{X_1}E_2\tilde{X_2} \dots \tilde{X_l}E_{l+1}\tilde{X}_{l+1}$. By Relation (13), for each letter x, $g_1(g_2g_3(x)) = \tilde{}(g_1(\tilde{}(g_2g_3(x))))$. Since $E_1fP^{-1}(x) = g_1g_2g_3$, and $E_1fP^{-1}(x) = \tilde{}(E_1fP^{-1}(x))$, we get $g_1(g_2g_3(x)) = g_1(\tilde{}(g_2g_3(x)))$. Relation (10) implies $g_1g_2(\tilde{}(g_3(x))) = g_1g_2(g_3(x))$. Since g_1g_2 is injective, we get $g_3(x) = \tilde{}(g_3(x))$. It follows that g_3 preserves palindromes. The result holds by induction. □

9 Unitarity

In this section, we generalize Corollary 2.3.9 in [2].

Proposition 9.1 *Given an alphabet A, the monoid of* Episturm(A) *is both left and right unitary, i.e., for two morphisms f and g*

- *If fg and f are episturmian, then g is also episturmian.*
- *If fg and g are episturmian, then f is also episturmian.*

The left unitarity can be proved by an induction using Lemma 7.4 (see also [10, Lemma 3.2] for epistandard morphisms). The two following lemmas can be proved respectively as Lemmas 7.3 and 7.4. We keep their proof for the reader. Lemma 9.2 is necessary to prove Lemma 9.3. Right unitarity of episturmian morphisms can be proved by an induction using Lemma 9.3.

Lemma 9.2 *Let $f_1, \dots, f_n$ be some morphisms in* $\mathrm{Exch(A)} \cup \{L, R\}$. *If $f_n \dots f_1$ is not a permutation, there exist some integers $k \geq 1$, $p \geq 2$ and some morphisms $g_1, \dots, g_p$ such that*

1. $g_{2i} \in \{L, R\}$ *for* $1 \leq i \leq k$,
2. $g_{2i-1} \in \{E_{ax} \mid x \in A\}$ *for* $1 \leq i \leq k$,
3. $g_i \in \mathrm{Exch(A)}$, *for* $2k + 1 \leq i \leq p$,
4. $g_p \dots g_1 = f_n \dots f_1$ *, and the equality can be stated using formulas (1), (2), (3), (7), (8).*

Remark that Lemma 9.2 can also be proved using Lemma 7.3 and Relations (1), (2) and (3).

Lemma 9.3 *Let $f, g, h_1, \ldots, h_n$ be $n+2$ morphisms ($n \geq 1$) in $\mathrm{Exch(A)} \cup \{L, R\}$. If $gf = h_1 \ldots h_n$, then there exists $p \geq 1$ morphisms $h'_1, \ldots, h'_p$ in $\mathrm{Exch(A)} \cup \{L, R\}$, such that $g = h'_1 \ldots h'_p$. Moreover the equality $h'_1 \ldots h'_p f = h_1 \ldots h_n$ can be stated using Relations (1), (2), (3), (7), (8) and (9).*

One main interest of Proposition 9.1 is to give some algorithms to determine if a morphism is episturmian. As in [2], we explain one algorithm using right unitarity.

If an episturmian morphism g is not a permutation, then by Lemma 9.2, there exists an episturmian morphism g', and a letter x such that $g = g'LE_{ax}$ or $g = g'RE_{ax}$. Note that such a decomposition of g into $g = g'LE_{ax}$ (resp. $g = g'RE_{ax}$) exists if and only if for each letter y, $g(x)$ is a prefix (resp. suffix) of $g(y)$. By Proposition 9.1, a morphism $g = g'LE_{ax}$ (or $g = g'RE_{ax}$) is episturmian if and only if g' is episturmian. Thus, we have

Proposition 9.4 *A morphism f is episturmian if and only if, with f as input, the following algorithm ends with $g = Id$. In this case, the output h is a decomposition of f over $\mathrm{Exch(A)} \cup \{L, R\}$.*

```
Algorithm:
        input: f morphism
        output: h morphism
        local: g morphism
begin
        g ← f
        h ← Id
        While (there exists a letter x such that
                        [for each letter y, g(x) is prefix of g(y)],
                        or [for each letter y, g(x) is suffix of g(y)]) do

                if g(x) is prefix of g(y) for each letter y then
                        for y in A \ {x} and g(y) = g(x)u
                                g(y) ← u
                        exchange g(a) and g(x)
                        h ← LE_{xa} h
                else
                        for y in A \ {x} and g(y) = ug(x)
                                g(y) ← u
                        exchange g(a) and g(x)
                        h ← RE_{xa} h
                end if
        end while
        if g is a permutation then
                h ← gh.
                g ← Id
        end if
end.
```

Replacing the affectations "$g(y) \leftarrow u$" by moving an index, we can implement this algorithm in time $O(||f||)$.

10 Connections with group morphisms

In [17], Wen and Wen prove that if A is a two-letter alphabet, the Sturmian morphisms on A are exactly the invertible morphisms on A. Their proof use some group morphisms (the so-called inner automorphisms) that are related to conjugacy. In this section, we study which connections between group morphisms, conjugacy and episturmian morphisms still hold when considering alphabet with at least three letters. In Section 10.1 we recall definitions. In Section 10.2, we show that inner automorphisms are closely related with conjugacy. Finally in Section 10.3, we show that episturmian morphisms are still invertible, but that an invertible morphism is not necessarily episturmian. More precisely, we show that the monoid of invertible morphisms on an alphabet A containing at least three letters is not finitely generated (whereas Episturm(A) is).

10.1 Group morphisms

Let us first recall (see for instance [2]) how the free monoid A^* can be naturally embedded into a free group, and how a morphism can be considered as a group morphism.

Given an alphabet A, let construct a new alphabet $\overline{A}$ disjoint from A, by choosing for each letter a in A, a new letter denoted by $\overline{a}$. The alphabet $\overline{A}$ have the same cardinal than A. Then, the free monoid $(A \cup \overline{A})^*$ can be equipped with an involution by defining $\overline{\overline{a}} = a$ for a in A, and $\overline{uv} = \overline{v}\ \overline{u}$ for $u, v \in (A \cup \overline{A})^*$. The *free group* $\mathrm{F}(A)$ over A is the quotient of $(A \cup \overline{A})^*$ under the congruence relation generated by the relation $a\overline{a} \equiv \overline{a}a \equiv \varepsilon$ for any $a \in A$.

Given a group G, a *group morphism* f on G is an application from G to G such that for each x, x' in G, $f(xx') = f(x)f(x')$ and $f(e) = e$ where e is the neutral element of G.

Given an alphabet A, any morphism f on A can be considered as a group morphism on $\mathrm{F}(A)$. Necessarily, one has $f(\overline{w}) = \overline{f(w)}$ for any w in $\mathrm{F}(A)$.

10.2 Conjugacy and inner automorphisms

In [17], Wen and Wen used the following group morphisms, called *inner automorphisms*, to study Sturmian morphisms. Given an element w in $F(A)$, ι_w is defined for all elements u in $F(A)$ by $\iota_w(u) = \overline{w}uw$. These morphisms are closely related to the notion of conjugacy as shown by

Lemma 10.1 *Let f be a morphism on A such that there exists a letter x which is the first letter of $f(y)$ for each letter y with $f(y) \neq \varepsilon$. Then the first right conjugate of f exists and is $\iota_x f$.*

Proof. By Corollary 3.10, the first conjugate g of f exists. It is defined for each letter y by $f(y)x = xg(y)$. We have $x(\iota_x f)(y) = x\overline{x}f(y)x = f(y)x$. □

The following relations show how inner automorphisms act on episturmian morphisms. We let the reader verify them.

$$\begin{aligned}
R &= \iota_a L = L\iota_a && (14)\\
L\iota_b &= \iota_b R \text{ for } b \in A \setminus \{a\} && (15)\\
\iota_a R &= R\iota_a && (16)\\
\iota_x E_{xy} &= E_{xy}\iota_y \text{ for } x, y \in A && (17)\\
\iota_z E_{xy} &= E_{xy}\iota_z \text{ for } x, y \in A, z \in A \setminus \{x, y\} && (18)
\end{aligned}$$

Using the last two relations, we can deduce

Corollary 10.2 *Given a permutation P on A and two letters x, y such that $x = P(y)$, we have*

$$\iota_x P = P \iota_y.$$

Proof. Using similar arguments than for the proof of Lemma 2.2, we can decompose P over Exch(A) in $P = E_1 \dots E_k E_{xy}$ with $k \geq 0$, $E_i(x) = x$ for each i, $1 \leq i \leq k$. Thus the relation follows by Relations (17) and (18). $\square$

Inner automorphisms can be used to give new proofs for many results on conjugacy. We give an example with

Another proof of Proposition 5.2

The beginning of the proof is the same. If f has a right conjugate different from itself then, by Corollary 3.11(2), $f \notin (\text{Exch(A)} \cup R)^*$. Thus there exists an integer k such that $f_k = L$. Conversely if such an integer exists, since $L \lhd R$, by Lemma 3.1(4) $f_1 \dots f_{k-1} R f_{k+1} \dots f_n$ is a right conjugate of f.

From here, we use inner automorphisms. For each integer i between 1 and k, let x_i be the first letter of $f_i \dots f_k(a)$. Since $f_k = L$, x_i is also the first letter of $f_i \dots f_k(w)$ for any word w. In particular, x_1 is the first letter of $f_1 \dots f_n(y)$ for each letter y in A. Thus by Lemma 10.1, $\iota_{x_1} f$ is the first right conjugate of f.

Assume now that for all i, $1 \leq i \leq k-1$, $f_i \neq L$ and let g_i $(1 \leq i \leq k-1)$ be the morphisms as in the hypotheses of Proposition 5.2. We show that for $1 \leq i \leq k-1$, $\iota_{x_i} f_i = g_i \iota_{x_{i+1}}$. Note that x_i is the first letter of $f_i(x_{i+1})$. If $f_i = R$ then $x_i = x_{i+1}$. If $x_i = a$, then $g_i = R$ and by Relation (16) $\iota_{x_i} f_i = g_i \iota_{x_{i+1}}$, else $g_i = L$ and the equality holds by Relation (15). If $f_i = E_{xy}$ and $x_i \in A \setminus \{x, y\}$, then $x_i = x_{i+1}$ and $f_i = g_i$. By Relation (18), $\iota_{x_i} f_i = g_i \iota_{x_{i+1}}$. Finally, if $f_i = E_{x_i y}$, then $x_{i+1} = y$ and $f_i = g_i$. By relation (17), $\iota_{x_i} f_i = g_i \iota_{x_{i+1}}$.

By an inductive use of the equalities $\iota_{x_i} f_i = g_i \iota_{x_{i+1}}$, we get $\iota_{x_1} f = g_1 \dots g_{k-1} \iota_{x_k} f_{k+1} \dots f_n$. The proof is ended since $x_k = a$ and $\iota_a L = R$ (Relation (14)). $\square$

10.3 Invertibility and episturmian morphisms

Given an alphabet A, a morphism f on A is *invertible* if considered as a group morphism, it is an automorphism, that is, there is a morphism f^{-1} called *inverse*, such that $ff^{-1} = f^{-1}f = Id$. It is well-known that permutations are invertible. Indeed, any permutation can be decomposed over exchange morphisms (see Lemma 2.2), and any exchange morphism is its own inverse (see Relation (1)).

We let the reader verify that the following morphisms L^{-1} and R^{-1} are respectively inverses of L and R.

$$L^{-1}: \begin{cases} a \to a \\ x \to \overline{a}x, & \forall x \in A \setminus \{a\} \end{cases} \qquad R^{-1}: \begin{cases} a \to a \\ x \to x\overline{a}, & \forall x \in A \setminus \{a\} \end{cases}$$

Consequently, we have the following fact which has been already mentionned without proof in [10].

Fact 10.3 *Any episturmian morphism is invertible*

Proof. Let f be an episturmian morphism. There exist some morphisms $f_1, \dots, f_n$ in Exch(A) $\cup$ $\{L, R\}$ such that $f = f_1 \dots f_n$. Each morphism f_i is invertible. Let f_i^{-1} be its inverse. It is now straightforward that $f_n^{-1} f_{n-1}^{-1} \dots f_1^{-1}$ is the inverse of f. $\square$

Note that the invertibility of episturmian morphisms, as said in [10], is another reason for the left and right cancellativity of the monoid Episturm(A) (Lemma 7.2).

Now let A be an alphabet containing at least three letters. If a and b are two different letters, the following morphism is not episturmian: $\begin{cases} a \to a \\ b \to ab \\ x \to x,\ \forall x \in A \setminus \{a, b\} \end{cases}$

But it is invertible and its inverse is $\begin{cases} a \to a \\ b \to \overline{a}b \\ x \to x,\ \forall x \in A \setminus \{a, b\} \end{cases}$

Thus an invertible morphism is not necessarily episturmian. The next theorem gives a deep reason for that.

Theorem 10.4 *Consider an alphabet A containing at least three letters. The monoid of invertible morphisms on A is not finitely generated.*

Proof. Let $p = \#A$. We denote the alphabet A by $\{a, b, c, a_4, \dots, a_p\}$. For any integer $n \geq 2$, we define the morphism f_n and the group morphism h_n by

$$\begin{cases} f_n(a) = ab \\ f_n(b) = acb \\ f_n(c) = ac^n ba \\ f_n(a_i) = a_i \ (\forall i, 4 \leq i \leq p) \end{cases} \qquad \begin{cases} h_n(a) = (\overline{b}a)^{n-1}\overline{b}c \\ h_n(b) = \overline{h_n(a)}a \\ h_n(c) = \overline{h_n(a)}b\overline{a}h_n(a) \\ h_n(a_i) = a_i \ (\forall i, 4 \leq i \leq p) \end{cases}$$

It is straigthforward to verify that $f_n h_n = h_n f_n = Id$. So f_n is invertible, and h_n is its inverse.

In order to end the proof or Theorem 10.4, we state the following fact which proves a kind of primitivity of the morphisms f_n.

Fact 10.5 *For each integer $n \geq 2$, if $f_n = \alpha\beta$ with α and β two invertible morphisms on A, then (exactly) one of the two morphisms α and β is a permutation.*

Before proving Fact 10.5, we make a remark.

Remark 10.6 *Given two morphisms α and β, a permutation P on A and its inverse P^{-1}, if $f_n = \alpha\beta$, then*

- $f_n = (\alpha P^{-1})(P\beta)$,
- αP^{-1} *is a permutation if and only if α is a permutation,*
- $P\beta$ *is a permutation if and only if β is a permutation.*

Proof of Fact 10.5.

By hypothesis, α is invertible. Thus it is non-erasing, that is, for each letter x in A, $f(x) \neq \varepsilon$.

For $4 \leq i \leq p$, one has $|\alpha\beta(a_i)| = 1$. Thus $|\beta(a_i)| = 1$. Since β is invertible, for $4 \leq i < j \leq p$, $\beta(a_i) \neq \beta(a_j)$. Consider a permutation P on A such that $P(\beta(a_i)) = a_i$, for each i, $4 \leq i \leq p$. By Remark 10.6, one can replace α by αP^{-1} and β by $P\beta$. In other words, one can assume $\beta(a_i) = a_i$ and $\alpha(a_i) = a_i$ for each i, $4 \leq i \leq p$.

Now, since for each i, $4 \leq i \leq p$, the letter a_i does not appear in the words $f_n(a)$, $f_n(b)$, $f_n(c)$, the three words $\beta(a)$, $\beta(b)$ and $\beta(c)$ belong to $\{a, b, c\}^*$. Since α is non-erasing, $|\beta(a)| = 1$ or $|\beta(a)| = 2$.

• If $|\beta(a)| = 1$, we prove that β is a permutation. By Remark 10.6, one can assume $\beta(a) = a$ (Consider a permutation P such that $P(a_i) = a_i$, $4 \leq i \leq p$, and $P(\beta(a)) = a$). In this case $\alpha(a) = ab$. Since ab is neither a subword of $f_n(b)$ nor of $f_n(c)$, the two words $\beta(b)$ and $\beta(c)$ belong to $\{b,c\}^*$. Once again, by Remark 10.6, one can assume that the first letter of $\beta(b)$ is b (Consider a permutation P such that $P(a_i) = a_i$, $4 \le i \le p$, $P(a) = a$ and $P(x) = b$ where x is the first letter of $\beta(b)$). Since α is non-erasing and since no letter appears twice in $f_n(b)$, one has $\beta(b) = b$ or $\beta(b) = bc$. If $\beta(b) = b$, one gets $\alpha(b) = acb$. Since acb is not a subword of $f_n(c)$ $(n \ge 2)$, $\beta(c) \in \{c\}^*$, and necessarily $\beta(c) = c$, that is β is a permutation. If $\beta(b) = bc$, one gets $(\alpha(b) = a$ and $\alpha(c) = cb)$ or $(\alpha(b) = ac$ and $\alpha(c) = b)$. In the two cases, $f_n(c)$ cannot be decomposed over $\{\alpha(b), \alpha(c)\}$: a contradiction with $\beta(c) \in \{b,c\}^*$.

• If $|\beta(a)| = 2$, we prove that α is a permutation. Note that the two letters of $\beta(a)$ are necessarily different. By Remark 10.6, one can assume $\beta(a) = ab$ (If $\beta(a) = xy$ with x, y two letters, consider a permutation P such that $P(x) = a$, $P(y) = b$, $P(a_i) = a_i$, $4 \le i \le p$). This implies $\alpha(a) = a$, $\alpha(b) = b$. Since $\beta(b), \beta(c) \in \{a,b,c\}^*$, the words $f_n(b), f_n(c)$ can be both decomposed on $\alpha(a) = a$, $\alpha(b) = b$ and $\alpha(c)$. Thus c occurs in $\alpha(c)$. But since cc occurs in $f_n(c)$, and since it does not occur in $f_n(b)$, $\alpha(c) = c$ □

Let now end the proof of Theorem 10.4. Let G be a set of generators of the monoid of invertible morphisms. For each $n \ge 0$, there exists an integer $p \ge 0$ and some morphisms $g_1, g_2, \ldots, g_p$ in G such that $f_n = g_1 g_2 \ldots g_p$. Since f_n is not a permutation, there is at least one integer i such that g_i is not a permutation. If for at least two integers i and j, $i < j$, g_i and g_j are not permutations of A, then $g_1 \ldots g_i$ and $g_{i+1} \ldots g_p$ are two invertible morphisms which are not permutations: a contradiction with Fact 10.5. Thus, i is the only integer between 1 and p such that g_i is not a permutation. But then $||g_i|| = ||f_n|| = n + \#A + 5$, which implies that for different values of n the g_i's are all different. Since $g_i \in G$ and n can be any natural integer different from 0 and 1, the set G is infinite. □

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