Some non finitely generated monoids of repetition-free endomorphisms

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Abstract

We answer a question raised by Mitrana in Information Processing Letters 64. Given an alphabet A with $\mathrm{Card}(A) \geq 2$, the monoid of primitive endomorphisms on A and the monoid of primitive uniform endomorphisms on A are not finitely generated. Moreover we show that it is also the case for the following monoids: the monoid of overlap-free (uniform) endomorphisms on A (when $\mathrm{Card}(A) \geq 3$), the monoid of k-power-free (uniform) endomorphisms on A (when $\mathrm{Card}(A) \geq 2$ and $k \geq 3$).

1 Introduction

Thue [17] (see also [3]) showed the existence of an infinite overlap-free word over the alphabet $\{a, b\}$ and the existence of an infinite 2-power-free word over the alphabet $\{a, b, c\}$. One way to generate such words is to use morphisms that preserve overlap-free words, or, 2-power-free words. Such morphisms are called respectively overlap-free morphisms and 2-power-free morphisms.

Since the works of Thue [17], it is known that the monoid of overlap-free endomorphisms on a binary alphabet is finitely generated. Trying to decompose a morphism on the set of generators gives an efficient algorithm to determine if a binary endomorphism is overlap-free. In Section 3, we prove that when considering alphabets with at least three letters, the monoid of overlap-free endomorphisms and the monoid of uniform overlap-free endomorphisms are not finitely generated. Let us note that Wlazinski and the author designed an algorithm to decide overlap-freeness for uniform morphisms [14]. Despite of other recent works on overlap-free endomorphisms (see for instance [1, 12, 16]), up to we know, no general algorithm to decide overlap-freeness of morphisms is known.

In Section 4, we extend the results of Section 3 to the case of k-power-free morphisms (with $k \geq 3$ an integer). This last result is true even for binary alphabets. This result was also known for the monoid of 2-power-free (uniform) endomorphisms on alphabets containing at least three letters [6]. When $k \geq 3$, the result was already proved in [15] but not in the case of uniform endomorphisms. The proof we propose here is new. The reader which is interested in checking whether an endomorphism if k-power-free can find partial answers for instance in [2, 5, 8, 9, 13, 15, 18].

In [10], Mitrana gives a study of primitive morphisms. Among other results, he proves that it is decidable whether a morphism is primitive, and he gives a characterization in case of binary uniform morphisms. These two results were already proved by Keränen [8] by a different way. At the end of his paper, Mitrana asked "Is the monoid of (uniform) primitive endomorphisms of A^+ finitely generated?". In Section 5, as a consequence of the previous results, we give a negative answer.

2 Preliminaries

In this section, we recall basic notions on words and morphisms, and we present the technics of proof used in the next sections.

Let A be a (finite) *alphabet*. We denote by $\mathrm{Card}(A)$ its *cardinality*. The free monoid A^* is the set of words over A including the empty word ε. The length $|u|$ of a word u is the number of occurrences of letters in u. A word u is a *factor* (resp. *prefix*, *suffix*) of a word v if $v = pus$ (resp. us, pv) for (possibly empty) words p and s.

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for any integer $n \geq 1$, $u^n = uu^{n-1}$. A word w is *primitive* if $w = u^n$ implies $n = 1$. For k an integer, a word w is *k-power-free* (resp. *overlap-free*) if it contains no factor on the form u^k with $u \neq \varepsilon$ (resp. no factor on the form $\alpha v \alpha v \alpha$ with $\alpha \in A$

and $v \in A^*$).

Let A, B be two alphabets. Let f be a *morphism* from A^* to B^*, that is a mapping from A^* to B^* such that for all $u, v \in A$, $f(uv) = f(u)f(v)$. Given a set X of words over A, let $f(X) = \{f(w) \mid w \in X\}$. The morphism f is entirely known by $f(A)$. When $B = A$, f is called an *endomorphism* on A. The morphism f is *uniform* if there exists an integer $L \geq 0$ such that $|f(a)| = L$ for each letter a in A.

Let P be a property of words (for instance k-power-free for an integer k, overlap-free, or primitive). The morphism f from A^* to B^* is said P if $f(w)$ is P for all words w over A verifying P. If f is primitive, then it is *non-erasing*, that is, $\varepsilon \notin f(A)$. If f is k-power-free (when $k \geq 2$) or overlap-free, it can be erasing, but then $f(A) = \{\varepsilon\}$. If f verifies for all a in A, $|f(a)| = 1$, and $\mathrm{Card}(f(A)) = \mathrm{Card}(A)$ then f is primitive, k-power-free and overlap-free. When $A = B$, such a morphism is called a *permutation*. Note that if f is a non-erasing overlap-free morphism, then f is strongly biprefix (see [7]), that is, $f(a)$ and $f(b)$ start with different letters and end with different letters, for all different letters a and b in A. If f is strongly biprefix, for $a \in A$, $f(a)$ is uniquely determined by its first letter, and also by its last letter.

We denote by juxtaposition the composition of two morphisms. Powers of an endomorphism f are defined by f^0 is the identity morphism, and $f^n = ff^{n-1}$ for $n \geq 1$. An endomorphism f on A is said *irreducible* if whenever $f = h_1h_2$ for endomorphisms h_1 and h_2 on A, h_1 or h_2 is a permutation.

If f and g are two overlap-free (resp. k-power-free, primitive) endomorphisms on an alphabet A, then fg is also overlap-free (resp. k-power-free, primitive). Thus the set of overlap-free (resp. k-power-free, primitive) endomorphisms on an alphabet A is a monoid with neutral element the identity morphism.

Let M be a monoid. A set of generators of M is a subset G of M such that any element of M can be decomposed over G. A monoid M is *finitely generated* if it has a finite set of generators. To show that a monoid M of endomorphisms is not finitely generated, it is sufficient to show that there exists an infinite family of irreducible endomorphisms in M. Indeed in this case for each morphism f in this infinite family, there exist two permutations p_1 and p_2 such that p_1fp_2 is a generator of M. Since we consider only finite alphabets, there exists a finite number of permutations. So the set of generators of M is infinite.

3 Monoids of overlap-free endomorphisms

The monoid of overlap-free endomorphisms on $A = \{a, b\}$ is known ([17], see also [3]) to be finitely generated by the exchange morphism E ($E(a) = b$ and $E(b) = a$) and by the so-called (since the works of Morse [11]) Thue-Morse morphism μ defined by $\mu(a) = ab$, $\mu(b) = ba$. This result is no longer true when considering larger alphabet:

Theorem 3.1 *For any alphabet A containing at least three letters, the monoid of overlap-free endomorphisms on A and the monoid of overlap-free uniform endomorphisms on A are not finitely generated.*

The proof of this theorem is given by the existence of the following infinite family of overlap-free (Lemma 3.3) irreducible (Lemma 3.4) uniform endomorphisms.

From now on, $p \geq 3$ is an integer. Let $A_p = \{a_0, \dots, a_{p-1}\}$ be a p-letter alphabet, and let $I_p = \{0, \dots, p-1\}$. For any integer i, we consider that $a_i = a_{i \bmod p}$. For an integer $n \geq 1$, we define the endomorphism f_n on A by:

$$\forall i \in I_p, \quad f_n(a_i) = a_i C_i(\mu^{2n}(a))$$

where C_i is the morphism from $\{a, b\}^*$ to $\{a_{i+1}, a_{i+2}\}^*$ such that $C_i(a) = a_{i+1}$ and $C_i(b) = a_{i+2}$. For $i \in I_p$, let x_i be the prefix of length 4 of $f_n(a_i)$: $x_i = a_i a_{i+1} a_{i+2} a_{i+2}$. Since $p \geq 3$, the following property of the morphisms f_n can be easily verified:

Lemma 3.2 *For $n \geq 1$, $i \in I_p$, $u, \beta, \gamma \in A_p^*$, we have $f_n(u) = \beta x_i \gamma$ if and only if there exist words u_1, u_2 such that $u = u_1 a_i u_2$, $f_n(u_1) = \beta$ and $f_n(a_i u_2) = x_i \gamma$.*

Lemma 3.3 *For $n \geq 1$, f_n is overlap-free.*

Proof. Let $n \geq 1$ be an integer. Let w, P, v, S be words and let α be a letter such that $f_n(w) = P\alpha v \alpha v \alpha S$. If for each $i \in I_p$, x_i is not a factor of $\alpha v \alpha v \alpha$, then $\alpha v \alpha v \alpha$ must be a factor of $C_j(\mu^{2n}(a)) a_k a_{k+1} a_{k+2}$ for some integers $j, k \in I_p$. Since $p \geq 3$, at least one of the three letters a_k, a_{k+1} or a_{k+2} occurs only once in $C_j(\mu^{2n}(a)) a_k a_{k+1} a_{k+2}$. Thus $\alpha v \alpha v \alpha$ must be a factor of $C_j(\mu^{2n}(a)) a_{j+1} a_{j+2} = C_j(\mu^{2n}(a)ab)$ or of $C_j(\mu^{2n}(a)) a_{j+2} = C_j(\mu^{2n}(a)b)$. Since $\mu^{2n}(a)ab$ (resp. $\mu^{2n}(a)b$) is a prefix of $\mu^{2n}(aa)$ (resp. $\mu^{2n}(ab)$), and since μ^{2n} and C_j are overlap-free morphisms, the words $C_j(\mu^{2n}(a)ab)$ and $C_j(\mu^{2n}(a)b)$ are overlap-free. A contradiction.

Thus for at least one i in I_p, x_i is a factor of $\alpha v \alpha v \alpha$, so of αv or of $v\alpha$. We treat only the first case (the second is similar). There exist two words v_1, v_2 such that $\alpha v = v_1 x_i v_2$: $f_n(w) = P v_1 x_i v_2 v_1 x_i v_2 \alpha S$. By Lemma 3.2, there exist words w_1, w_2, w_3 such that $w = w_1 a_i w_2 a_i w_3$, $f_n(w_1) = P v_1$, $f_n(a_i w_2) = x_i v_2 v_1$, $f_n(a_i w_3) = x_i v_2 \alpha S$.

If $v_1 = f_n(w_4)$ for a word w_4 then, since f_n is strongly biprefix, w_4 is a suffix of w_1 and of $a_i w_2$. Let w_5, w_6 be the words such that $w_1 = w_5 w_4$, and $a_i w_2 = w_6 w_4$. We have $f_n(w_6) = x_i v_2$. Thus w_6 is a prefix of $a_i w_3$. For a word w_7, $a_i w_3 = w_6 w_7$. We have $f_n(w_7) = \alpha S$, and $f_n(w_4 w_6) = \alpha v$. So, by definition of f_n, α is the first letter of both w_7 and $w_4 w_6$. Since w contains $w_4 w_6 w_4 w_6 w_7$ as a factor, w is not overlap-free.

If v_1 is not an image of a word, then there exist a letter a and words w_4, w_5, $v_3 \neq \varepsilon$, $v_4 \neq \varepsilon$ such that $a_i w_2 = w_4 a w_5$, $x_i v_2 = f_n(w_4) v_3$, $f_n(a) = v_3 v_4$, $v_1 = v_4 f_n(w_5)$. From f_n strongly biprefix, we deduce that $w_4 a$ is a prefix of $a_i w_3$ and that $a w_5$ is a suffix of w_1. Thus the word w contains the factor $a w_5 w_4 a w_5 w_4 a$, and so is not overlap-free. $\square$

Lemma 3.4 *For $n \geq 1$, f_n is irreducible.*

Proof. Let $n \geq 1$ be an integer. Let h_1, h_2 be two endomorphisms on A_p such that $f_n = h_1 h_2$. Then, for each i in I_p, the word $f_n(a_i)$ can be decomposed over $h_1(A_p)$. By definitions of f_n and h_1, $h_1(A_p)$ contains exactly p words $u_1, \ldots, u_p$ such that, for $i \in I_p$, u_i is the only word in $h_1(A_p)$ starting with a_i, and $f_n(a_i)$ starts with u_i.

Fact A. For each $i \in I_p$, $u_i \neq a_i a_{i+1}$.
Proof. Assume that $u_i = a_i a_{i+1}$ for some $i \in I_p$. By definition of f_n and h_1, two different words in $h_1(A_p)$ must end with two different letters. But $f_n(a_i)$ ends with $a_{i+2} a_{i+1}$ since $n \geq 1$. Thus u_i which is the only word of $h_1(A_p)$ ending with a_{i+1} must end with $a_{i+2} a_{i+1}$. A contradiction since $p \geq 3$. $\square$

Fact B. If for one $i \in I_p$, $u_i = a_i$ then $h_1(A_p) = A_p$.
Proof. Assume that $u_i = a_i$ for some $i \in I_p$. We show that $u_{i+1} = a_{i+1}$. The proof ends by induction since $h_1(A_p) = \{u_1, \ldots, u_p\}$ and $a_p = a_0$. From $u_i = a_i$, we get that $C_i(\mu^{2n}(a))$ starts with u_{i+1}. It is also the case for $f_n(a_{i+1})$. Since $n \geq 1$, $C_i(\mu^{2n}(a))$ starts with $a_{i+1} a_{i+2} a_{i+2}$. Moreover $f_n(a_{i+1})$ starts with $a_{i+1} a_{i+2} a_{i+3}$. Thus $u_{i+1} = a_{i+1}$ or $u_{i+1} = a_{i+1} a_{i+2}$. By Fact A, $u_{i+1} = a_{i+1}$. $\square$

Fact C. For $i \in I_p$, if $u_i \neq a_i$ then $u_i = f_n(a_i)$.

Proof. Assume that $u_i \neq a_i$ with $i \in I_p$. By Fact A, the prefix u_i of $f_n(a_i)$ starts with $a_i a_{i+1} a_{i+2}$ and ends with $a_{i+1}a_{i+1}$, $a_{i+1}a_{i+2}$, $a_{i+2}a_{i+1}$, or, $a_{i+2}a_{i+2}$. Remember that two different words in $h_1(A_p)$ must end with two different letters. The word $f_n(a_{i+1})$ ends with $a_{i+3}a_{i+2}$. Since $p \geq 3$, $a_{i+3} \notin \{a_{i+1}, a_{i+2}\}$. Thus u_i ends with a_{i+1}. Since a_{i+1} is the last letter of $f_n(a_i)$, u_i is a suffix of $f_n(a_i)$. We cannot have $f_n(a_i) = u_i v u_i$ for a word v, since a_i occurs only once in $f_n(a_i)$. So $u_i = f_n(a_i)$. $\square$

Now we end the proof of Lemma 3.4. If $u_1 = a_1$, then by Fact B, $h_1(A_p) = A_p$ which means that h_1 is a permutation. If $u_1 \neq a_1$, then, also by Fact B, we deduce that $u_i \neq a_i$ for each $i \in I_p$. It follows by Fact C that $h_1(A_p) = f_n(A_p)$. This implies that h_2 is a permutation. So f_n is irreducible. $\square$

4 Monoids of k-power-free endomorphisms

We extend Theorem 3.1 to the case of k-power-free endomorphisms.

Theorem 4.1 *For any alphabet A containing at least two letters, and for any integer $k \geq 3$, the monoid of k-power-free endomorphisms on A and the monoid of k-power-free uniform endomorphisms on A are not finitely generated.*

When $\mathrm{Card}(A) \geq 3$, the proof of this theorem follows by the fact that the family of endomorphisms f_n $(n \geq 1)$ introduced in the previous section is an infinite family of irreducible k-power-free morphisms as shown by Lemma 3.4 and

Lemma 4.2 *For $n \geq 1$ and $k \geq 3$, f_n is k-power-free.*

Proof. Let $n \geq 1$ and $k \geq 3$ be integers. Let w, P, v, S be words such that $f_n(w) = Pv^kS$ and $v \neq \varepsilon$. We prove that w is not k-power-free. Since any overlap-free word is k-power-free, the words $\mu^{2n}(aa)$ and $\mu^{2n}(ab)$ are k-power-free. As in the proof of Lemma 3.3, we can see that x_i is a factor of v^k for at least one integer i in I_p. We choose the first occurrence in v^k of such an x_i. Two cases can hold. In both cases, we show that w is not k-power-free.

Case 1: $v = v_1 x_i v_2$ for words v_1, v_2. In this case, $|v_1| < 1 + |\mu^{2n}(a)|$ and $f_n(w) = Pv_1(x_i v_2 v_1)^{k-1} x_i v_2 S$. By Lemma 3.2, there exist words w_1, w_2, w_3 such that $w = w_1 w_2^{k-1} w_3$, $f_n(w_1) = Pv_1$, $f_n(w_2) = x_i v_2 v_1$, $f_n(w_3) = x_i v_2 S$

(remember that f_n is strongly biprefix and so injective). Let a be the last letter of w_2. Let w'_2 be the word such that $w_2 = w'_2 a$. Let v_3 be the word such that $f_n(a) = av_3v_1$. We have $x_iv_2 = f_n(w'_2)av_3$. Since f_n is strongly biprefix, w'_2a is a prefix of w_3: $(w'_2a)^k$ is a factor of w.

Case 2: $x_i = y_iz_i$, $v = z_iuy_i$ for words u, $y_i \neq \varepsilon$, $z_i \neq \varepsilon$. In this case, $f_n(w) = Pz_iu(x_iu)^{k-1}y_iS$. There exist words w_1, w_2, w_3 such that $w = w_1w_2^{k-1}w_3$, $f_n(w_1) = Pz_iu$, $f_n(w_2) = x_iu = y_iz_iu$, $f_n(w_3) = y_iS$. Since f_n is strongly biprefix, we can see as above that w_2 is a suffix of w_1: w_2^k is a factor of w. $\square$

To end this section, we prove Theorem 4.1 when $\mathrm{Card}(A) = 2$. We present a new infinite family of irreducible k-power-free endomorphims on $A_2 = \{a, b\}$. For $n \geq 0$, let g_n be the endomorphism on A_2 defined by $g_n(a) = g(f_n(b))$ and $g_n(b) = g(f_n(c))$, where f_n is the endomorphism defined in Section 3 on A_3 with $a_0 = a$, $a_1 = b$ and $a_2 = c$, and g is the morphism from $\{a, b, c\}^*$ to A_2^* defined by $g(a) = aababb$, $g(b) = aabbab$, $g(c) = abbaab$.

The morphism g was introduced by Brandenburg [4] who showed that g is 3-power-free. Actually g is k-power-free for any integer $k \geq 3$ [15]. Let C be the morphism from A_2^* to $\{b, c\}^*$ defined by $C(a) = b$, and $C(b) = c$. We have for all $w \in A_2^*$, $g_n(w) = g(f_n(C(w)))$. Since C is k-power-free, by Lemma 4.2, we have

Lemma 4.3 *For $n \geq 1$, $k \geq 3$, g_n is k-power-free.*

Lemma 4.4 *For $n \geq 1$, g_n is irreducible.*

Proof. Let $n \geq 1$. Let h_1 and h_2 be two endomorphisms on A_2 such that $g_n = h_1h_2$. The words $g_n(a)$ and $g_n(b)$ can be decomposed over $\{h_1(a), h_1(b)\}$.

If $h_1(a) = a$, since $g_n(a)$ ends with ab and $g_n(b)$ ends with bb, we must have $h_1(b) = b$. If $h_1(a) = b$, since $g_n(a)$ starts with aa, and $g_n(b)$ starts with ab, we must have $h_1(b) = a$. The case $|h_1(b)| = 1$ similarly implies $|h_1(a)| = 1$ and $h_1(a) \neq h_1(b)$. So if $|h_1(a)| = 1$ or $|h_1(b)| = 1$, then h_1 is a permutation.

From now on, we assume $|h_1(a)| \geq 2$ and $|h_1(b)| \geq 2$. Let us observe that $g_n(a)$ starts with $g(bca) = aabbababbaabaababb$ and $g_n(b)$ starts with $g(cab) = abbaabaababbaabbab$. Looking at the first two letters of $g_n(a)$ and $g_n(b)$, we can see that one of the two words $h_1(a)$ or $h_1(b)$ starts with aa, and the other starts with ab. Without loss of generality, we assume that $h_1(a)$ starts with aa (it is a prefix of $g_n(a)$), and that $h_1(b)$ starts with ab (it is a prefix of $g_n(b)$). Since the third letter of $g_n(a)$ and $g_n(b)$ is b, $h_1(a) \neq aa$ and

$h_1(b) \neq ab$. Similarly $h_1(a) \neq aab$. So $aabb$ is a prefix of $h_1(a)$. Since $aabb$ has only one occurrence as a factor in $g_n(a)$, we have $g_n(a) = h_1(ab^i)$ for some $i \geq 0$. If $i \geq 1$, then $aabb$ is not a factor of $h_1(b)$, $h_1(b)$ starts with abb (since $|h_1(b)| \geq 2$ and $h_1(b) \neq ab$), and ends with aab (since it is a suffix of $g_n(a)$). Since $|h_1(b)| < |g_n(a)| = |g_n(b)|$, $h_1(b)aabb$, or $h_1(b)abb$ is a prefix of $g_n(b)$. From what preceeds, we must have $h_1(b) = abbaabaab$: a contradiction with $h_1(b)$ suffix of $g_n(a)$. So $g_n(a) = h_1(a)$. Since g_n is uniform, $h_1(a)$ cannot be a factor of $g_n(b)$. Since the prefix $abbaabaab$ of $g_n(b)$ occurs only once in $g_n(b)$, we must have $g_n(b) = h_1(b)$. Thus $|h_2(a)| = |h_2(b)| = 1$: h_2 is a permutation.

□

5 Monoids of primitive morphisms

In [9], it is shown that for any integer $k \geq 4$, any k-power-free morphism is primitive. Thus Lemmas 4.2 and 4.3 show that, $\forall n \geq 1$, f_n and g_n are primitive. So Lemmas 3.4 and 4.4 imply:

Theorem 5.1 *For any alphabet A containing at least two letters, the monoid of primitive endomorphisms on A and the monoid of primitive uniform endomorphisms on A are not finitely generated.*

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