Conjugacy of morphisms and Lyndon decomposition of standard Sturmian words

An answer to a question of G. Melançon

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Outline

- Preliminaries
  - Standard Sturmian words
  - Melançon's result: decomposition in Lyndon words
  - Melançon's question
- Tools
  - Sturmian morphisms
  - Strong conjugacy
  - Key result
- Main result: answer to Melançon's question
- Another consequence of Strong conjugacy
- Conclusion

Standard Sturmian words

- Definition: An infinite word w over $\{a, b\}$ is called standard Sturmian if and only if there exists a sequence $(d_n)_{n\geq 1}$ of integers such that $d_1 \geq 0$, $d_k \geq 1$ for $k \geq 2$ and

$$w = \lim_{n\to\infty} s_n$$

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- General remark:

  for $n \geq 1$, s_{2n} ends with a.

  Notation: $s_{2n}a^{-1} = s_{2n}$ without its last a

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- Melançon (2000): decomposition of standard Sturmian words
  - $w((d_n)_{n\geq 1}) = \prod_{n\geq 0} {\ell_n}^{d_{2n+1}}$ where for $n \geq 0$, $\ell_n = a s_{2n}^{d_{2n+1}-1} s_{2n-1} s_{2n} a^{-1}$
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- Remark on the example (Melançon 2000):
  for any $n \geq 1$, $\ell_n = f(\ell_{n-1})$
  where $f : \begin{cases} a \mapsto aaabaab \\ b \mapsto aab \end{cases}$

Melançon's question (2000)

- When is the sequence $(\ell_n)_{n\geq 0}$ morphic?
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- Remark (still Melançon (2000)):
  Such a morphism exists when the sequence $(d_n)_{n\geq 1}$ is constant.

Sturmian morphisms

- Sturmian morphisms: elements of $\{L_a, L_b, R_a, R_b, E\}^*$

$$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \quad L_b : \begin{cases} a \mapsto ba \\ b \mapsto b \end{cases} \quad E : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases}$$

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- Remark (See Berstel, Séébold in Lothaire II):

for any $n \geq 0$, $\begin{cases} s_{2n} = f_n(a) \\ s_{2n-1} = f_n(b) \end{cases}$

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- Remark: $\ell_n a = a s_{2n}^{d_{2n+1}-1} s_{2n-1} s_{2n} = a f_n(a^{d_{2n+1}-1} ba)$

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- $f_n = L_a^{d_1} L_b^{d_2} \dots L_a^{d_{2n-1}} L_b^{d_{2n}}$ is strongly conjugated to $g_n = L_a^{d_1} R_b^{d_2} \dots L_a^{d_{2n-1}} R_b^{d_{2n}}$

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	$= a f_n(a^{d_{2n+1}}b) u_n a$	$u_n \mid f_n(a) = u_n a$
	$= a u_n g_n(a^{d_{2n+1}}b) a$	f_n et g_n u_n-conjugated
	$= g_n(a^{d_{2n+1}}b) a$	$g_n(a) = a u_n$

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- Remark. A direct proof that: $\ell_n \succeq \ell_{n-1}$.

Since g_n preserves the lexicographic order (Richomme 2003)

Answer to Melançon's question

There exists a morphism f such that

for all $n \geq 1$, $\ell_n = f(\ell_{n-1})$

if and only if one of the two following cases hold

Case 1. $\begin{cases} 1 \leq d_1 \leq d_3 \\ (d_n)_{n\geq 1} = (d_1, d_2, d_3, d_2, d_3, \ldots) \end{cases}$

In this case $\begin{cases} \ell_0 = a^{d_1} b \\ f = L_a^{d_1} R_b^{d_2} L_a^{d_3-d_1} \end{cases}$

Case 2. $\begin{cases} d_1 = 0,\ 1 \leq d_2 \leq d_4 \\ (d_n)_{n\geq 1} = (0, d_2, d_3, d_4, d_3, d_4, \ldots) \end{cases}$

In this case $\begin{cases} \ell_0 = b \\ f = R_b^{d_2} L_a^{d_3} R_b^{d_4-d_2} \end{cases}$

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- So the word $aw((d_n)_{n\geq 1})$ has an infinity of prefixes that are Lyndon words: it is an infinite Lyndon word.

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- But g_n preserves Lyndon word (Richomme 2003)
- So the word $aw((d_n)_{n\geq 1})$ has an infinity of prefixes that are Lyndon words: it is an infinite Lyndon word.
- This proves a result of Borel and Laubie (1993)

  For any standard Sturmian word w,
  aw is an infinite Lyndon word

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- Further work: for any Sturmian word, using its decomposition over Sturmian morphisms, we are looking for its decomposition in Lyndon words.
- Remark: recently, with F. Levé, we have obtained a characterization of the Sturmian words that are infinite Lyndon words : they are the non-quasiperiodic Sturmian words.