On a conjecture about finite fixed points of morphisms (extended abstract)

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Abstract

A conjecture of M. Billaud is: Given a word w, if for each letter x occurring in w, the word obtained by erasing all the occurrences of x in w is a fixed point of a non-trivial morphism f_x, then w is also a fixed point of a non-trivial morphism. We prove that this conjecture is equivalent to a similar one on sets of words. Using this equivalence, we solve these conjectures in the particular case where each morphism f_x has only one expansive letter.

1 Introduction

One kind of combinatorial problems is to determine some conditions for which an object is characterized by partial informations. For instance, in traces theory [8, 9], a trace is entirely determined by its projections on sets $\{a, b\}$ of dependent letters [7, 10]. In combinatorics on words, the exact length of factors needed to reconstruct a word has been recently stated [3] (see also [4, 5, 14]).

The conjecture we examine here belongs to this kind of problems. It concerns fixed points of morphisms. A lot of studies deal with infinite fixed points of morphisms (see, e.g., [1, 6, 15, 16]). There exist also studies of biinfinite words (see, e.g., [17]). Here we consider only finite words that are fixed points of a morphism. T. Head [12] characterized the language of finite fixed points of a morphism. In Section 3, we recall this result considering constant, mortal and expansive letters of a morphism.

In 1993, in newsgroup comp.theory, M. Billaud conjectured:

> *Given a word w, if for each letter x occurring in w, there exists a non-trivial morphism f_x such that the word obtained by erasing all the occurrences of x in w is a fixed point of f_x, then there exists a non-trivial morphism f such that w is a fixed point of f.*

P. Zimmermann [18] solved the conjecture on three-letter alphabets. Moreover A. Geser [11] suggested to use idempotent morphisms.

The conjecture is a particular case of the following one:

> *Given a set S of words, if for each letter x occurring in words of S, there exists a non-trivial morphism f_x such that, for each word w in S, the word obtained by erasing all the occurrences of x in w is a fixed point of f_x, then there exists a non-trivial morphism f such that each word of S is a fixed point of f.*

In Section 4, we prove that the two conjectures are equivalent. In Section 5, we generalize the result of P. Zimmermann to the case where each morphism f_x has no constant letter and only one expansive letter. Results of Sections 4 and 5 are used in Section 6 to prove the conjectures by induction when each morphism f_x has only one expansive letter (the morphisms f_x can have constant letters).

2 Generalities

We assume the reader is familiar with combinatorics on words and morphisms (see, e.g., [6, 15, 16]). We precise our notations.

Given a finite set X, we denote by $\mathrm{Card}(X)$ its cardinality.

Given an alphabet A (a non-empty set of letters), A^* is the set of words over A including the empty word ε. We denote the concatenation on words by juxtaposition: uv is the word obtained by concatenation of u and v. If X is a set of words, we denote by X^* (resp. X^+) the set of all words that are finite (resp. non-empty finite) concatenation of words of X: $\varepsilon \in X^*$, $X^* = X^+ \cup \{\varepsilon\}$.

Let w be a word. We denote by $\mathrm{alph}(w)$ the set of letters occurring in w. We extend this notation to any set of words S: $\mathrm{alph}(S) = \bigcup_{w \in S} \mathrm{alph}(w)$. The *number of occurrences* of a letter x in w is denoted by $|w|_x$ and the total length of w is denoted by $|w|$. We denote by $\mathrm{minLetters}(w)$ the set $\{a \in \mathrm{alph}(w) \mid \forall x \in \mathrm{alph}(w), |w|_a \leq |w|_x\}$, that is, the set of letters with minimal number of occurrences in w. *Powers* of a word u are the words defined by $u^0 = \varepsilon$, $u^n = uu^{n-1}$ for any integer $n \geq 1$.

Given an alphabet A, a(n endo)*morphism* f on A is an application from A^* to A^* such that $f(uv) = f(u)f(v)$ for any words u, v over A. A morphism on A is entirely defined by the images of elements of A. A particular morphism on A is the *identity* morphism denoted by Id_A or simply Id.

A morphism f is *erasing* if there exists a letter a such that $f(a) = \varepsilon$. Examples of erasing morphisms are the projections. Let B be a subset of an alphabet A. The *projection* on B is the morphism π_B from A^* to B^* defined by $\pi_B(a) = a$ for $a \in B$ and $\pi_B(a) = \varepsilon$ for $a \notin B$. When $B = A \setminus \{x\}$ for a letter x, we denote by δ_x the projection π_B.

Given a morphism f, *powers* of f are defined inductively by $f^0 = Id$, $f^i = f \circ f^{i-1}$ for integers $i \geq 1$.

3 Finite fixed points of morphisms

A word w is a *fixed point* of a morphism f on A if $f(w) = w$. Given a morphism f, T. Head [12] has characterized the language of finite fixed points of f. After some notions [13, 17] and notations, we recall this result.

Let f be a morphism on the alphabet A. A letter a is *mortal* for the morphism f if there exists an integer $i \geq 1$ such that $f^i(a) = \varepsilon$. We will denote by M_f the set of mortal letters of f. The *mortality exponent* of f, denoted by $\exp(f)$, is the least integer i such that $f^i(a) = \varepsilon$ for all $a \in M_f$. The existence of this exponent can be stated showing that $\exp(f) \leq \mathrm{Card}(M_f)$.

A letter a is said *monorecursive* (for the morphism f) if there exist two words x, y in M_f^* such that $f(a) = xay$. For any monorecursive letter a, we have $|f(a)| \geq 1$. In order to ease future developments, we split the set of monorecursive letters into the two following subsets:

$$C_f = \{a \in A \mid f(a) = a\},$$

$$E_f = \{a \in A \mid a \text{ monorecursive}, |f(a)| \geq 2\}.$$

Letters in C_f are *constant* by f. Letters in E_f are called *expansive* letters. When $\mathrm{Card}(E_f) = 1$, we denote by e_f the unique element of E_f.

By definition, $C_f \cap E_f = \emptyset$. Moreover for any letter a in $C_f \cup E_f$, we can see by induction that $|f^n(a)|_a = 1$ for any integer $n \geq 1$. Consequently $(C_f \cup E_f) \cap M_f = \emptyset$. We can also observe that $f(C_f) = C_f$ and $f(E_f) \subseteq (E_f \cup M_f)^*$.

Theorem 3.1 [12] *Let f be a morphism on A. The set of finite fixed points of f is the set $(f^{\exp(f)}(C_f \cup E_f))^*$.*

Consider, for instance, the morphism f defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by $f(1) = 2$, $f(2) = 1$, $f(3) = 34$, $f(4) = 435$, $f(5) = 6$, $f(6) = 7$, $f(7) = \varepsilon$, $f(8) = 6787$, $f(9) = 966$. For this morphism, we have $M_f = \{5, 6, 7\}$, $C_f = \emptyset$, $E_f = \{8, 9\}$, $\exp(f) = 3$. The fixed points of f are the words in $\{76787, 96677\}^*$.

Most of time (as in Theorem 3.1), given a morphism f, one studies words that are fixed points of f. Here we study intrinsic properties of (finite) words that are fixed points of morphisms (independently of these morphisms), that is we study words in the set

$$\mathrm{FixedWords} = \left\{ w \text{ word} \;\middle|\; \begin{array}{l} \exists f : \mathrm{alph}(w)^* \to \mathrm{alph}(w)^* \text{ morphism}, \\ \text{such that } f \neq Id \text{ and } f(w) = w \end{array} \right\}$$

For $w \in \mathrm{FixedWords}$, as suggested by A. Geser [11], we call *witness* of w any morphism f on $\mathrm{alph}(w)$ such that $f(w) = w$ and $f \neq Id_{\mathrm{alph}(w)}$. A witness is an erasing morphism.

We can observe that $\mathrm{FixedWords} \cap \{a\}^* = \emptyset$, and $abba, aabb \notin \mathrm{FixedWords}$. Moreover given an alphabet A containing at least two letters, if a is a letter in A, we can see that any word containing exactly one occurrence of the letter a

belongs to FixedWords, so that this set is infinite (even restricted to a two-letter alphabet): a word w with $|w|_a = 1$ has a witness f defined by $f(a) = w$ and $f(b) = \varepsilon$ for all $b \in A \setminus \{a\}$. Of course an element of FixedWords can have several witnesses (see for instance the word ab).

A consequence of Theorem 3.1 is:

Corollary 3.2 *For $w \in \mathrm{FixedWords}$, and f a witness of w,*

$$C_f \cup E_f \cup M_f = \mathrm{alph}(w).$$

The first part of the next proposition (which is a consequence of Corollary 3.2) was proved by A. Geser [11]. It shows that it is possible to focus only on idempotent witnesses of words in FixedWords (A morphism f is *idempotent* if $f^2 = f$).

Proposition 3.3 *Any word w in FixedWords has a witness which is idempotent. More precisely, given a witness f of w, there exists an idempotent witness g of w with $C_f = C_g$, $E_f = E_g$ and $M_f = M_g$.*

Since for an idempotent morphism f, $\exp(f) = 0$, Theorem 3.1 can be simplified into:

Corollary 3.4 *Let f be an idempotent morphism on A. The set of finite fixed points of f is the set $(f(C_f \cup E_f))^* = (C_f \cup f(E_f))^*$.*

4 The conjecture

As already said, in 1993, in newsgroup comp.theory, M. Billaud [2] proposed:

Conjecture 4.1 *Let A be an alphabet with at least three letters. Let w be a word with $\mathrm{alph}(w) = A$. If for all $a \in A$, $\delta_a(w) \in \mathrm{FixedWords}$, then $w \in \mathrm{FixedWords}$.*

Conjecture 4.1 assumes $\mathrm{Card}(A) \geq 3$, since $\mathrm{FixedWords} \cap \{b\}^* = \emptyset$ for any letter b. The converse of this conjecture does not hold. There exist words w in FixedWords such that $\delta_a(w) \notin \mathrm{FixedWords}$ for at least one letter a in $\mathrm{alph}(w)$. For instance, one can consider the word $w = acbba$. This word belongs to FixedWords since it has a witness f defined on $\{a, b, c\}$ by $f(a) = \varepsilon$, $f(b) = \varepsilon$, $f(c) = w$. Nevertheless the word $\delta_c(w) \notin \mathrm{FixedWords}$.

Given an alphabet with at least three letters, we can find $w \in \mathrm{FixedWords}$ such that for each letter x in $\mathrm{alph}(w)$, $\delta_x(w) \notin \mathrm{FixedWords}$. For instance, if $A = \{a_1, \ldots, a_n\}$ with $n \geq 3$, the word

$$w = \prod_{i=1}^{n-1}(a_i a_n) \prod_{i=1}^{n-1}(a_{n-i} a_n)$$

belongs to FixedWords. It has a witness f defined by $f(a_i) = a_i a_n$ for $1 \leq i < n$ and $f(a_n) = \varepsilon$. But for all x in $\mathrm{alph}(w)$, $\delta_x(w)$ does not belong to FixedWords.

Maybe one reason of the difficulty to solve Conjecture 4.1 is that it is equivalent to another one that seems more complex. We define this new conjecture and show the equivalence. For this let us define

$$\text{FixedSets} = \left\{ S \text{ set of words} \;\middle|\; \begin{array}{l} \exists\, f\colon \text{alph}(S)^* \to \text{alph}(S)^* \text{ morphism} \\ \text{such that } f \neq Id \text{ and} \\ \text{for all } w \text{ in } S,\ f(w) = w \end{array} \right\}$$

For instance,

$$\{ab, ba\} \notin \text{FixedSets}$$

$$\{\varepsilon, abcb, aaa, bcb, bcbabcb\} \in \text{FixedSets}$$

Conjecture 4.2 *Let A be an alphabet with at least three letters. Let $S \subseteq A^*$ with $\text{alph}(S) = A$. If for all $a \in A$, $\delta_a(S) \in \text{FixedSets}$, then $S \in \text{FixedSets}$.*

Theorem 4.3 *Conjecture 4.1 is true if and only if Conjecture 4.2 is.*
More precisely, given an alphabet A, Conjecture 4.1 is true for words w with $\text{alph}(w) = A$ if and only if Conjecture 4.2 is true for sets S of words with $\text{alph}(S) = A$.

Conjecture 4.1 is a restriction of Conjecture 4.2 to singletons. Conversely Proposition 4.5 states that we can reduce Conjecture 4.2 to Conjecture 4.1 using inductively the next lemma.

Lemma 4.4 *Let f be an idempotent morphism on an alphabet A. Given two words u and v over A, the two following assertions are equivalent:*

1. *$f(u) = u$ and $f(v) = v$,*
2. *$f(uvvu) = uvvu$.*

Proposition 4.5 *Let A be an alphabet with at least three letters. Let S be a set of words such that $\text{alph}(S) = A$ and for all $a \in A$, $\delta_a(S) \in \text{FixedSets}$. For $a \in A$, let f_a be a morphism on $A \setminus \{a\}$ such that $f_a \neq Id$ and $\forall x \in \delta_a(S)$, $f_a(x) = x$.*

There exists a word w such that $\text{alph}(w) = \text{alph}(S)$ and:

1. *for all $a \in A$, $f_a(\delta_a(w)) = \delta_a(w)$, and so $\delta_a(w) \in \text{FixedWords}$.*
2. *$w \in \text{FixedWords}$ if and only $S \in \text{FixedSets}$.*
   More precisely for any morphism $f \neq Id$ on A, $f(w) = w$ if and only if $\forall x \in S$, $f(x) = x$.

Proof. The proof holds by induction on $\text{Card}(S)$.

The result is immediate if $\text{Card}(S) = 1$.

Assume $\text{Card}(S) \geq 2$. Let u, v be two different elements of S. Let $S' = (S \setminus \{u, v\}) \cup \{uvvu\}$. For $a \in A$ and $x \in S'$, $f_a(\delta_a(x)) = \delta_a(x)$, and so $\delta_a(S') \in \text{FixedSets}$. Moreover $\text{Card}(S') < \text{Card}(S)$. By induction hypothesis, there exists a word w such that $\text{alph}(w) = \text{alph}(S')$ and

1. for all $a \in A$, $f_a(\delta_a(w)) = \delta_a(w)$, that is $\delta_a(w) \in$ FixedWords.
2. $w \in$ FixedWords if and only $S' \in$ FixedSets.
   More precisely for any morphism $f \neq Id$ on A, $f(w) = w$ if and only if $\forall x \in S'$, $f(x) = x$.

By construction, $\text{alph}(S') = \text{alph}(S) = \text{alph}(w)$. To end the proof, we have to state, for any morphism $f \neq Id$, the equivalence between

- $\forall x \in S$, $f(x) = x$ and
- $\forall x \in S'$, $f(x) = x$.

This is a direct consequence of Lemma 4.4. $\square$

In order to show more clearly the utility of Proposition 4.5, let us give the **Proof of Theorem 4.3.** As already said, the "if part" is immediate. Assume now Conjecture 4.1 is true for any words w with $\text{alph}(w) = A$. Let $S \subseteq A^*$ with $\text{alph}(S) = A$ such that $\forall a \in A$, $\delta_a(S) \in$ FixedSets. By Proposition 4.5, there exists a word w such that:

1. for all $a \in A$, $\delta_a(w) \in$ FixedWords.
2. $w \in$ FixedWords if and only $S \in$ FixedSets.

From Assertion 1 and the fact that Conjecture 4.1 is true, we know $w \in$ FixedWords. So $S \in$ FixedSets. $\square$

5 A particular case

As already said, Conjecture 4.1 was solved by P. Zimmermann [18] in case of words over 3-letter alphabets. In this section, we extend this result.

Proposition 5.1 *Let w be a word such that* $\text{Card}(\text{alph}(w)) \geq 3$. *Assume there exist three pairwise distinct letters a, b, c in* $\text{alph}(w)$ *such that, for each letter $x \in \{a, b, c\}$, there exists a witness f_x of $\delta_x(w)$ such that* $\text{Card}(E_{f_x}) = 1$ *and* $\text{Card}(C_{f_x}) = 0$. *Then $w \in FixedWords$.*
Moreover w has a witness f with $\text{Card}(E_f) = 1$ *and* $\text{Card}(C_f) = 0$.

Scheme of proof of Proposition 5.1.
Taking α in minLetters(w), we can first prove:

$|w|_x$ is a multiple of $|w|_\alpha$ for each letter x in $\text{alph}(w)$.

So we can consider the words $u_a = f_a(e_a)^{\frac{|w|_{e_a}}{|w|_\alpha}}$, $u_b = f_b(e_b)^{\frac{|w|_{e_b}}{|w|_\alpha}}$ and $u_c = f_c(e_c)^{\frac{|w|_{e_c}}{|w|_\alpha}}$. Since $C_{f_a} = C_{f_b} = C_{f_c} = \emptyset$, we have $\delta_x(w) = u_x^{|w|_\alpha}$ for each x in $\{a, b, c\}$.

Now let $w_1, \ldots, w_{|w|_\alpha}$ be the words of length $\frac{|w|}{|w|_\alpha}$ such that $w = \prod_{i=1}^{|w|_\alpha} w_i$. We can prove

$$|w_i|_x = |u_\beta|_x,\ \forall \beta \in \{a,b,c\},\ \forall x \in \mathrm{alph}(w) \setminus \{\beta\},\ \forall i \in \{1,\dots,|w|_\alpha\}$$

and consequently

$$\pi_{\{x,y\}}(w_i) = \pi_{\{x,y\}}(w_1),\ \forall i \in \{1,\dots,|w|_\alpha\},\ \forall x,y \in \mathrm{alph}(w).$$

Proposition 5.1 appears then as a consequence of the following particular case of a more general result in traces theory (see [8, 9] for more information about traces theory).

Lemma 5.2 [7] *Let u and v be two words. Are equivalent:*

1. $u = v$.
2. $\forall x, y \in A$, $\pi_{\{x,y\}}(u) = \pi_{\{x,y\}}(v)$.

From previous lemma, we deduce $\forall i = 1,\dots,|w|_\alpha$, $w_i = w_1$. So $w = w_1^{|w|_\alpha}$. Moreover, $|w_1|_\alpha = 1$. The next lemma allows to conclude that $w \in$ FixedWords. This ends the proof of Proposition 5.1.

Lemma 5.3 *A word w belongs to* FixedWords *and has a witness f such that* $\mathrm{Card}(E_f) = 1$ *if and only if there exist a set C of letters, a letter b, and a word u, such that $|u| \geq 2$, $|u|_b = 1$, $\mathrm{alph}(u) \cap C = \emptyset$ and $w \in (C \cup \{u\})^*$.*

A first consequence of Proposition 5.1 is the particular case proved by P. Zimmermann:

Corollary 5.4 [18] *Conjecture 4.1 is true for words over a 3-letter alphabet.*

Proof. Let w be a word over a 3-letter alphabet such that for each $x \in \mathrm{alph}(w)$, $\delta_x(w) \in$ FixedWords. Let $x \in \mathrm{alph}(w)$ and let f_x be a witness of $\delta_x(w)$. Since $f_x \neq Id$, $f_x(\delta_x(w)) = \delta_x(w)$ and $\mathrm{Card}(\mathrm{alph}(\delta_x(w))) = 2$, necessarily $\mathrm{Card}(E_{f_x}) = 1 = \mathrm{Card}(M_{f_x})$ and $\mathrm{Card}(C_{f_x}) = 0$. So by Proposition 5.1, $w \in$ FixedWords. $\square$

The use of projections on sets of two letters leads to consider reconstruction of fixed points from such projections. We get (similarly than in Corollary 5.4) the following result.

Corollary 5.5 *Let w be a word such that* $\mathrm{Card}(\mathrm{alph}(w)) \geq 3$. *If for all $a, b \in A$, $\pi_{\{a,b\}}(w) \in$* FixedWords*, then $w \in$* FixedWords*. More precisely, w has a witness f such that* $\mathrm{Card}(E_f) = 1$ *and* $\mathrm{Card}(C_f) = 0$.

6 Main result

Actually we solved Conjecture 4.1 in a more general case than Proposition 5.1. We have a result that does not depend on the inexistence of constant letters.

Theorem 6.1 *Let w be a word with* $\mathrm{Card}(\mathrm{alph}(w)) \geq 3$*. Assume that, for each x in* $\mathrm{alph}(w)$*, $\delta_x(w)$ belongs to* FixedWords *and has a witness f_x with* $\mathrm{Card}(E_{f_x}) = 1$*. Then w belongs to* FixedWords *and has a witness f with* $\mathrm{Card}(E_f) = 1$*.*

The proof of this theorem is done by induction. The inductive step uses the following proposition, the proof of which, long (more than 10 pages) and technical, is omitted.

Proposition 6.2 *Let w be a word such that* $\mathrm{Card}(\mathrm{alph}(w)) \geq 4$*. If for each letter x, the word $\delta_x(w)$ has a witness f_x with* $\mathrm{Card}(E_{f_x}) = 1$*, then (at least) one of the two following assertions is verified:*

1. *$w \in$* FixedWords *and it has a witness f with* $\mathrm{Card}(E_f) = 1$*.*
2. *there exist two non-empty sets B and C with* $\mathrm{alph}(w) = B \cup C$*, $B \cap C = \emptyset$ such that $\forall x \in B$, $\forall y \in C$, $y \in C_{f_x}$.*

The second assertion of this proposition allows to reduce the problem to an equivalent one on a set of words over strictly less letters. Then Proposition 4.5 reduces the problem back to the problem on words. For more details we give the

Proof of Theorem 6.1. From Proposition 3.3, we assume that the morphisms f_x in the hypotheses are idempotent. We proceed by induction on $\mathrm{Card}(\mathrm{alph}(w))$.

When $\mathrm{Card}(\mathrm{alph}(w)) = 3$, Corollary 5.4 states that $w \in$ FixedWords. Moreover the proof of this corollary can be continued to show that w has a witness f with $\mathrm{Card}(E_f) = 1$.

From now on, assume $\mathrm{Card}(\mathrm{alph}(w)) \geq 4$. By Proposition 6.2, if we have not directly the result, there exist two non-empty sets B and C such that $\mathrm{alph}(w) = B \cup C$, $B \cap C = \emptyset$, and for all α in B, $C \subseteq C_{f_\alpha}$. There exists an integer $n \geq 1$ and words $x_i \in C^*$ $(i = 0, \ldots, n)$, and $y_i \in B^+$ $(i = 1, \ldots, n)$ such that $x_i \neq \varepsilon$ if $i \in \{1, \ldots, n-1\}$ and

$$w = x_0 \prod_{i=1}^{n} y_i x_i.$$

Let $S = \{y_1, \ldots, y_n\}$. We have $\mathrm{alph}(S) = B$. Let $x \in B$. Let us denote by g_x the restriction of f_x to the set B. Since $C \subseteq C_{f_x}$, for $y \in S$, we have $f_x(\delta_x(y)) = \delta_x(y)$, that is $g_x(\delta_x(y)) = \delta_x(y)$. From $B \cup C = \mathrm{alph}(w) = \{x\} \cup C_{f_x} \cup M_{f_x} \cup E_{f_x}$, $B \cap C = \emptyset$ and $C \subseteq C_{f_x}$, we deduce $\{x\} \cup M_{f_x} \cup E_{f_x} \subseteq B$. So $M_{g_x} = M_{f_x}$, $E_{g_x} = E_{f_x}$, $\mathrm{Card}(B) \geq 3$ and $g_x \neq Id_B$.

So S verifies the hypotheses of Proposition 4.5. There exists a word w' such that $\mathrm{alph}(w') = \mathrm{alph}(S)$ and:

1. for all $a \in \text{alph}(w')$, $f_a(\delta_a(w')) = \delta_a(w')$.

2. $w' \in$ FixedWords if and only $S \in$ FixedSets.
   More precisely for any morphism $f \neq Id$ on $\text{alph}(w')$, $f(w') = w'$ if and only if $\forall y \in S$, $f(y) = y$.

Since $\text{alph}(w') = \text{alph}(S) = \text{alph}(B) < \text{alph}(w)$, by induction hypothesis, there exists a morphism $f \neq Id$ on B such that $\text{Card}(E_f) = 1$ and $f(w') = w'$. So $\forall y \in S$, $f(y) = y$.

For $c \in C$, let define $f(c) = c$. We get $f(w) = w$. $\square$

7 Conclusion

We have partially solved Conjecture 4.1. Actually we proved more. In Proposition 5.1, Corollary 5.5 and Theorem 6.1, when, for each letter x occurring in a word w, $\delta_x(w) \in$ FixedWords, we show a connection between the minimal number of expansive letters of witnesses of w and the number of expansive letters of witnesses of $\delta_x(w)$.

We wonder if there exists a more general connection. For w a word, let $\text{minCardExp}(w)$ be the value $\min\{\text{Card}(E_f) \mid f \text{ witness of } w\}$. When Conjecture 4.1 holds, do we have : $\text{minCardExp}(w) \leq \max\{\text{minCardExp}(\delta_x(w)) \mid x \in \text{alph}(w)\}$?

There exist examples where the previous inequality becomes an equality. For instance, for $w = dababecbcbecbcbdabab$, we have $\text{Card}(E_f) = 2$ for each witness f of w and for each witness f of $\delta_x(w)$ whatever is x in $\text{alph}(w)$. We do not know any example of equality for a word w over a 4-letter alphabet with $\text{minCardExp}(w) = 2$.

In most cases, the previous inequality is certainly strict. For instance, with $w = bdabdcbdcbda$, $\text{minCardExp}(w) = 1$ but $\max\{\text{minCardExp}(\delta_x(w)) \mid x \in \text{alph}(w)\} = 2$.

To find other examples, the reader can focus on primitive words. Indeed a consequence of Theorem 3.1 is: for w a word and $n \geq 1$ an integer, $f(w) = w$ if and only if $f(w^n) = w^n$.

Previous examples are not concerned by our results. Indeed there exists x in $\text{alph}(w)$ such that $\text{minCardExp}(E_{f_x}) = 2$. This shows that Conjecture 4.1 is still open on 4-letter alphabets.

Another question is: what becomes the conjecture for (bi)infinite words?

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