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Episturmian morphisms and Conjugacy

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Sturmian words

- widely and still studied [Morse/Hedlund (1940), ..., Lothaire II (200?)]
- many different equivalent definitions
- many different generalizations (..., Arnoux-Rauzy 1991)

Episturmian words

Droubay, Justin, Pirillo 1999-2001

- one generalization based on a De Luca property (1997)
- def = infinite words which set of factors is closed under reversal and that have a right special factor of each length
- aperiodic A-strict episturmian word = Arnoux-Rauzy sequences

(endo)morphisms

- alphabet A
- free monoid A^*
- f is a (endo)morphism = application from A^* to A^* such that $\forall u, v, \quad f(uv) = f(u)f(v)$
- A morphism is entirely defined by the images of the letters

Sturmian morphisms

- def = morphisms that preserve sturmian words
- (Séébold 1991): on $\{a, b\}$, they are the compositions of

$$E : \begin{cases} E(a) = b \\ E(b) = a \end{cases} \qquad L : \begin{cases} L(a) = a \\ L(b) = ab \end{cases} \qquad R : \begin{cases} R(a) = a \\ R(b) = ba \end{cases}$$

- a tool to work on Sturmian words

Episturmian morphisms

Justin, Pirillo 2000 : a tool to work on episturmian words

- def = composition of morphisms that generalize E, L, R.
- preserve episturmian words.
- preserve Arnoux-Rauzy sequences.

Notations

- $\text{Episturm}(A) = \{E_{xy/x,y\in A}, \Psi_{\alpha/\alpha\in A}, \overline{\Psi}_{\alpha/\alpha\in A}\}^*$

for $x, y \in A$, (Θ_{xy} in Justin Pirillo)

$$E_{xy} : \begin{cases} E_{xy}(x) = y \\ E_{xy}(y) = x \\ E_{xy}(z) = z \text{ for } z \in A \setminus \{x, y\} \end{cases}$$

Exchange morphisms

for $\alpha \in A$,

$$\Psi_\alpha : \begin{cases} \Psi_\alpha(\alpha) = \alpha \\ \Psi_\alpha(x) = \alpha x \end{cases} \qquad \overline{\Psi}_\alpha : \begin{cases} \overline{\Psi}_\alpha(\alpha) = \alpha \\ \overline{\Psi}_\alpha(x) = x\alpha \text{ for } x \in A \setminus \{\alpha\} \end{cases}$$

- case $A = \{a, b\}$,

$$E_{ab} = E, \qquad E_{aa} = Id, \qquad \Psi_a = L, \qquad \overline{\Psi}_a = R.$$

Notations (continue)

From now

- a is a particular letter of A,
- $L = \Psi_a$
- $R = \overline{\Psi}_a$

We have:

- for $\alpha \in A$, $\quad \Psi_\alpha = E_{a\alpha} L E_{a\alpha}$
- for $\alpha \in A$, $\quad \overline{\Psi}_\alpha = E_{a\alpha} R E_{a\alpha}$
- $\text{Episturm}(A) = \{E_{xy/x,y \in A}, L, R\}^*$

Presentation

Theorem. The monoid Episturm(A) has the following presentation (x, y, z, t are pairwise different letters):

$$\begin{aligned}
E_{xy}E_{xy} &= Id, && (1)\\
E_{xy}E_{yz} &= E_{yz}E_{zx}, && (2)\\
E_{xy}E_{zt} &= E_{zt}E_{xy}, && (3)\\
E_{xy}L &= LE_{xy} \qquad \text{when } a \notin \{x,y\}, && (4)\\
E_{xy}R &= RE_{xy} \qquad \text{when } a \notin \{x,y\}, && (5)\\
LE_1LE_2\ldots LE_kR &= RE_1RE_2\ldots RE_kL && (6)
\end{aligned}$$

with $k \geq 1$, $E_1, \ldots, E_k$ exchange morphisms such that

$E_1 \ldots E_k(a) = a$,

for $2 \leq i \leq k$, $\quad E_i \ldots E_k(a) \neq a$.

Presentation (continue)

- (1), (2) and (3) is a presentation of the set of permutations.

- Previous theorem generalizes the presentation of Sturmian morphisms (Séébold 1991).

- (1) becomes $EE = Id$
- (2), (3), (4) and (5) are impossible
- (6) becomes for $k = 1$, $LR = RL$
- (6) becomes for $k \geq 2$, $LEL^{k-1}ER = RER^{k-1}EL$

Connections with palindromes

- The morphism $LE_1LE_2\dots LE_kR$ **preserves palindromes** when $E_1,\dots,E_k$ exchange morphisms as in (6).

$$\begin{cases} E_1\dots E_k(a)=a, \\ \text{for } 2\le i\le k, \quad E_i\dots E_k(a)\neq a). \end{cases}$$

- **Proposition.**

An episturmian morphisms **preserves palindromes**
if and only if
it is a composition of exchange morphisms and
morphisms $LE_1LE_2\dots LE_kR$ as in (6).

Conjugacy

- Conjugation in [Séébold 1998], Conjugacy in [Lothaire II].
- Def. A morphism g is a *right conjugate* of a morphism f if
  $$\exists w \in A^*, \quad \forall x \in A^*, \; f(x)w = wg(x).$$
- Notation: $f \lhd g \qquad \text{or} \qquad f \lhd (w)g$
- $L \lhd (\varepsilon)L, \qquad\qquad L \lhd (a)R.$
- $\mathrm{NbR}(f) =$ number of right conjugates of f.

Characterization of Epistandard morphisms

Definitions.

- (Epi)standard morphisms = standard episturmian = morphisms obtained by composition of exchange morphisms (E_{xy}) and L.
- $||f|| = \sum_{x \in A} |f(x)|$.

Proposition.

Let f be a morphism such $A = \{\mathrm{last}(f(x)) / x \in A\}$

1. $\mathrm{NbR}(f) \leq \frac{||f||-1}{\#A-1}$
2. $\mathrm{NbR}(f) = \frac{||f||-1}{\#A-1}$ *if and only if* f is epistandard.

Definition

- A morphism is periodic if $\exists w,\ f(A) \subseteq w^*$
- standard = epistandard in binary case

Corollary. Binary case.

If f is a *non-periodic* morphism defined on $\{a, b\}$

1. $\mathrm{NbR}(f) \leq \frac{||f||-1}{\#A-1}$
2. $\mathrm{NbR}(f) = \frac{||f||-1}{\#A-1}$ *if and only if* f is standard.

Corollary. Property of Episturmian morphisms

For any episturmian morphism f, there are exactly $\frac{||f||-1}{\#A-1}$ morphisms g with $f \lhd g$ or $g \lhd f$.

Characterization of episturmian morphism

Theorem.

1. A morphism f is episturmian if and only if it is a right conjugate of a *unique* epistandard morphism.
2. *This epistandard morphism is obtained from any decomposition* of f over exchange morphisms, L and R *by replacing all the occurrences of* R *by* L

Remark.

- A generalization of a result by Séébold (1998) for binary case.
- Example. $LE_{ab}LLE_{ab}L \triangleleft LE_{ab}RLE_{ab}R$.

General result on conjugacy

- The conjugates of a morphism f can be naturally ordered:

$$g_0, g_1 \ldots, g_{\mathrm{NbR}(f)-1} \text{ such that}$$
$$g_0 = f$$
$$f \triangleleft (w_i) g_i \text{ for a (unique) word } w_i, \quad |w_i| = i.$$

- g_1 (when exists) is called the *first conjugate* of f

First conjugate of an episturmian morphism

Proposition

Data. $f_1, \ldots, f_n$ exchange morphisms or L or R.

Let

$$\begin{cases} k \text{ the least integer with } f_k = L. \\ \qquad\qquad \text{(it exists if and only if the first conjugate of } f \text{ exists)} \\ \\ g_i = L \text{ if } i < k,\ f_i = R \text{ and first}(f_{i+1} \ldots f_k(a)) \neq a \\ g_i = f_i \text{ otherwise.} \end{cases}$$

The first conjugate of $f_1 \ldots f_n$ is $\quad g_1 \ldots g_{k-1} R f_{k+1} \ldots f_n.$

Example. Inverse computations

	$LE_{ac}LLE_{ab}L$	
$\triangleleft$	$RE_{ab}LLE_{ab}L$	$RE_{ab}LLE_{ab}L$
$\triangleleft$	$LE_{ab}RLE_{ab}L$	$LE_{ab}LRE_{ab}L$
$\triangleleft$	$RE_{ab}RLE_{ab}L$	$RE_{ab}LRE_{ab}L$
$\triangleleft$	$LE_{ab}RRE_{ab}L$	$LE_{ab}RRE_{ab}L$
$\triangleleft$	$RE_{ab}RRE_{ab}L$	$LE_{ab}LLE_{ab}R$
$\triangleleft$	$RE_{ab}LLE_{ab}R$	$RE_{ab}LLE_{ab}R$
$\triangleleft$	$LE_{ab}RLE_{ab}R$	$LE_{ab}LRE_{ab}R$
$\triangleleft$	$RE_{ab}RLE_{ab}R$	$RE_{ab}LRE_{ab}R$
$\triangleleft$	$LE_{ab}RRE_{ab}R$	$LE_{ab}RRE_{ab}R$
$\triangleleft$	$RE_{ab}RRE_{ab}R$	

Application: back on presentation theorem

$$\text{When} \begin{cases} k \geq 1,\ E_1, \ldots, E_k \text{ exchange morphisms such that} \\ \qquad E_1 \ldots E_k(a) = a, \\ \qquad \text{for } 2 \leq i \leq k, \quad E_i \ldots E_k(a) \neq a. \end{cases}$$

$LE_1LE_2 \ldots LE_kR$ is the first conjugate of $LE_1RE_2 \ldots RE_kL$

$LE_1RE_2 \ldots RE_kL$ has as first conjugate $RE_1RE_2 \ldots RE_kL$

Thus (6) $LE_1LE_2 \ldots LE_kR = RE_1RE_2 \ldots RE_kL$.

Miscelleanous properties

1. The monoid of episturmian morphisms is both unitary.

   If fg and f are epistumian then g is episturmian

   If gf and f are epistumian then g is episturmian

   $\Rightarrow$ Give ways to determine both

   if a morphism is episturmian and

   one decomposition over exchange morphisms, L and R.

   (like in Lothaire II for sturmian morphisms)

2. (Justin, Pirillo 2000) The monoid of episturmian morphisms is both left and right cancellative.
   If $hf = hg$ then $f = g$
   If $fh = gh$ then $f = g$

3. Episturmian morphisms are invertible.
   Binary case: Sturmian = invertible [Wen and Wen 1994].

   Theorem.
   When considering alphabet with at least three letters
   The monoid of invertible endomorphisms is not finitely generated.