Global-local graph decompositions via coverings

RD, Montpellier 2022

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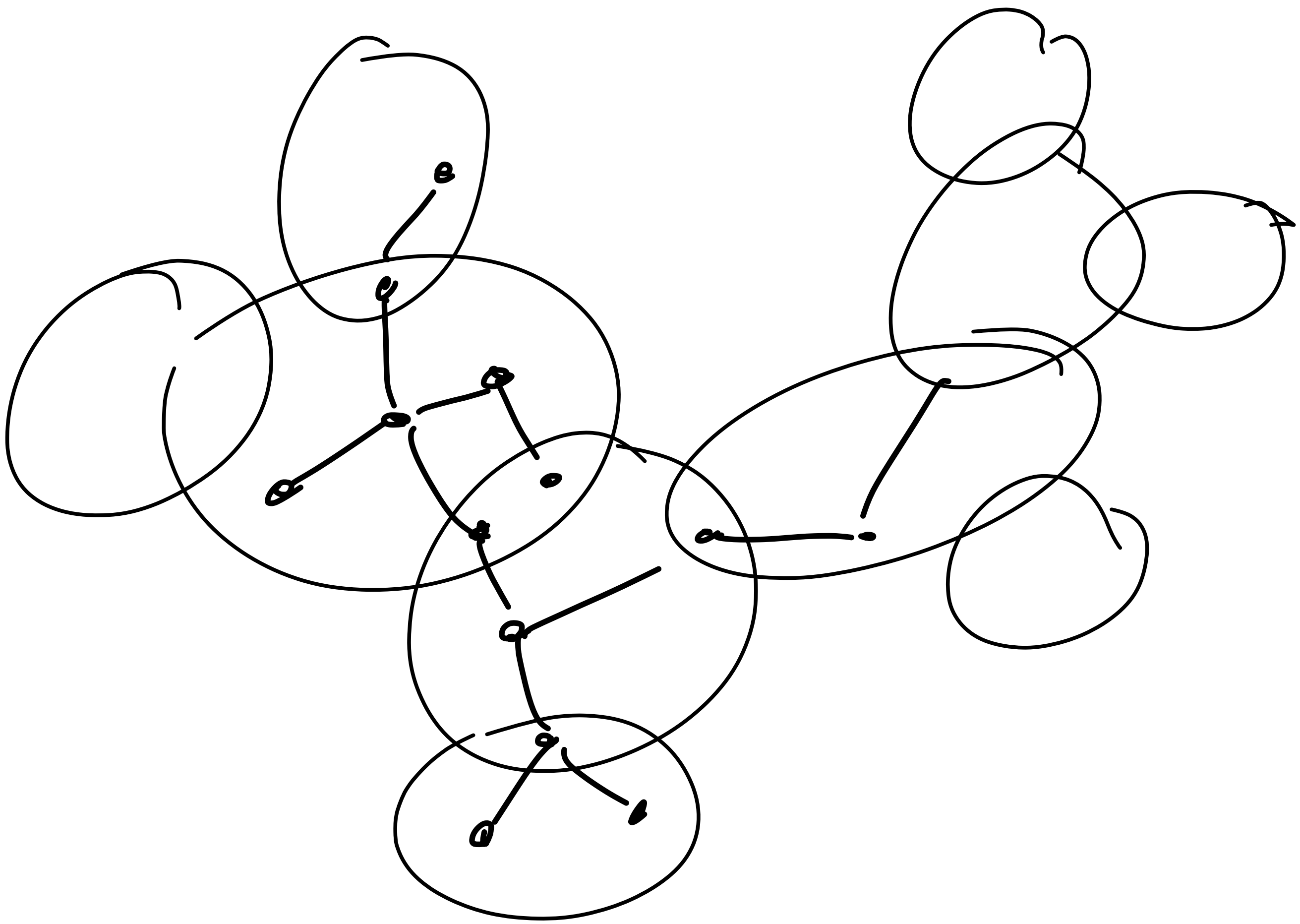
arXiv: http://arxiv.org/abs/2207.04855

Tree-decompositions aim to display the (tree-like) global structure of a graph:



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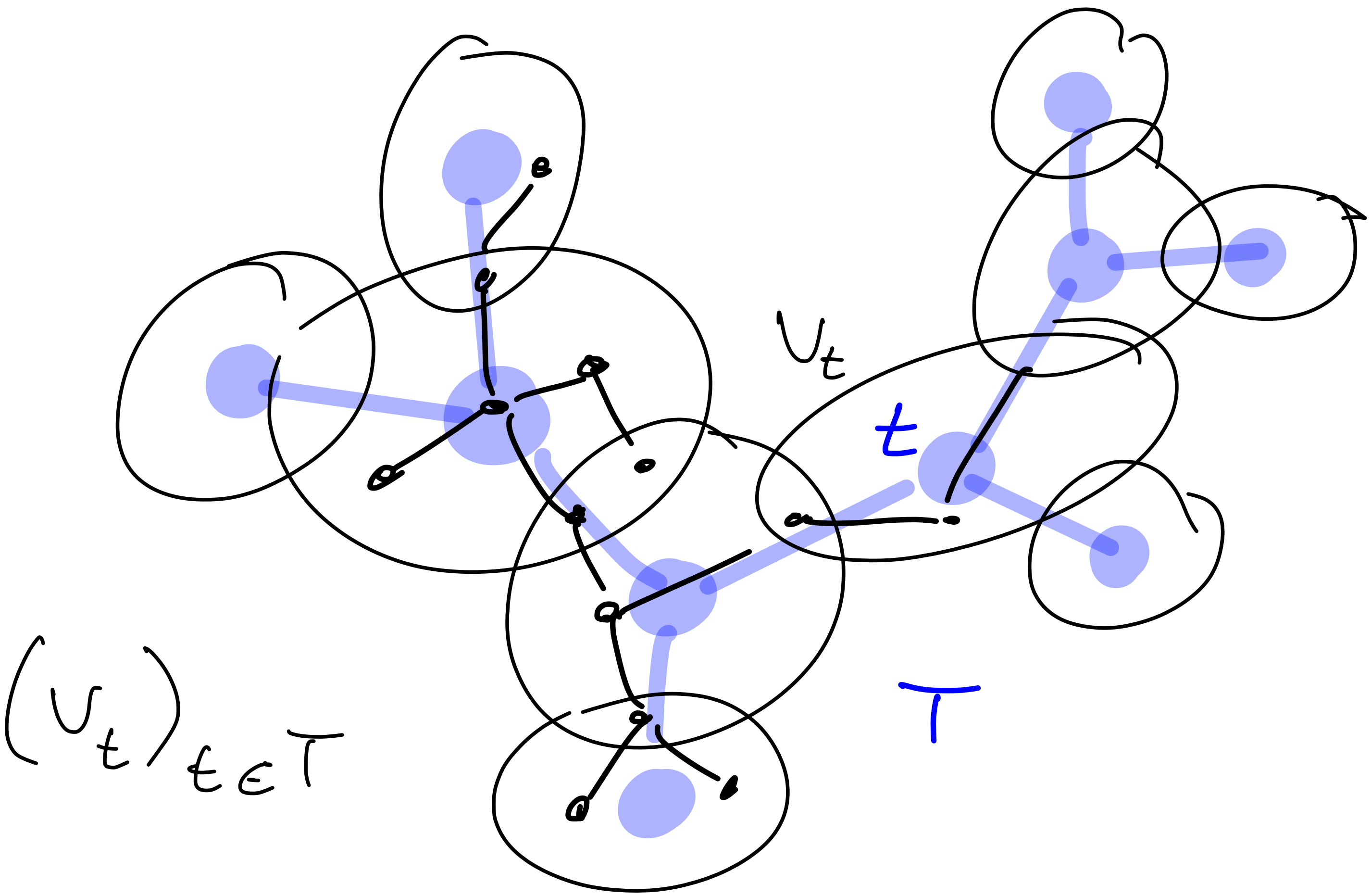

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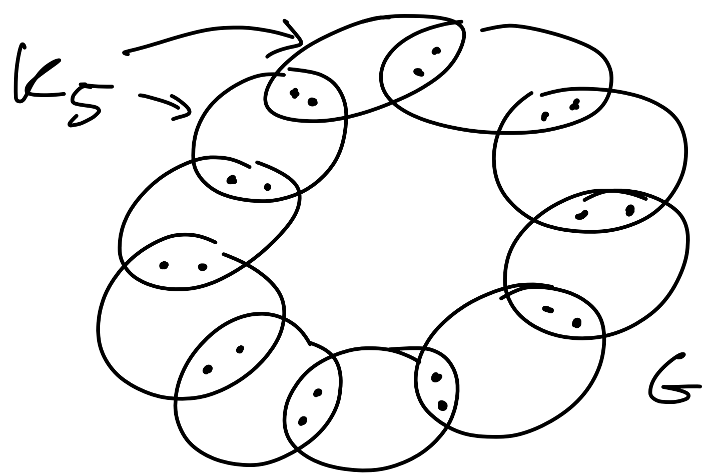
Problem: What if that structure is not tree-like?



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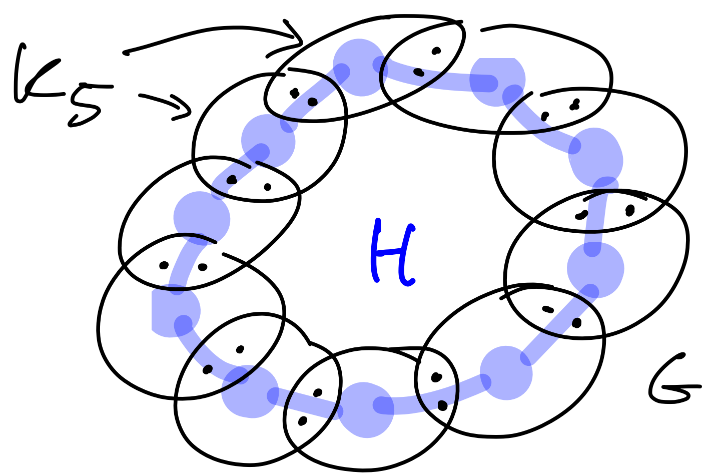
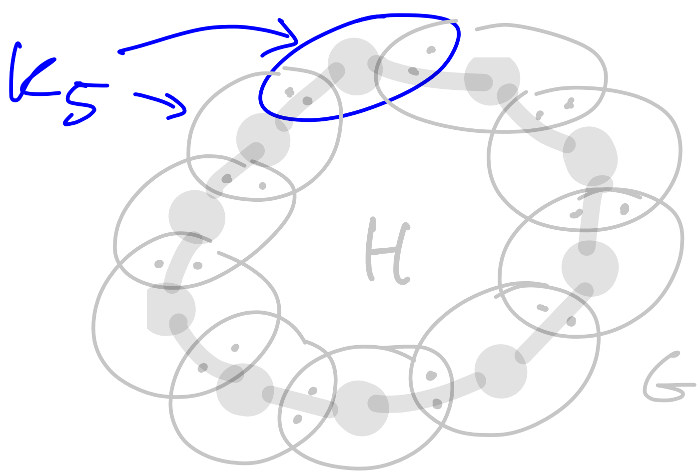


An H-decomposition of G with H a cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

↳ H found, not imposed

how the r-local parts hang together



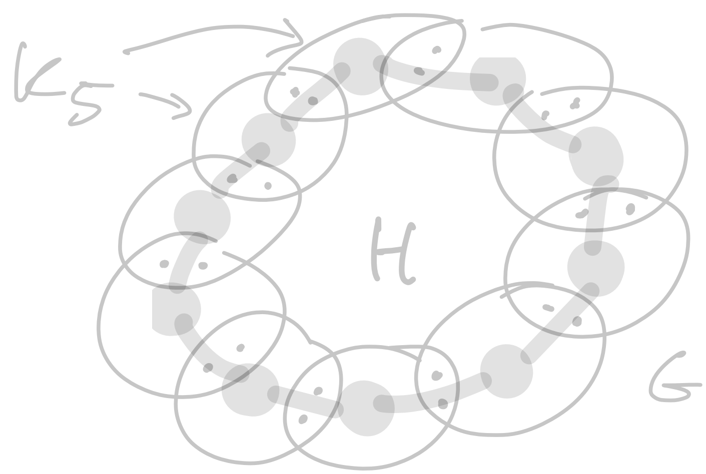
An H-decomposition of G with H a cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

Idea: $r = 1$: local $= \ast$ is tree-like; global $H = G$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$r \geqslant |G|$: local $= G$; global H is tree-like: $td(G)$



An H-decomposition of G with H a cycle

$1 \ll r \ll |G|$

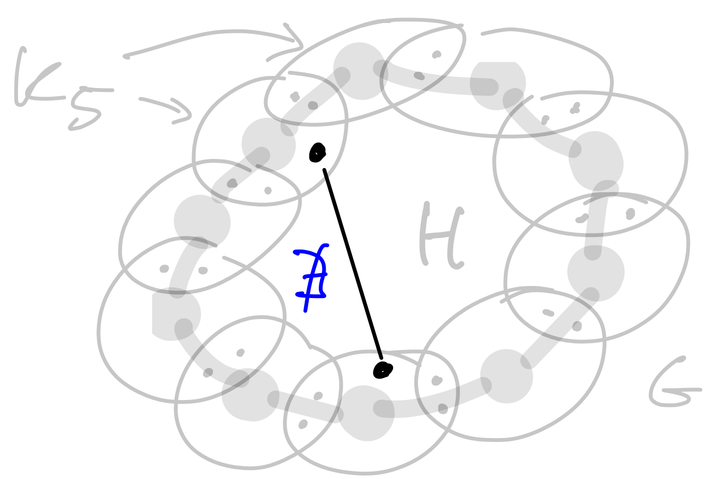
$\Rightarrow H =$ cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical

H-decomposition that displays its r-global structure

$\hookrightarrow$ family $(V_h)_{h \in H}$ satisfying:

$$\text{(H1)} \quad \bigcup_{h \in H} G[V_h] = G$$



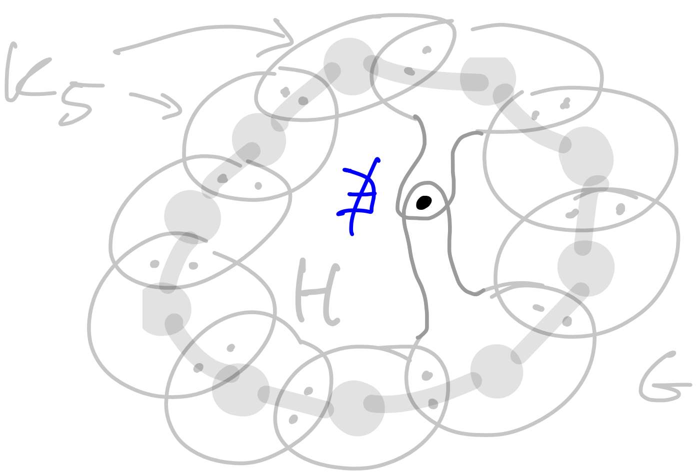
An H-decomposition of G with H a cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

↳ family $(V_h)_{h \in H}$ satisfying

(H1) $\bigcup_{h \in H} G[V_h] = G$

(H2) $\forall v \in G$: $\{h \in H \mid v \in V_h\}$ is conn^d in H.



An H-decomposition of G with H a cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

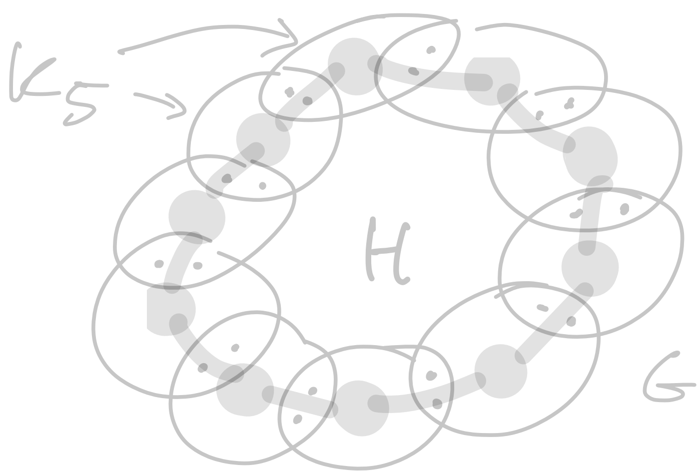
- proof constructs unique H and $(V_h)_{h \in H}$

formally:

- $H = H(G,r)$ is a graph invariant:

$$
\begin{tikzcd}
\underline{\forall \sigma:} \quad G \arrow[r, "\sigma", "\simeq"'] \arrow[d] & G' \arrow[d] \\
H(G,r) = H \arrow[r, "\simeq"] & H' = H(G',r)
\end{tikzcd}
$$

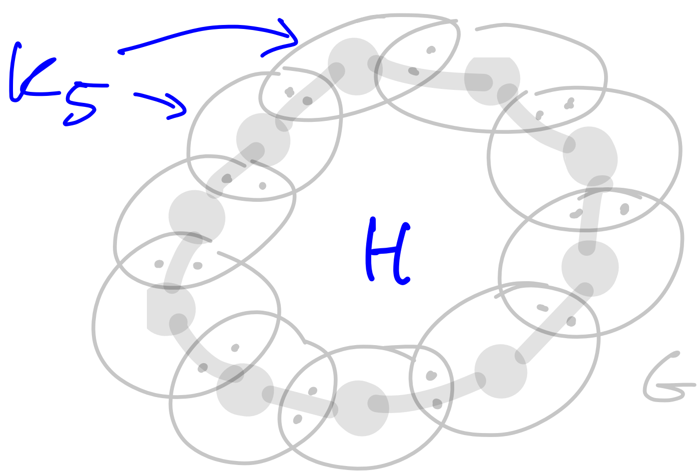
Every isom. σ as maps parts V_h to parts $V_{h'}$ so as to induce isomorphism $H \to H'$, $h \mapsto h'$



An H-decomposition of G with H a cycle

Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

Task: define H and the V_h canonically, for all G & r



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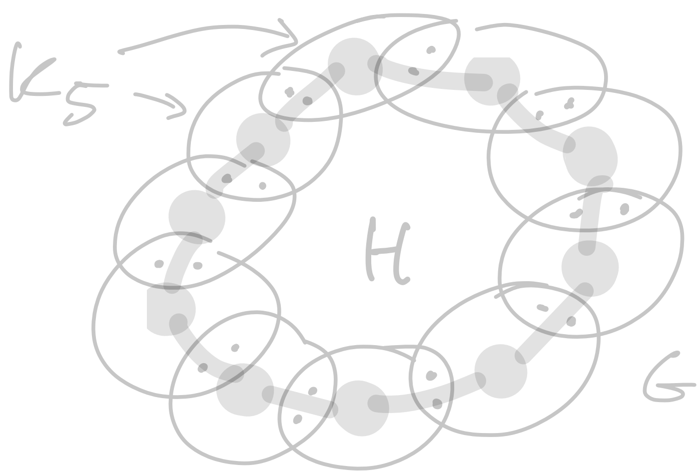
To get started, let's look at what is known for the base case of $r \geq |G|$, where $(V_h)_{h \in H}$ should be a td of G:



An H-decomposition of G with H a cycle

A <u>k-block</u> in G is a maximal set of $\geqslant k$ vertices no two of which are separated in G by $< k$ other vertices.

$k=2$: block-cutvertex tree into 2-blocks

$k=3$: Tutte-td into 3-blocks and cycles

A k-block in G is a maximal set of $\geq k$ vertices no two of which are separated in G by $< k$ other vertices.

Thm (Carmesin, D, Hamann, Stein; Comb. '14)

$\forall$ finite G $\exists$ canonical td that distinguishes all the blocks ($\forall k$) in G efficiently.

Thm (D, Hundertmark, Lemanczyk; Comb. '19)

... tangles ...

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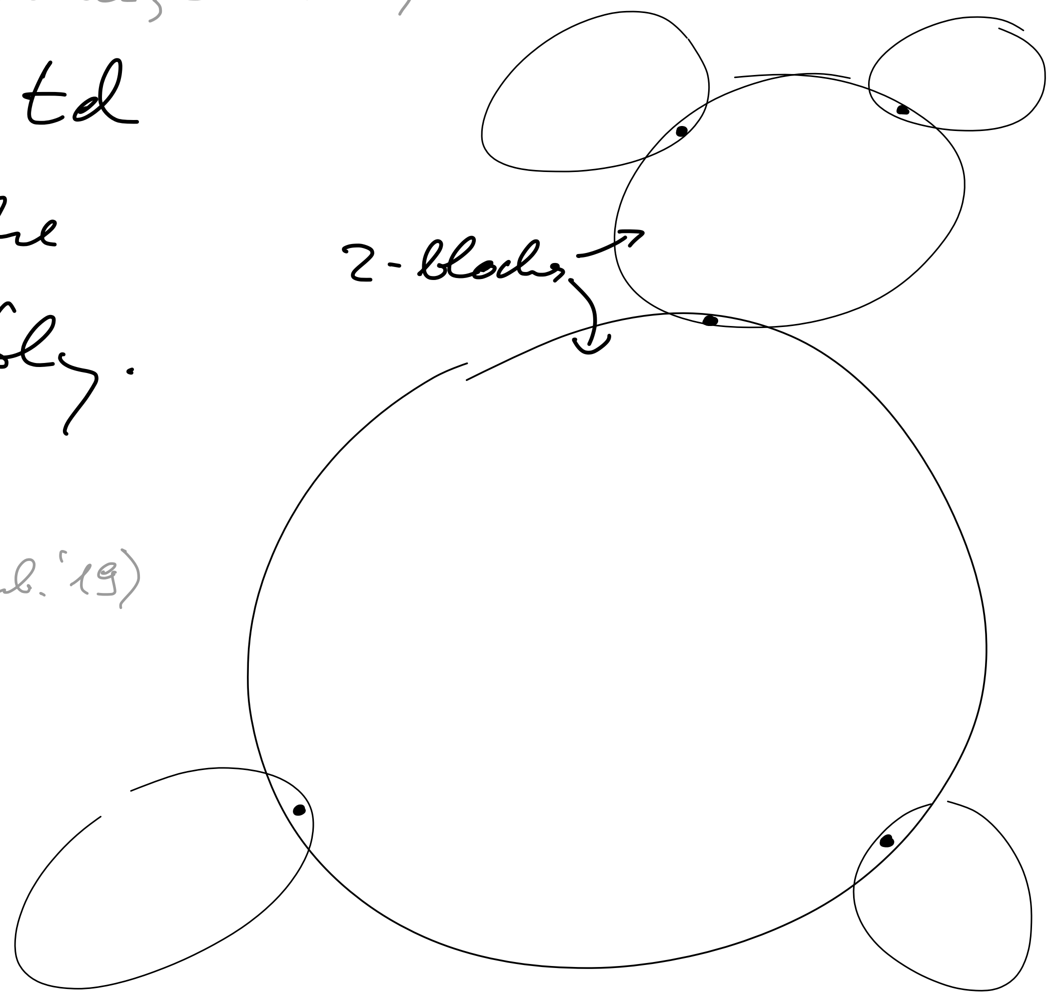

Tree of tangles for $k=2$

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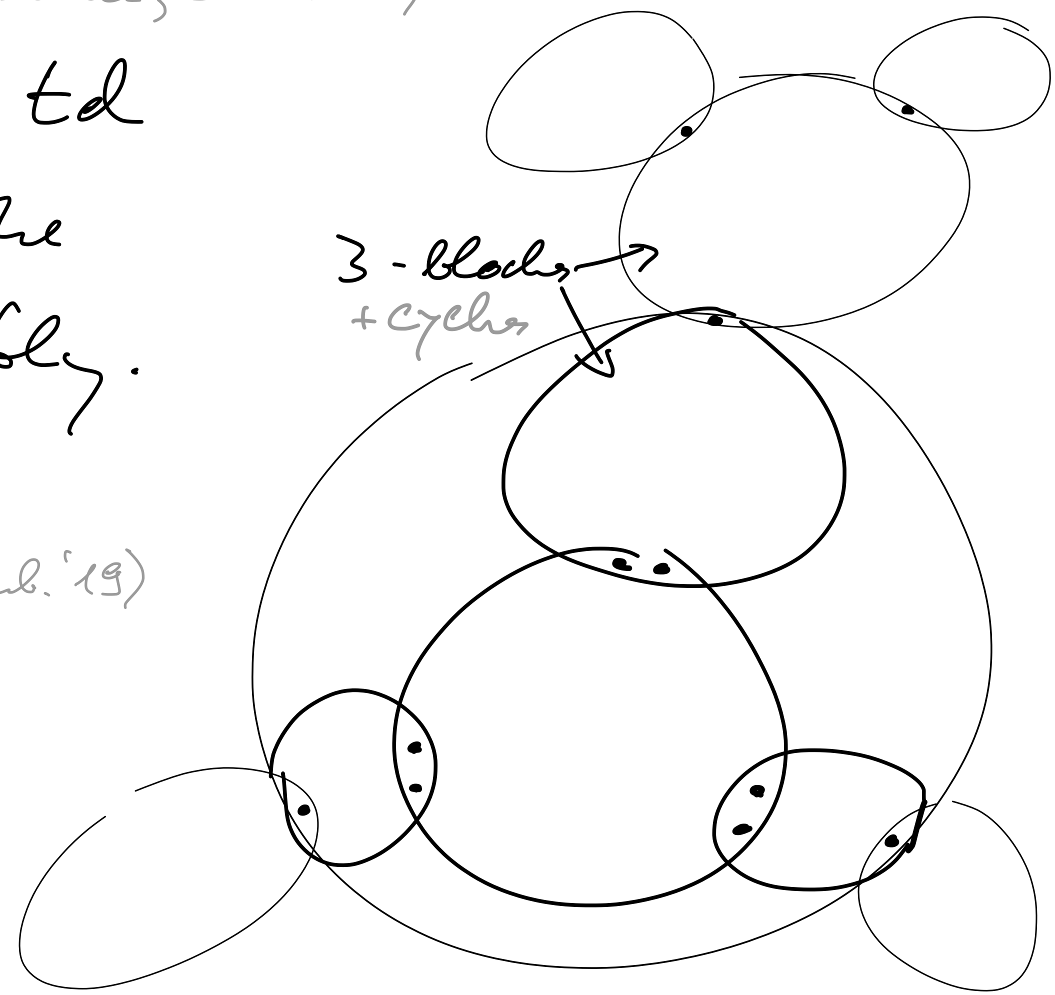

Tree of tangles for $k=3$

A *k-block* in G is a maximal set of $\geq k$ vertices no two of which are separated in G by $< k$ other vertices.

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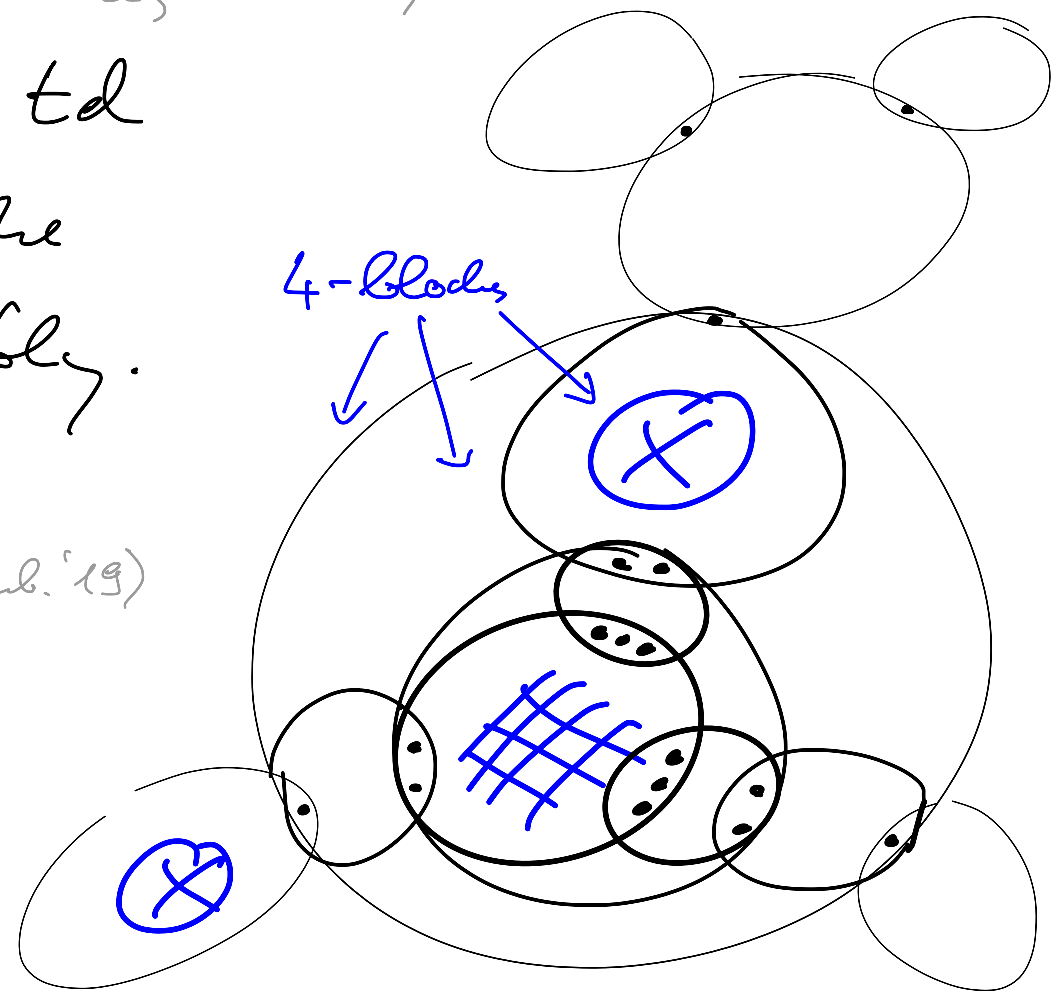

Tree of tangles for $k=4$

A *k-block* in G is a maximal set of $\geq k$ vertices no two of which are separated in G by $< k$ other vertices.

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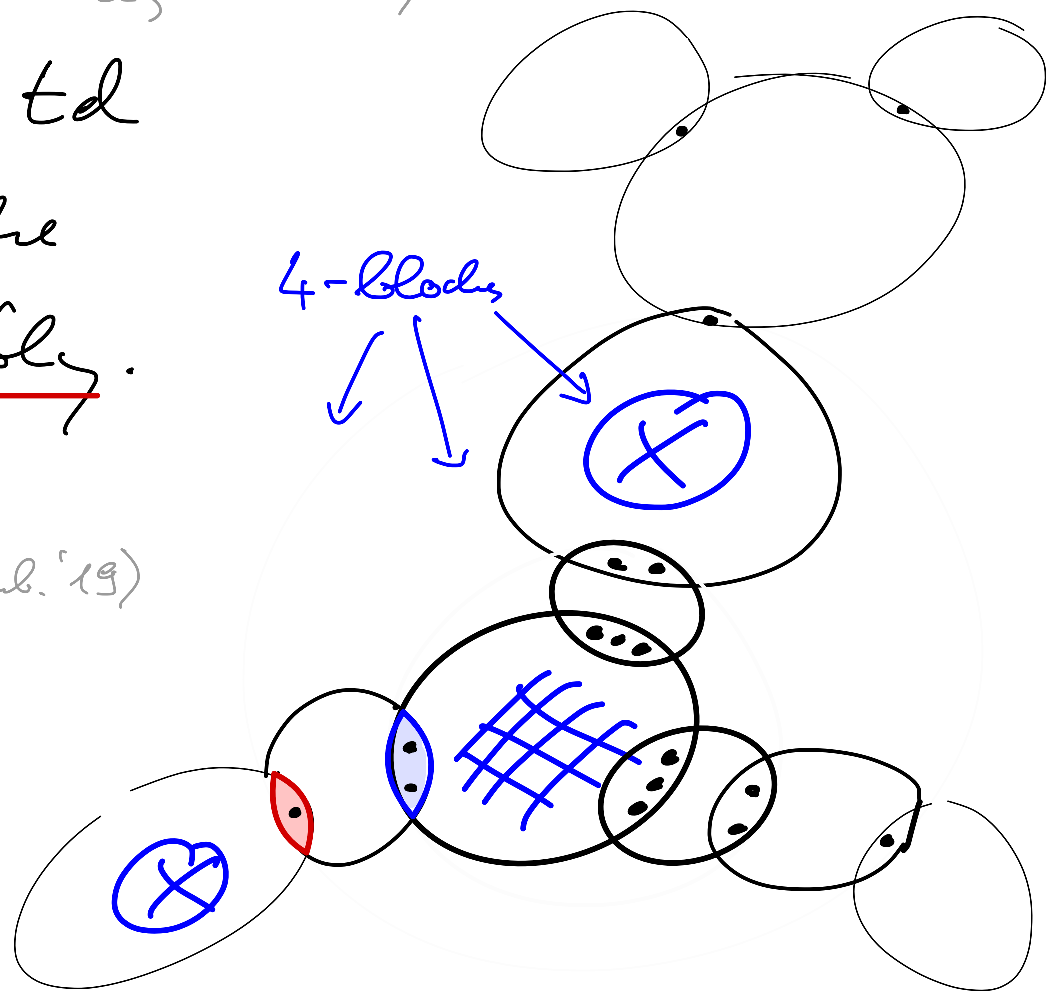


Tree of tangles (ToT)

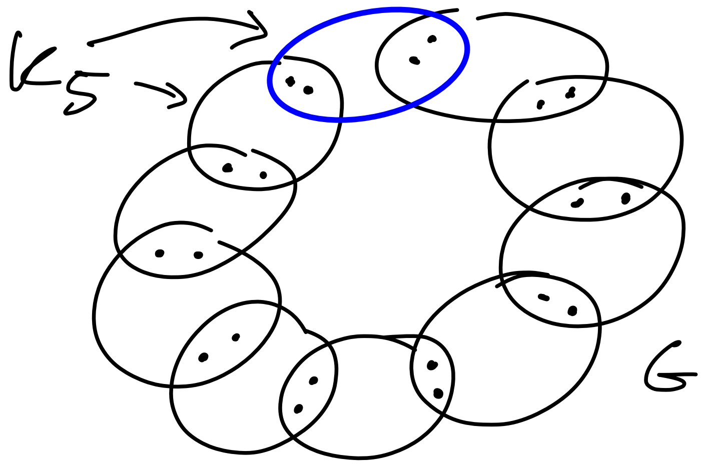
For $r = |G|$, the only r-global structure of G is tree-like,
displayed by its ToT

no connectivity added
'outside our r-focus'

For $r = |G|$, the only r-global structure of G is tree-like, displayed by its ToT

For $1 \ll r \ll |G|$:

- <u>intended local blobs</u> are the K_5:



- ToT (G) would give us:

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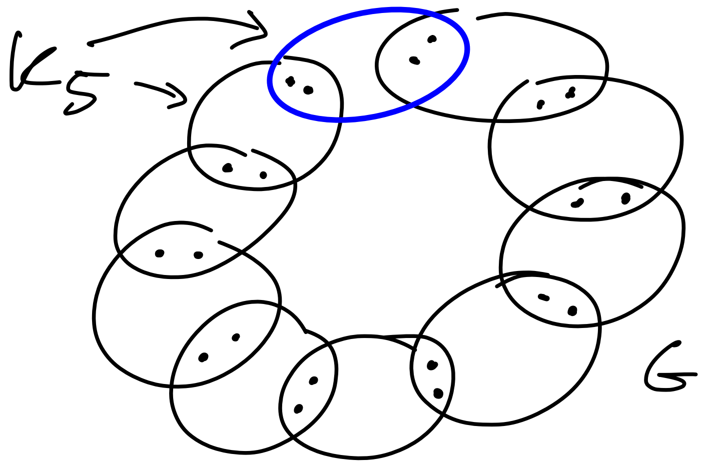

- ToT (G) would give us:

$k \leq 4$: K_1

G = 4-block 'despite' ☹

For $r = |G|$, the only r-global structure of G is tree-like, displayed by its ToT

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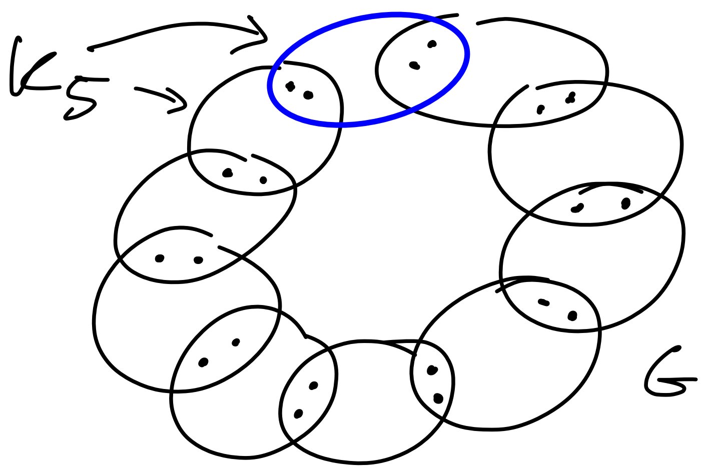
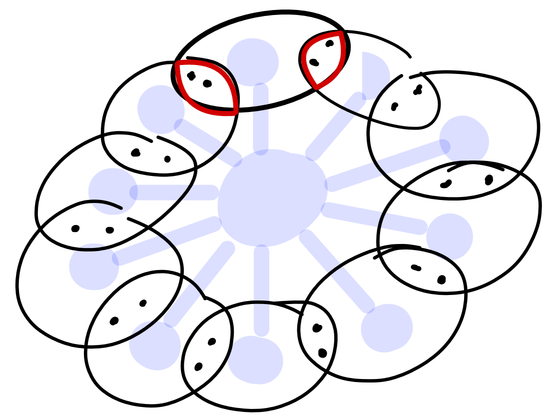


- ToT (G) would give us:

$k \le 4$: K_1

$k = 5$:



star

For $r = |G|$, the only r-global structure of G is tree-like, displayed by its ToT

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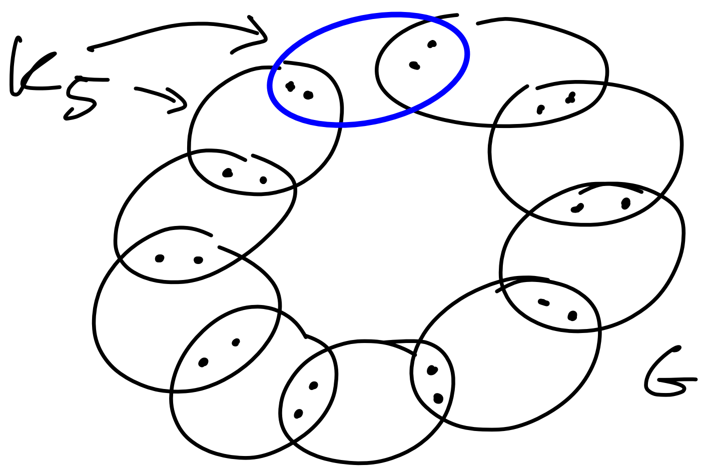

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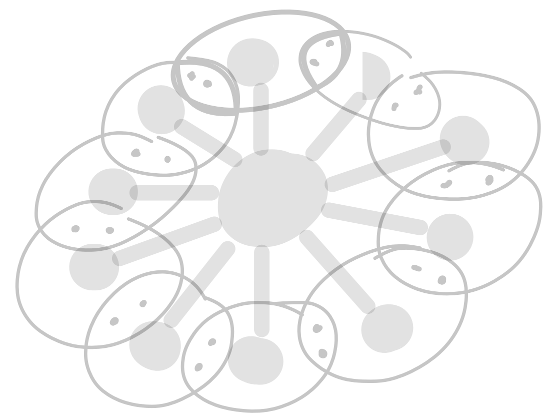

$\Rightarrow$ to make the K_5 into blocks, abstract from global cyclic structure

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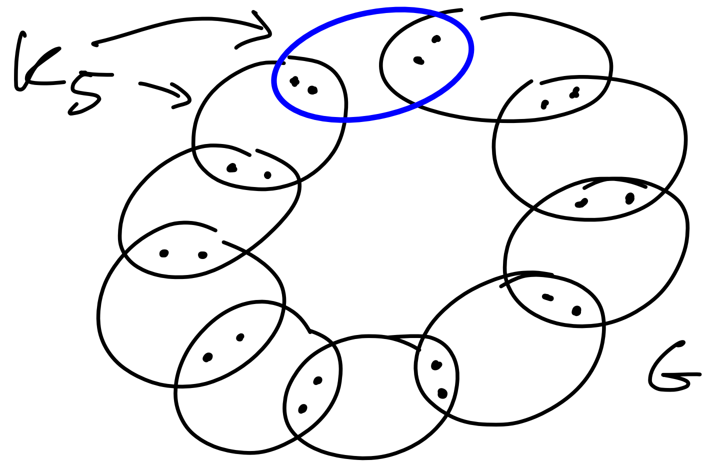
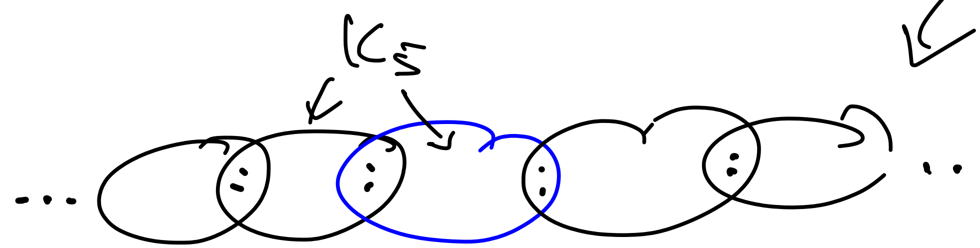


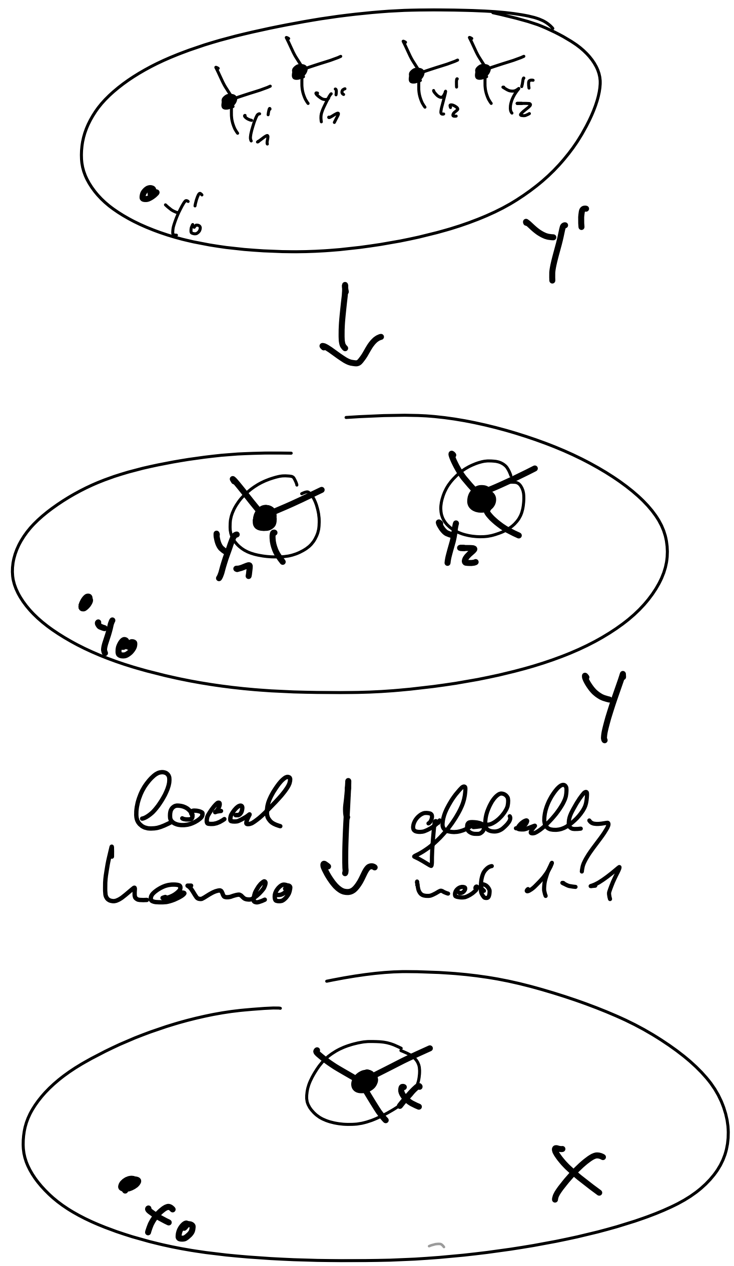
Idea:

local blobs := bags of ToT of



$\Rightarrow$ to **make** the K_5 into blocks, abstract from global cyclic structure

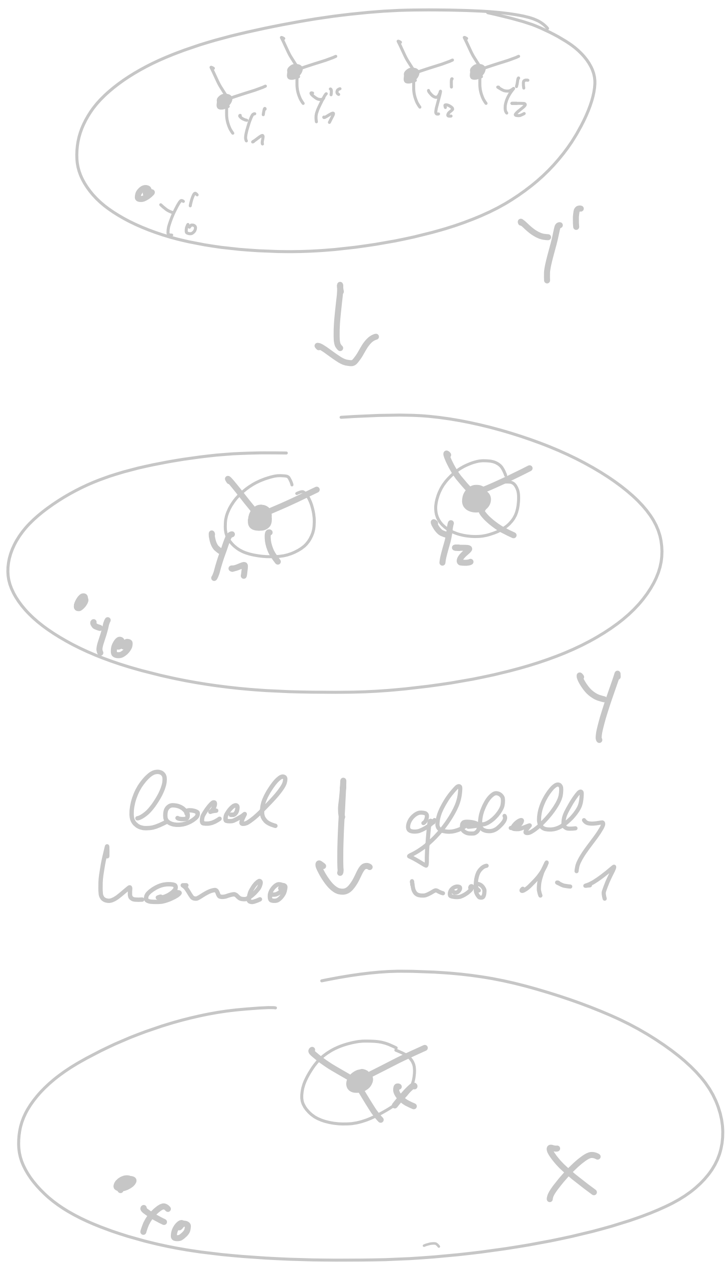
'Unfolding' long cycles $\longrightarrow$ covering spaces

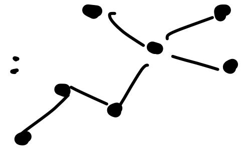


y_1'
y_1''
y_2'
y_2''
y_0'
Y'
y_1
y_2
y_0
Y
local homeo
globally not 1-1
x
x_0
X

$$
\begin{tikzcd}
\pi_1(Y', y_0') \arrow[d, hook] \\
\pi_1(Y, y_0) \arrow[d, hook, two heads] \\
S \arrow[d, phantom, "\cap"] \\
\pi_1(X, x_0)
\end{tikzcd}
$$

may choose S first



$\pi_1(U)=\{1\} \Rightarrow$ tree:  U

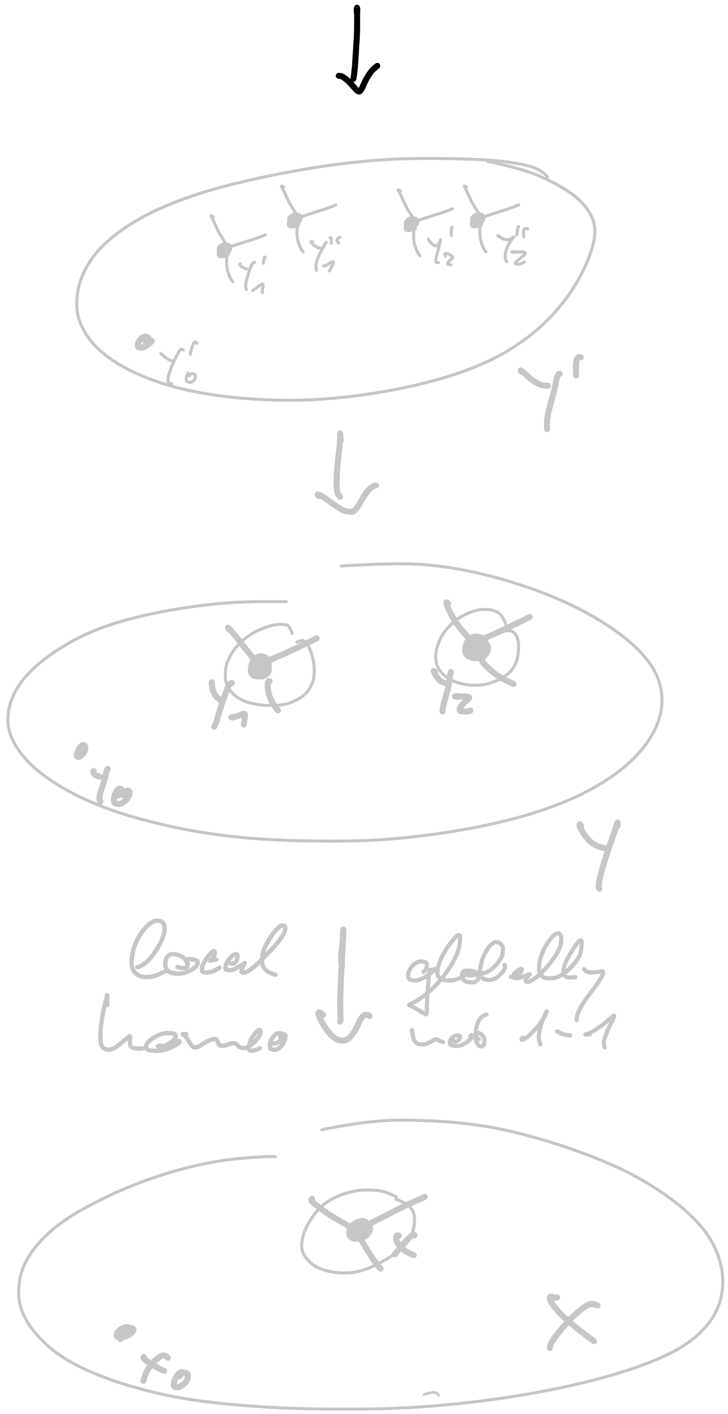
$\pi_1(Y',y'_0)$

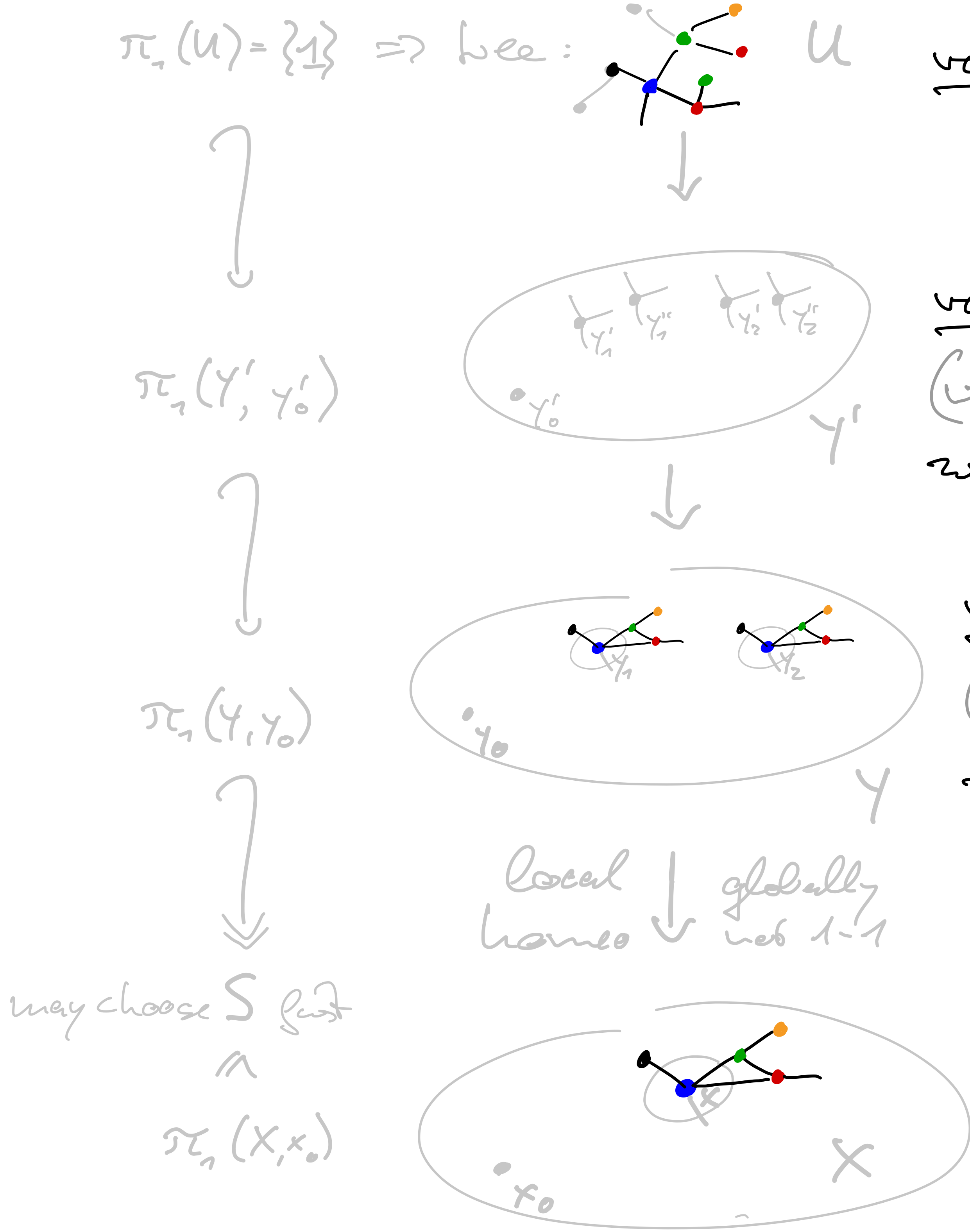
$\pi_1(Y,y_0)$

may choose S first

$\cap$

$\pi_1(X,x_0)$

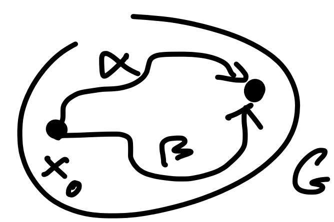


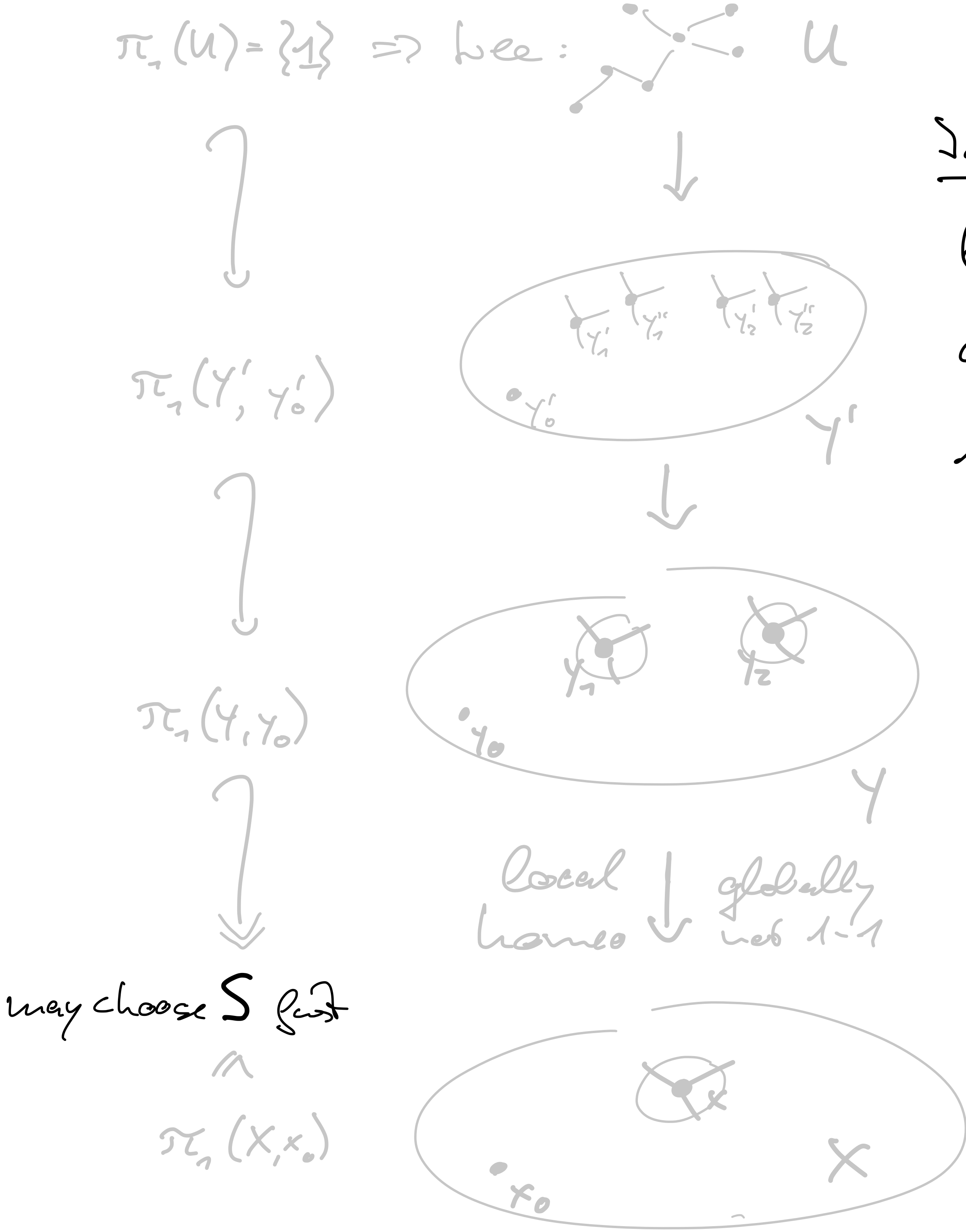


vertices: 'reduced' walks in G from x_0

vertices:
(walks in G from x_0)$/\sim'$
with fewer identifications than

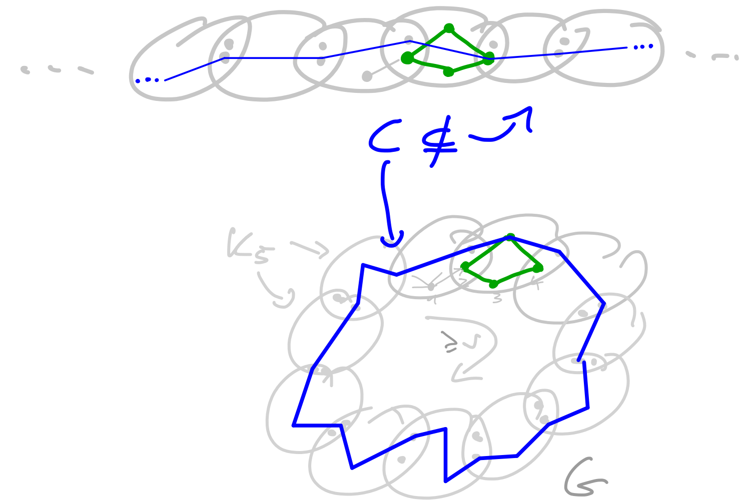
vertices:
(walks in G from x_0)$/\sim$
where $\alpha \sim \beta :\Leftrightarrow \underbrace{[\alpha\beta^{-1}]}\in S$



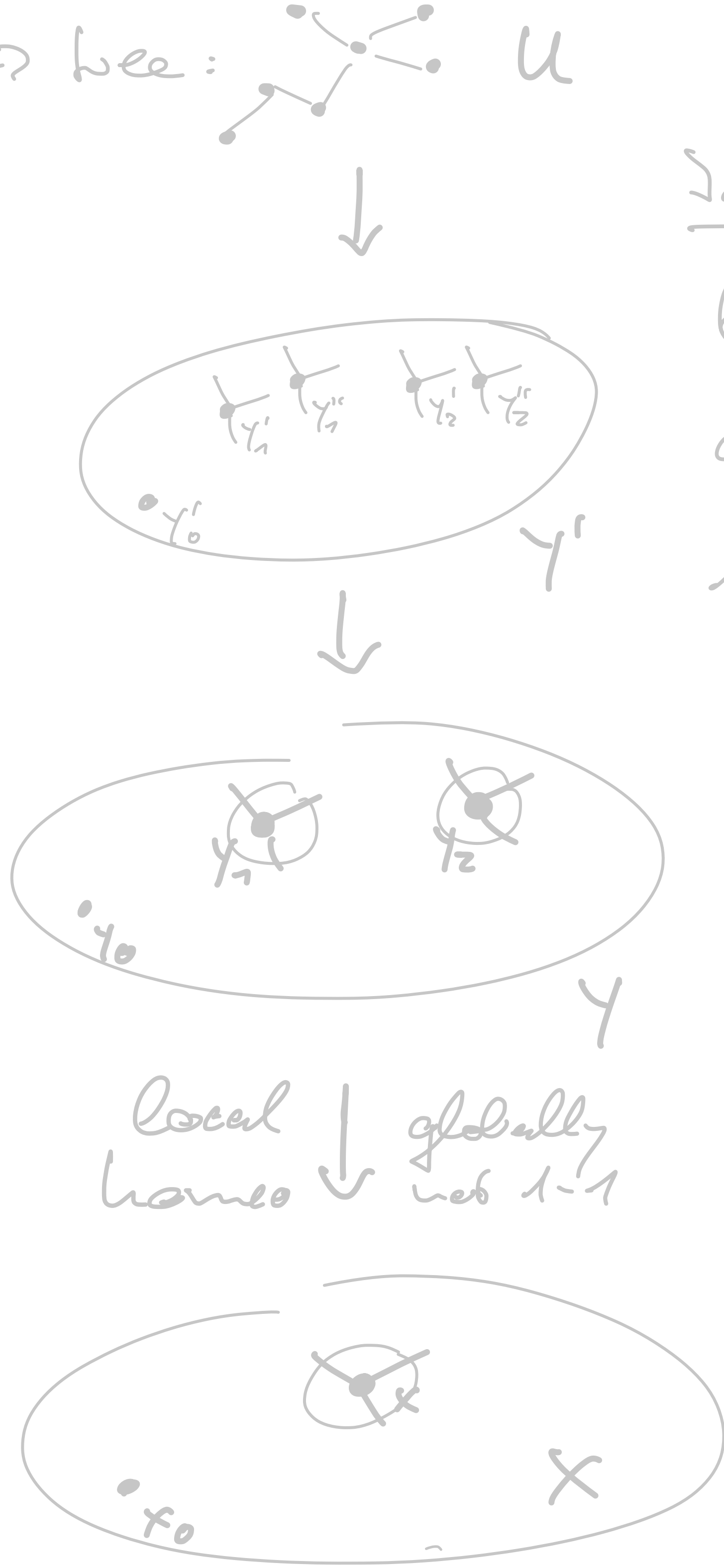


<u>Idea</u>: Choose S to encode the local structure in G, and use ToTs to describe its global aspects, made tree-like in G_S

$$\begin{array}{ccc} G_S & & \pi_1(G_S) \\ \downarrow & & \wr \\ G & & S_r \le \pi_1(G) \end{array}$$



$\pi_1(U) = \{1\} \Rightarrow$ tree:



$\pi_1(U)$

$\downarrow$

$\pi_1(Y', y_0')$

$\downarrow$

$\pi_1(Y, y_0)$

$\downarrow$

may choose S first

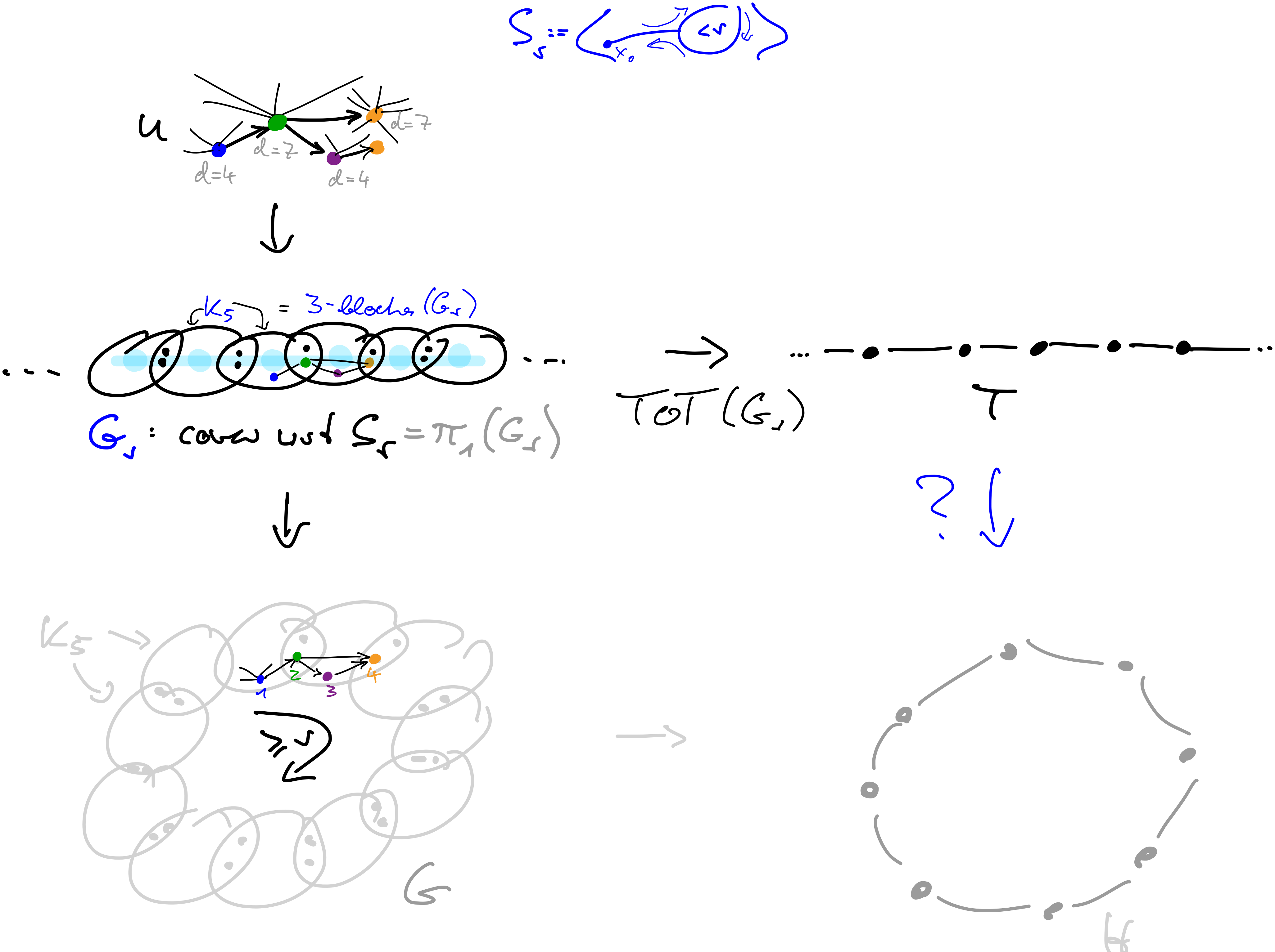
$\wedge$

$\pi_1(X, x_0)$

<u>Idea</u>: Choose S to encode the local structure in G, and use π_1's to describe its global aspects, made tree-like in G_S

$$\begin{array}{cc} G_S & \pi_1(G_S) \\ \downarrow & \downarrow \\ G & S_S \le \pi_1(G) \end{array}$$

$S_s := \Big\langle$  $\Big\rangle \le \pi_1(G)$

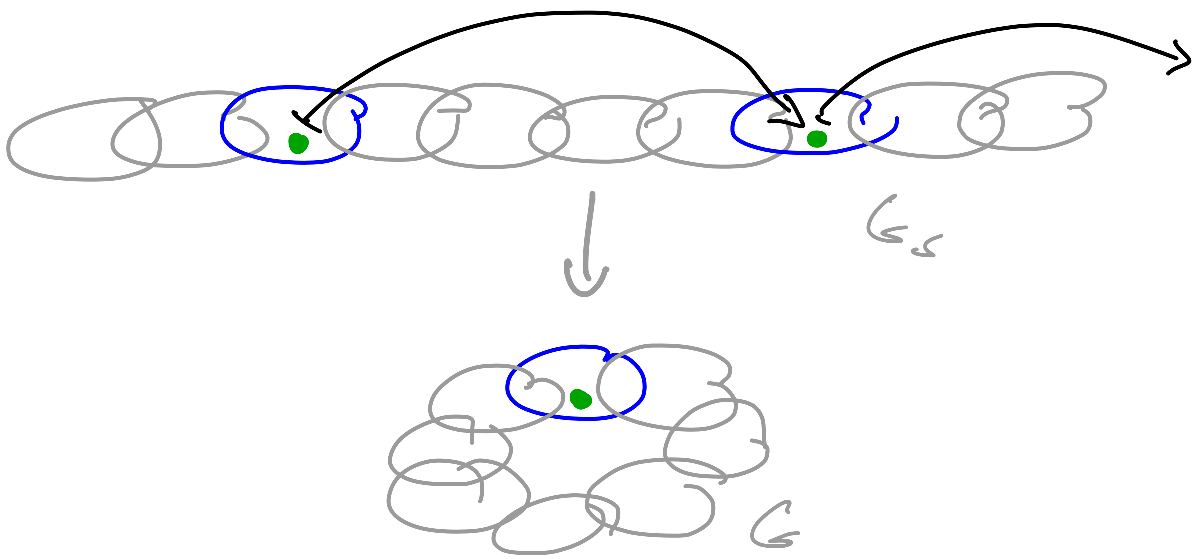

$S_r := \langle x_0, < r \rangle$
u
d=4
d=7
d=7
d=4
K_5 = 3-blocks(G_r)
G_r : cover wrt $S_r = \pi_1(G_r)$
Tor(G_r)
T
?
K_5
1
2
3
4
$\geq r$
G
H

Properties of the r-covering $\begin{matrix} G_r \\ \downarrow \\ G \end{matrix}$ and S_r:

Properties of the r-covering $\begin{array}{c} G_r \\ \downarrow \\ G \end{array}$ wrt S_r:

Deck-transformations (group D of autos of G_r over G
$$\begin{tikzcd} G_r \arrow[rr] \arrow[dr] & \cong & G_r \arrow[dl] \\ & G & \end{tikzcd}$$
) act...

- transitively on each of the fibres of $\begin{array}{c} G_r \\ \downarrow \\ G \end{array}$



Properties of the r-covering $G_r \downarrow G$ wrt S_r:

Deck-transformations (group D of autos of G_r over G) act...

$$\begin{tikzcd} G_s \arrow[r] \arrow[dr] & G_r \arrow[d] \\ & G \end{tikzcd}$$

- transitively on each of the fibres of $G_r \downarrow G$
- on the bags of the $\overline{\mathrm{ToT}}(G_s) =: \overline{T}$

canonical

Properties of the r-covering $\begin{matrix} G_r \\ \downarrow \\ G \end{matrix}$ wrt S_r:

Deck-transformations (group D of autos of G_r over G) act...

$$\begin{tikzcd} G_r \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\cong"] & G_r \arrow[dl] \\ & G & \end{tikzcd}$$

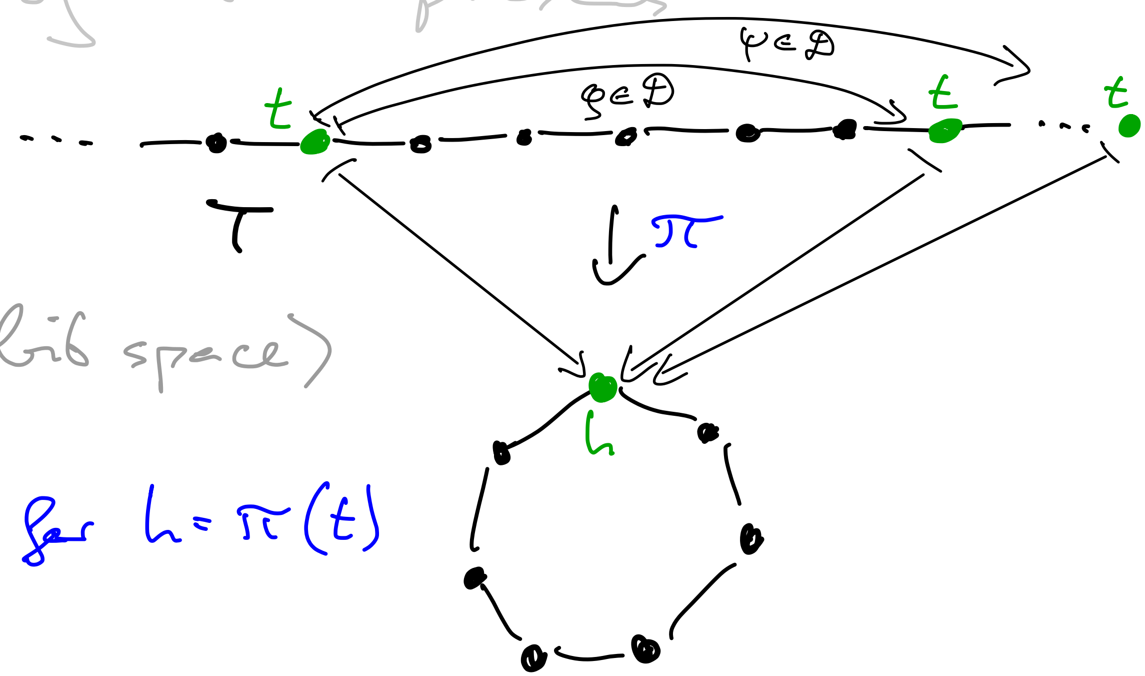
- transitively on each of the fibres of $\begin{matrix} G_r \\ \downarrow \\ G \end{matrix}$
- on the bags of the $\mathrm{ToT}(G_r) =: T$
- on T as a group of automorphisms (T canonical)

Properties of the r-covering $\begin{array}{c} G_r \\ \downarrow p_r \\ G \end{array}$ wrt S_r:

Deck-transformations (group $\mathcal{D}$ of autos of G_r over G) act...

- transitively on each of the fibres of $\begin{array}{c} G_r \\ \downarrow \\ G \end{array}$ $\quad$ \begin{tikzcd} G_s \arrow[rr] \arrow[dr] & \arrow[d, phantom, "///"] & G_r \arrow[dl] \\ & G & \end{tikzcd}
- on the bags of the $\mathrm{TOT}(G_s) =: T$
- on T as a group of automorphisms

$\Rightarrow$ May define

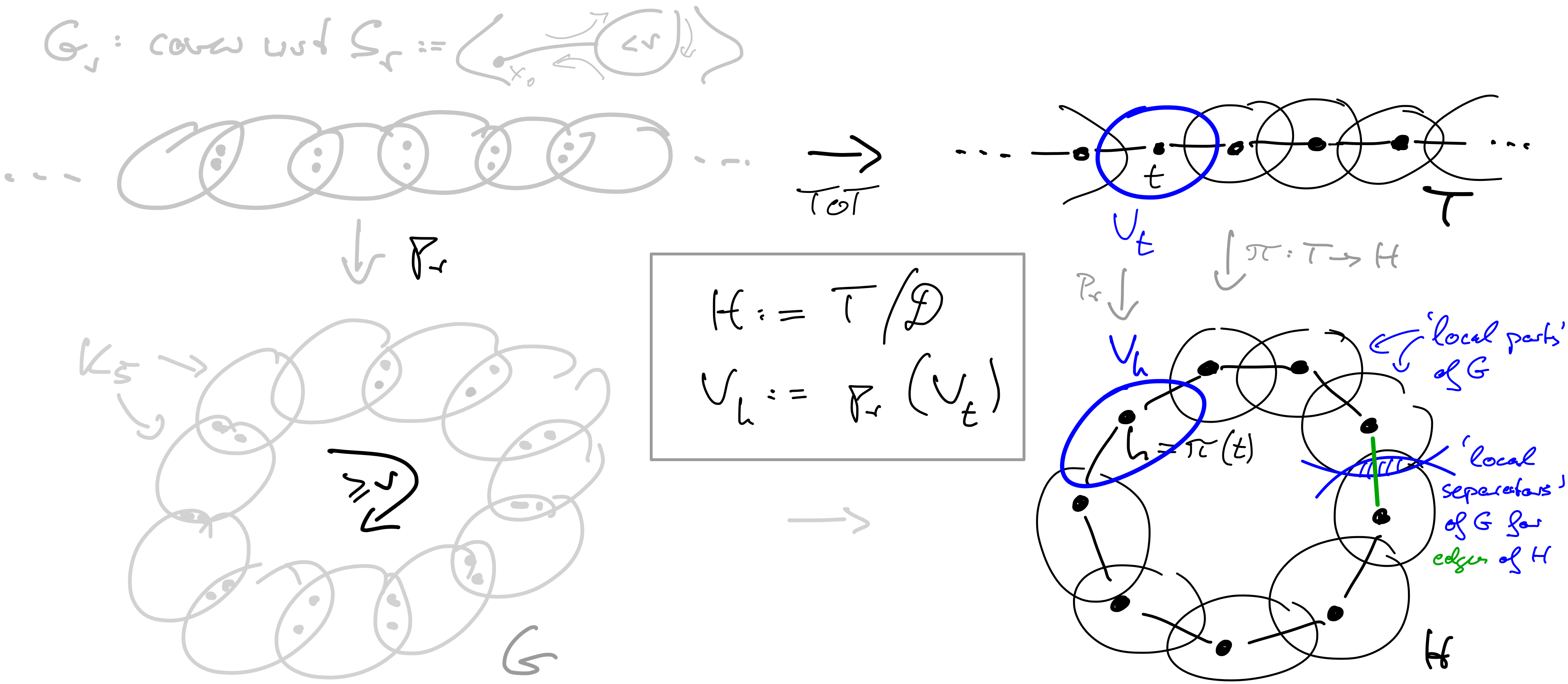


$H := T/\mathcal{D}$ (orbit space)

$V_h := p_r(V_t)$ for $h = \pi(t)$

H: r-global structure of G

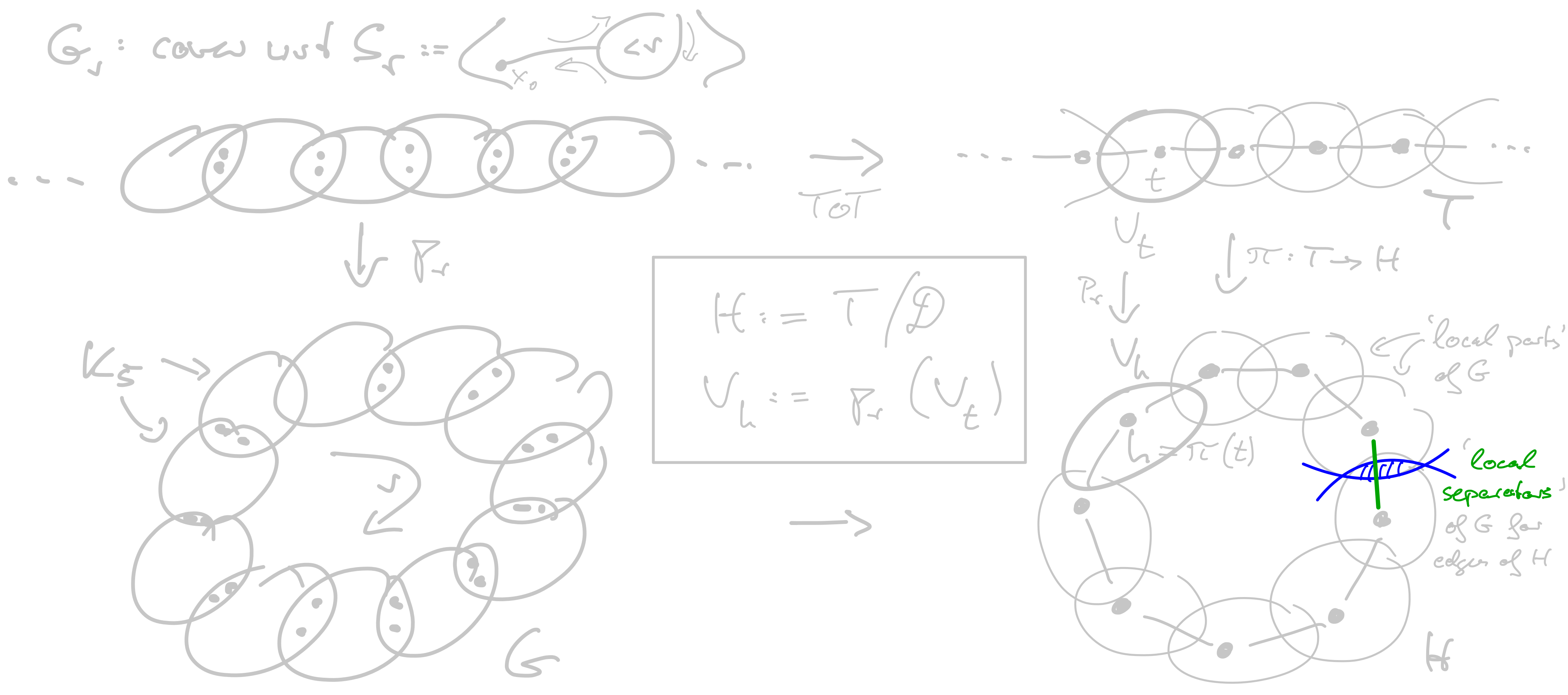
Thm: $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure



<u>Thm:</u> $\forall r \in \mathbb{N}$, every finite graph G has a canonical H-decomposition that displays its r-global structure

$$\exists \text{ minors } H_1 \preccurlyeq H_2 \preccurlyeq \dots \preccurlyeq H_n = H \text{ with}$$

H_k: contract edges meeting local $(\geq k)$-separations



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Cor (pf): H distinguishes the r-local blocks of G efficiently

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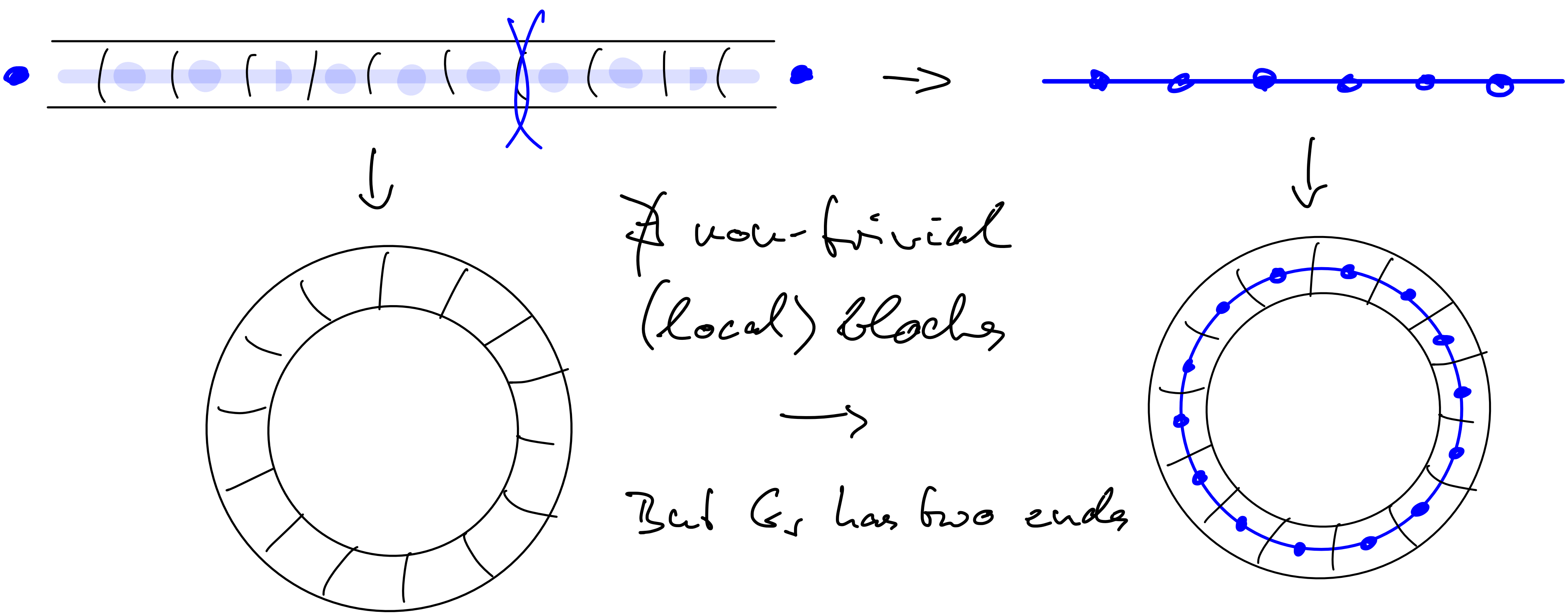
Example:



Summary

- Structure theory exists for tree-like graphs only: ToTs.

⇒ <u>make</u> global structure tree-like, apply ToT, project back to G

coverings

- r-global structure H of G, and H-dec^{n} $(V_h)_{h \in H}$, are <u>found</u> on input G, r — not imposed

- Defaults to tangle-structure of G for $r = |G|$

- Main challenge in proof: extend known tangle theory to infinite G_r to obtain ToT without limits. This fails in general but is needed to get graphs $T \to H$

- Applies to finitely generated groups

Open problems:

- $H(G, r)$ = new invariant: relate to others
- Find alg^s to compute G_r, H, posets V_h
- Compute local blocks & sep^s locally
- Define 'local tangles' $\longrightarrow$ new Car