Stack and Queue Layouts

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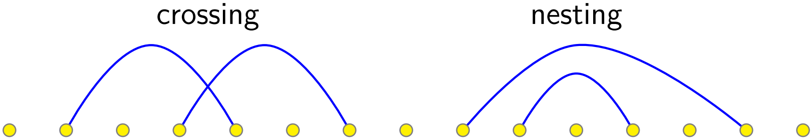
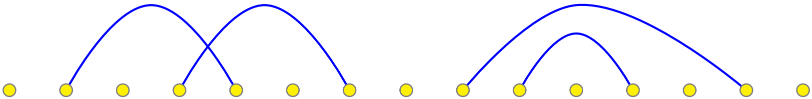
stacks and queues

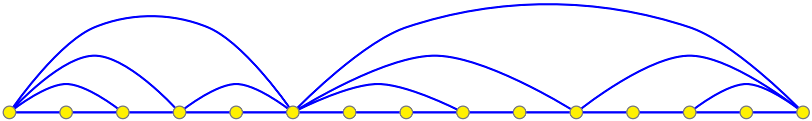


stacks and queues

crossing nesting



stack = set of non-crossing edges



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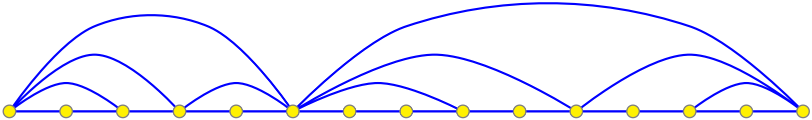
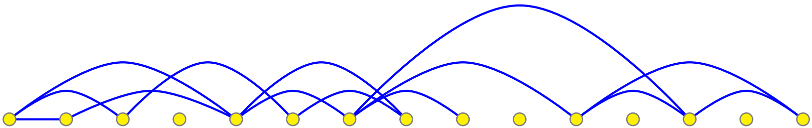
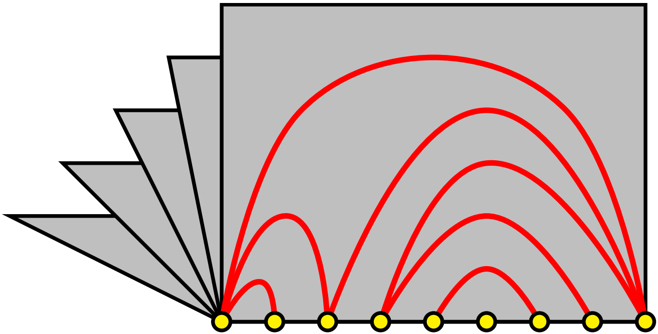


queue = set of non-nesting edges



stack-number (page-number, book thickness)

$\mathrm{sn}(G) :=$ minimum number of stacks in a stack layout of G



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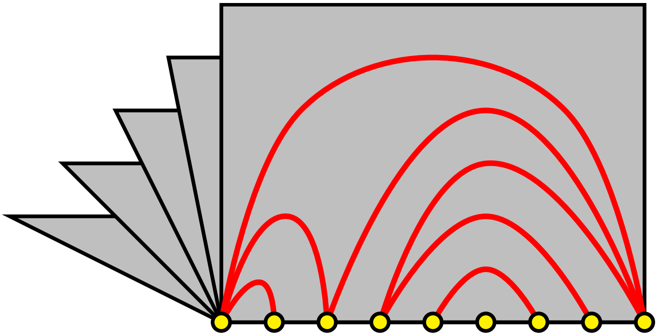
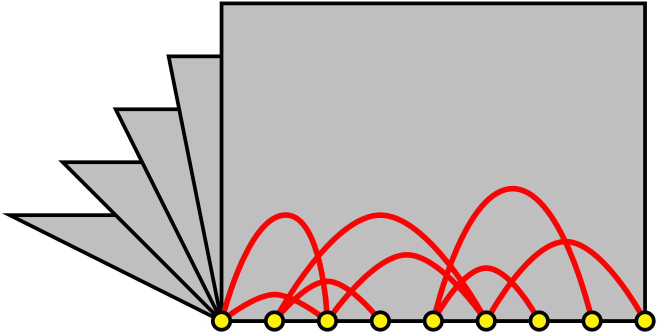


$\mathrm{sn}(\mathcal{G}) := \max\{\mathrm{sn}(G) : G \in \mathcal{G}\}$

queue-number

qn(G) := minimum number of queues in a queue layout of G



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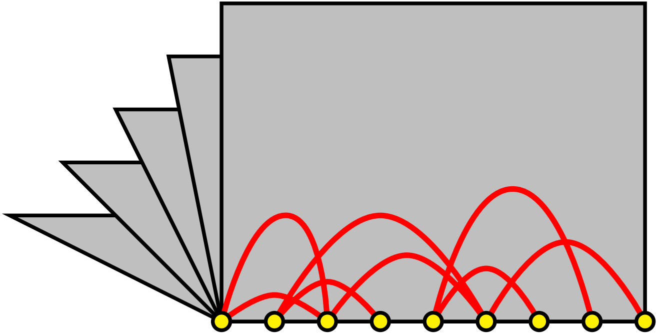

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some applications

- computational complexity [Galil, Kannan, Szemerédi '89] [Bourgain, Yehudayoff '13; Dujmović, Sidiropoulos, Wood]
- RNA folding [Haslinger, Stadler '99]
- graph drawing [Baur, Brandes '04; Angelini et al. '12; etc.]
- three-dimensional graph drawing [Dujmović, Morin, Wood '05; Dujmović, Por, Wood '05]
- fault-tolerant multiprocessing [Rosenberg '83; Chung, Leighton, Rosenberg '87]
- traffic light control [Kainen '90]

what is more powerful?

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queues would be considered more powerful than stacks if:

- queue-number is bounded by stack-number
- stack-number is not bounded by queue-number

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  i.e. $\forall \mathcal{G} \quad \mathrm{sn}(\mathcal{G}) \leqslant c \quad \Longrightarrow \quad \mathrm{qn}(\mathcal{G}) \leqslant c'$

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[Heath, Leighton, Rosenberg '92]

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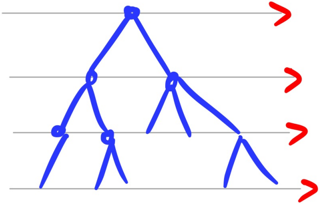
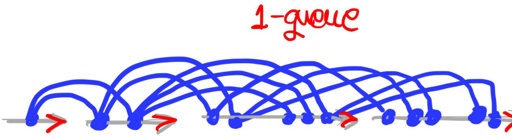
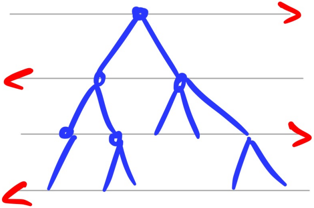
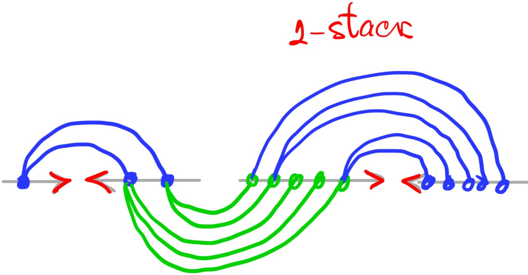
What classes of graphs (if any) separate them?

examples: trees



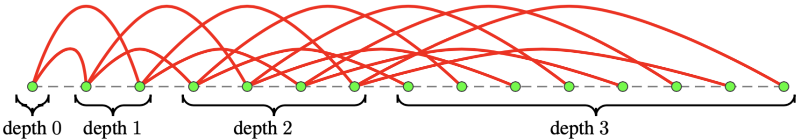




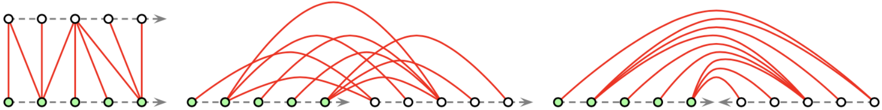


examples: trees

example $\mathsf{qn}(\mathsf{tree}) = 1$

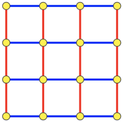


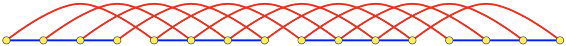
small tool



examples: stack and queue layout of 2-dimensional grid

example $\mathrm{qn}(\mathrm{grid}) \leqslant 2$





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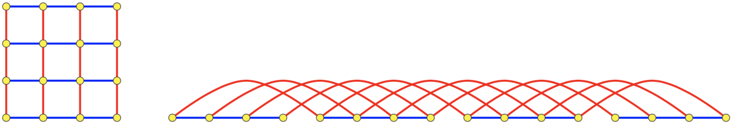

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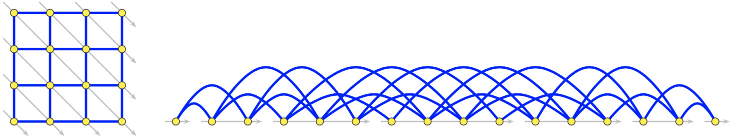

1-stack, 1-queue characterization

[Bernhart, Kainen '79] A graph has a 1-stack layout iff it is outer-planar.

[Heath, Leighton, Rosenberg '92] A graph has a 1-queue layout iff it is "leveled-planar graph".

A graph has a 2-stack layout iff it is sub-Hamiltonian.

density, sparseness

Graphs with bounded stack/queue number have $O(n)$ edges.

Thus **all** graphs with $w(n)$ edges have unbounded bounded stack/queue number.

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Thus **all** graphs with $w(n)$ edges have unbounded bounded stack/queue number.

[Bourgain, Yehudayoff '13] There are $O(1)$-monotone bipartite expander

Thus, there are (bounded degree) expanders with $O(1)$ stack/queue layout.

Used to know much more about stack number

All of the following graphs classes have bounded stack number:

bounded treewidth
planar
genus
proper minor closes
subdivsions

What did we know about queue number?

[Pemmaraju '92]: conjectured that there exists planar 3-trees with unbounded queue-number

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Disproved by [Dujmović, Morin, Wood '05]:

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What about planar graphs

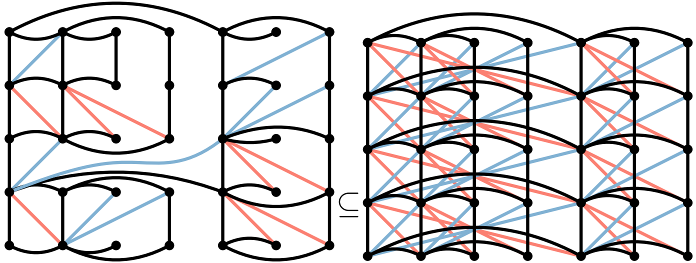
planar graphs

e.g. sn(planar graphs) $= 4$ [Yannakakis '89 '20]
[Kaufmann, Bekos, Klute, Pupyrev, Raftopoulou, Ueckerdt '20]

open problem [Heath, Leighton, Rosenberg '92]
do planar graphs have bounded queue-number?

structure of planar graphs

theorem [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19]
every planar graph G is a subgraph of $H \boxtimes P$
for some graph H with treewidth $\leqslant 8$ and some path P



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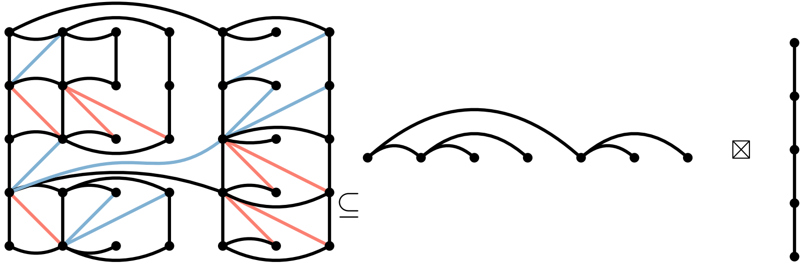

strong product

For two graphs A and B, the *strong product* $A \boxtimes B$ is a graph:

- $V(A \boxtimes B) := V(A) \times V(B)$
- (a_1, b_1) and (a_2, b_2) are adjacent if and only if:
  - $a_1 = a_2$ and $b_1 b_2 \in E(B)$;
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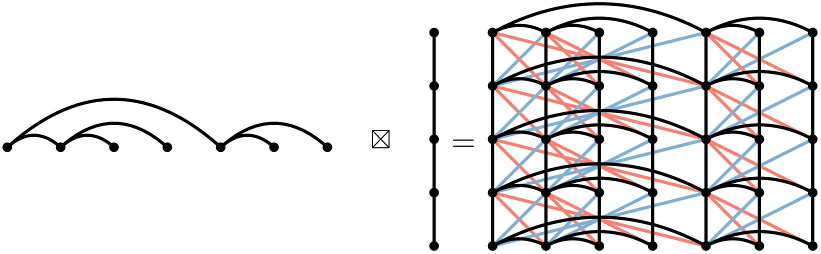

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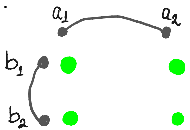

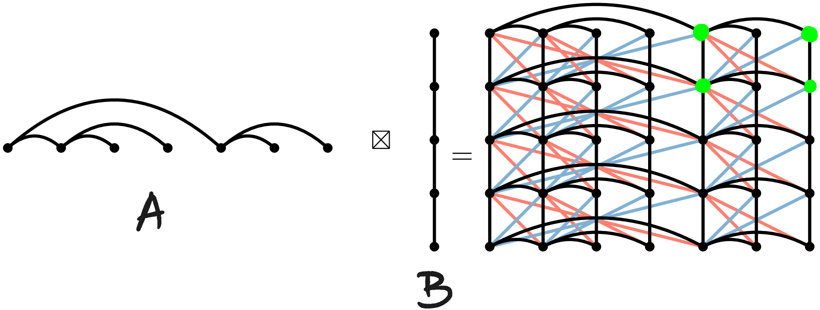

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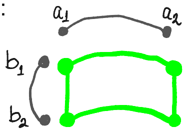

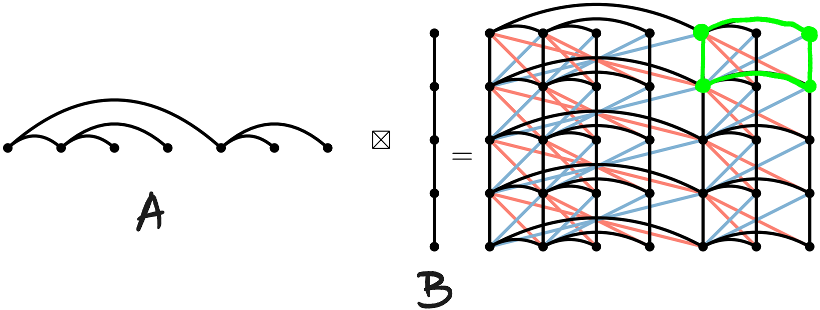

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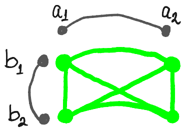
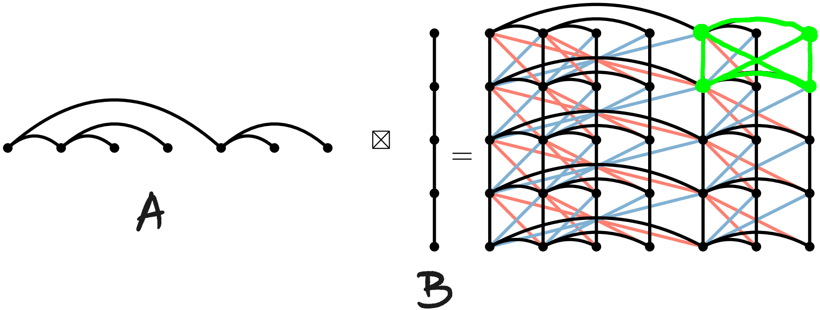
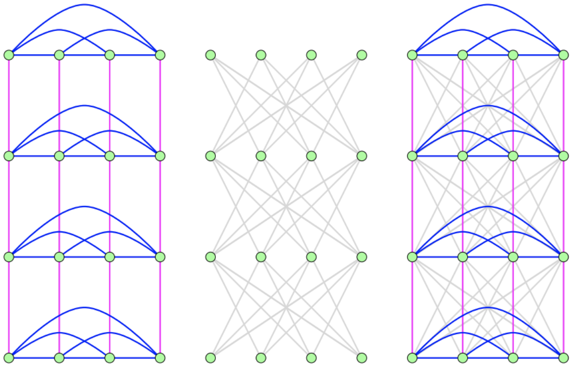




cartesian, direct, strong product



(a) $P_4 \,\square\, K_4$ (b) $P_4 \times K_4$ (c) $P_4 \boxtimes K_4$

Fig. 4: Examples of graph products: (a) cartesian, (b) direct, (c) strong.

structure of planar graphs

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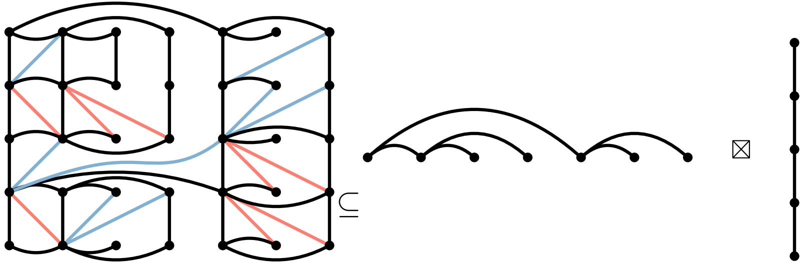
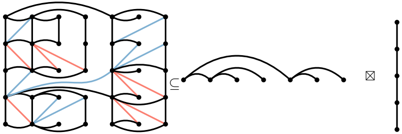


what is it good for?

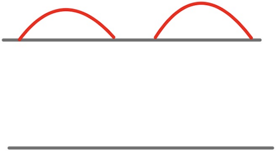
Why?

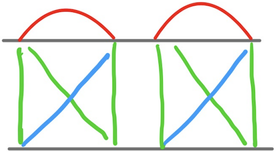
$G \subseteq H \boxtimes P$

- H is a graph of treewidth at most 8
- Many problems are easy for H
- Extending a solution from H to $H \boxtimes P$ is sometimes easy
- Examples:
  - queue number
  - nonrepetitive colouring
  - p-centered colouring
  - ℓ-vertex ranking
  - adjacency labelling (universal graphs)



stack/queue layouts of products





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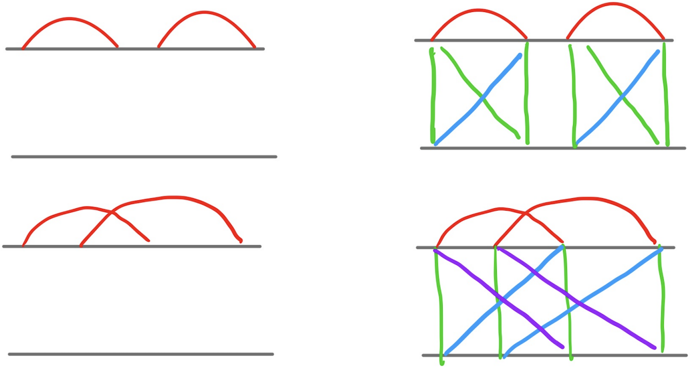

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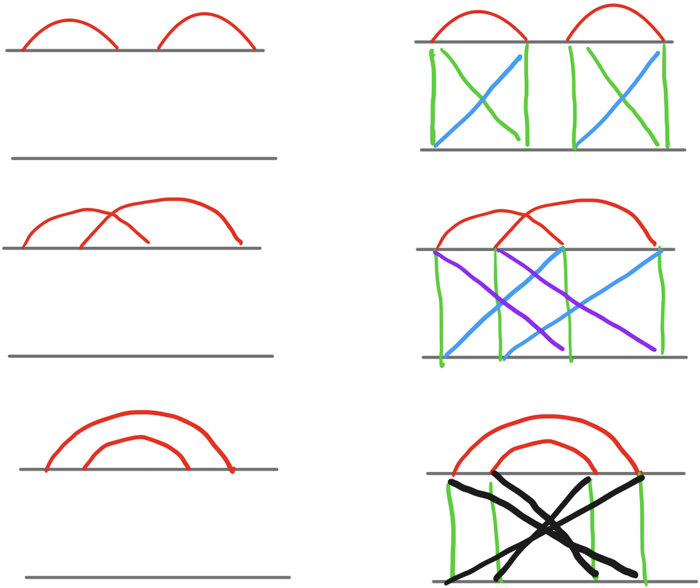

queue layouts of products

lemma [Wood '04] $\mathsf{qn}(H \boxtimes P) \leqslant 3\,\mathsf{qn}(H) + 1$

queue-number of planar graphs

- $\mathrm{qn}(H \boxtimes P) \leqslant 3\,\mathrm{qn}(H) + 1$ [Wood '04]

queue-number of planar graphs

- $\mathrm{qn}(H \boxtimes P) \leqslant 3\,\mathrm{qn}(H) + 1$ [Wood '04]
- graphs of bounded treewidth have bounded queue-number [Dujmović, Morin, Wood '05]
- $\mathrm{qn}(H) \leqslant 2^{\mathrm{tw}(H)} - 1$ [Wiechert '18]

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- $\mathrm{qn}(H) \leqslant 2^{\mathrm{tw}(H)} - 1$ [Wiechert '18]

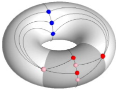
$$\begin{aligned} \mathrm{qn}(\text{planar } G) &\leqslant \mathrm{qn}(H \boxtimes P) && \text{where } \mathrm{tw}(H) \leqslant 8 \\ &\leqslant 3\,\mathrm{qn}(H) + 1 \\ &\leqslant 3(2^8 - 1) + 1 \\ &= 766 \end{aligned}$$

[Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19]

generalizations of product structure theorem

Similar product structure theorems for:
$G \subseteq H \boxtimes P$, only the treewidth of H changes

- graphs of bounded genus and apex-minor free graphs
  [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19]



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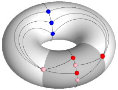

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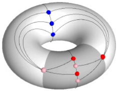

- bounded degree graphs that exclude a fixed graph as a minor
  [Dujmović, Esperet, Morin, Walczak, Wood '20]
- k-planar graphs and (g, k)-planar graphs (**non-minor closed families)**
  [Dujmović, Morin, Wood '20]

stacks vs queues: which do we know more about?

bounded stack number trees, bounded treewidth, planar, bounded genus, all proper minor closed, d-monotone bipartite graphs, **1-planar**

stacks vs queues: which do we know more about?

bounded stack number trees, bounded treewidth, planar, bounded genus, all proper minor closed, d-monotone bipartite graphs, **1-planar**

bounded queue number trees, bounded treewidth, planar, bounded genus, all proper minor closed, d-monotone bipartite graphs, **k-planar**, **graph products***, and other non-minor closed families.

what is more powerful?

queues would be considered more powerful than stacks if:

$$\forall \mathcal{G} \quad \mathrm{sn}(\mathcal{G}) \leqslant c \quad \Longrightarrow \mathrm{qn}(\mathcal{G}) \leqslant c' \text{ and}$$
$$\exists \mathcal{G} \quad \mathrm{qn}(\mathcal{G}) \leqslant c \quad \text{and} \quad \mathrm{sn}(\mathcal{G}) \to \infty$$

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ubounded stack/queue number

all graphs with $w(n)$ edges, some sparse expanders

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$$\forall \mathcal{G} \quad \mathrm{qn}(\mathcal{G}) \leqslant c \quad \Longrightarrow \mathrm{sn}(\mathcal{G}) \leqslant c' \text{ and}$$
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ubounded stack/queue number

all graphs with $w(n)$ edges, some sparse expanders

Only tools for unbounded stack/queue number we had: **county and density**

What is a good candidate for unbounded stack number?

What did we learn?

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Are products good for stacks?

layouts of cartesian products

lemma [Wood '04]
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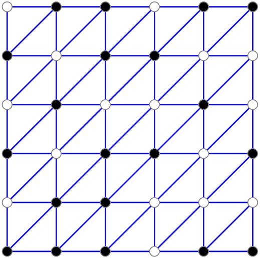
- H_2 to be 'far from' bipartite

hex game

hex game



H_n

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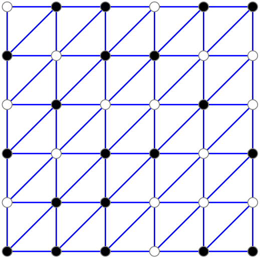

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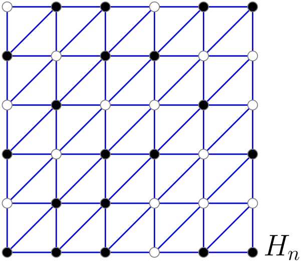

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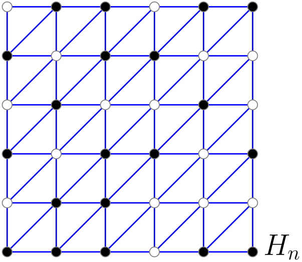


lemma [Nash '49; Pierce '61; Gale '79]
every 2-colouring of H_n has a monochromatic path of length n

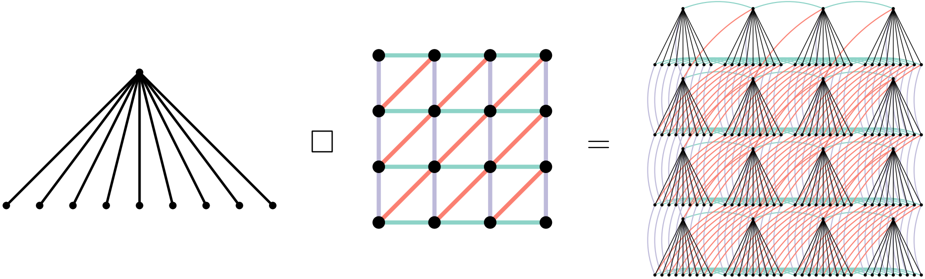
so take $H_2 = H_n$ and take $H_1 =$ star S_n

main theorem

theorem [Dujmović, Eppstein, Hickingbotham, Morin, Wood '20]

if S_n is the n-vertex star and H_n is the $n \times n$ Hex grid graph, then

$$\mathrm{qn}(S_n \,\square\, H_n) \leqslant 4 \quad \text{and} \quad \mathrm{sn}(S_n \,\square\, H_n) \to \infty$$



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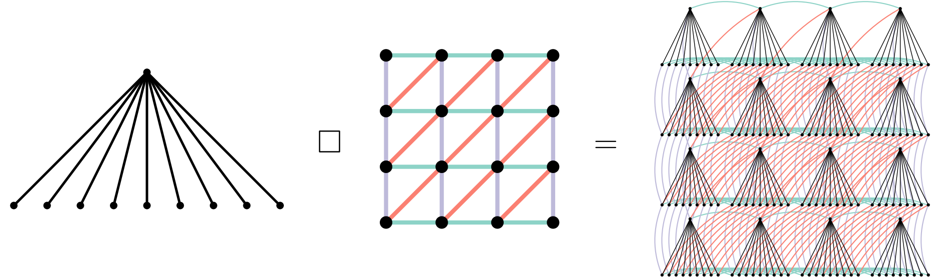

proof pigeon-hole, Erdős-Szekeres lemma, Hex lemma theorem [Dujmović, Eppstein, Hickingbotham, Morin, Wood '20]

there exists $\mathcal{G}$ with $\mathrm{qn}(\mathcal{G}) \leqslant 4$ and $\mathrm{sn}(\mathcal{G}) \to \infty$

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also answers questions about

- graph subdivisions [Blankenship, Oporwoski '99]
- twin-width [Bonnet, Geniet, Kim, Thomassé, Watrigant '20]

Bounded degree products and unbounded stack number

$\mathrm{sn}(S_n \boxtimes H_n) \to \infty$

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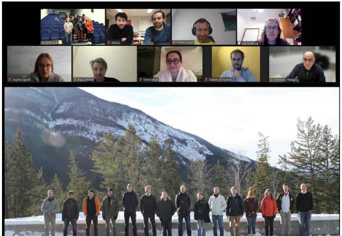

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theorem
[Eppstein, Hickingbotham, Merker, Norin, Seweryn, Wood'22]

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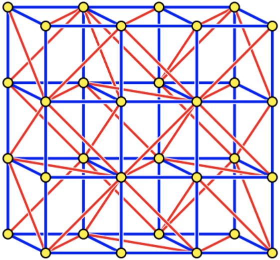

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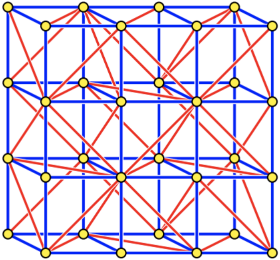


Main tool: topological overlap theorem of Gromov, 2010.

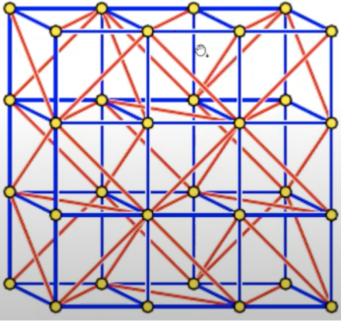
Sergey's slides

Brambles in grids

Let Grid(n) be a 2-dimensional simplicial complex obtained from the Cartesian product $P_n \square P_n \square P_n$ of three paths, by adding a diagonal to every four cycle and adding a face corresponding to every triangle.

Lemma

Grid(n) *contains a bramble* $\mathcal{B}$ *such that* $\|\mathcal{B}\| \geq n$.



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Corollary

$$\text{overlap}(\text{Grid}(n), \mathbb{R}^2) \geq n.$$

Theorem

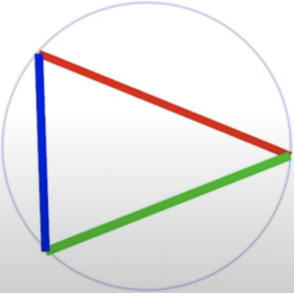
Let X and Y be 2-dimensional simplicial complexes such that Y is collapsible. Let $f : X \to Y$ be continuous, and let $\mathcal{B}$ be a bramble in X. Then

$$\bigcap_{B \in \mathcal{B}} f(B) \neq \emptyset.$$

From overlap to stacks

Lemma (EHMNSW)

Let $T_1, \ldots, T_m$ be pairwise vertex-disjoint pairwise intersecting triangles in $\mathbb{R}^2$ with all the vertices on a circle S. Assume that the edges of $T_1, \ldots, T_m$ can be partitioned into k non-crossing sets. Then $m \leq k^3$.



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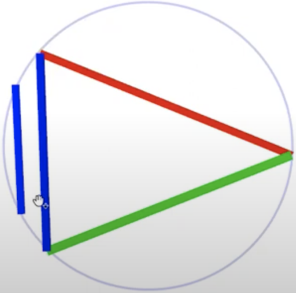

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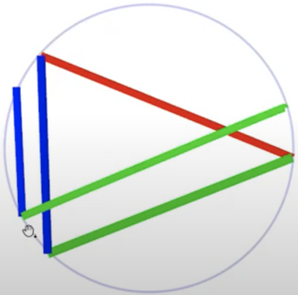

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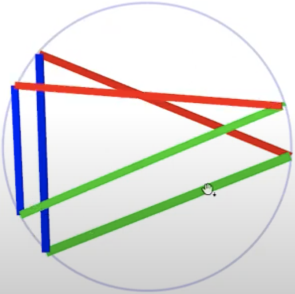

stacks vs queues: which do we know more about?

bounded stack number trees, bounded treewidth, planar, bounded genus, all proper minor closed, d-monotone bipartite graphs, **1-planar**

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bounded queue number trees, bounded treewidth, planar, bounded genus, all proper minor closed, d-monotone bipartite graphs, **k-planar**, **graph products***, and other non-minor closed families.

ubounded stack/queue number all graphs with $w(n)$ edges, some sparse expanders

ubounded stack number $P_n \boxtimes P_n \boxtimes P_n$

Open problem

1. do k-planar graphs have bounded stack-number? True for $k = 1$.
2. does $H \boxtimes P$ with H of bounded **treewidth** have bounded stack-number? If 2. is true, so is 1.

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   What is a good candidate?
   Problem: Graphs that we know to have bounded queue number also have bounded stack number.