Exact Matching in Graphs of Bounded Independence Number[1]

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Abstract

In the *Exact Matching Problem* (EM), we are given a graph equipped with a fixed coloring of its edges with two colors (red and blue), as well as a positive integer k. The task is then to decide whether the given graph contains a perfect matching exactly k of whose edges have color red. Despite EM being quite well-known, attempts to devise deterministic polynomial algorithms have remained illusive during the last 40 years and progress has been lacking even for very restrictive classes of input graphs. In this paper we finally push the frontier of positive results forward by proving that EM can be solved in deterministic polynomial time for input graphs of bounded independence number, and for bipartite input graphs of bounded bipartite independence number. This generalizes previous positive results for complete (bipartite) graphs which were the only known results for EM on dense graphs.

1 Introduction

In 1982, Papadimitriou and Yannakakis [12] studied a decision problem related to perfect matchings in edge-colored graphs as follows: Given as input a graph G whose edges come with a given fixed two-edge coloring (say, with colors red and blue), then the task is to decide whether for a given integer k there exists a perfect matching M of G such that exactly k of the edges in M are red. Only few years after its introduction, Mulmuley, Vazirani and Vazirani [11] showed that EM can be solved by a randomized polynomial time algorithm, i.e. it is contained in **RP**. This makes it unlikely to be **NP**-hard. In fact, deciding whether **RP**=**P** remains one of the big challenges in complexity theory. This means that problems such as EM, for which we know containment in **RP** but are not aware of deterministic polynomial time algorithms, are interesting candidates for testing the hypothesis **RP**=**P**. Indeed, due to this, EM is cited in several papers as an open problem. This includes recent breakthrough papers such as the seminal work on the parallel computation complexity of the matching problem [13], works on planarizing gadgets for perfect matchings [8], works on more general constrained matching problems [1, 10] and on multicriteria optimization problems [6] among others. Even though EM has caught the attention of many researchers from different areas, there seems to be a substantial lack of progress on the problem even when restricted to very special subclasses of input graphs as we will see next. This highlights the surprising difficulty of the problem given how simple it may seem at first glance.

1.1 Previous results for EM on restricted classes of graphs.

It may surprise some readers that EM is even non-trivial if the input graphs are complete or complete bipartite graphs: In fact, at least four different articles have appeared on resolving these two special cases of EM [9, 14, 5, 7], which are now known to be solvable in deterministic polynomial time. Another positive result follows from the existence of Pfaffian orientations and their analogues on planar graphs and $K_{3,3}$-minor free graphs [15], EM is solvable in polynomial time on these classes via a derandomization of the techniques used in [11]. Considering a generalization of

[1]This is a short version of [3]

Pfaffian orientations, it was further proved in [4] that EM can be solved in polynomial time for graphs embeddable on a surface of bounded genus. Finally, from the well-known meta-theorem of Courcelle [2], one easily obtains that EM can be efficiently solved on classes of bounded tree-width.

1.2 Our contribution.

In this paper, we generalize the known positive results for EM on very dense graphs such as complete and complete bipartite graphs to graphs of independence number at most α and to bipartite graphs of bipartite independence number[2] at most β, for all fixed integers $\alpha, \beta \geq 1$.

Theorem 1. *There is a deterministic algorithm for EM on graphs of independence number α running in time $n^{O(f(\alpha))}$, for $f(\alpha) = 2^{O(\alpha)}$.*

Theorem 2. *There is a deterministic algorithm for EM on bipartite graphs of bipartite independence number β running in time $n^{O(f(\beta))}$, for $f(\beta) = 2^{O(\beta)}$.*

For the full proofs we refer the reader to the full version of the paper [3]. In Section 2 we give an overview of the proof of Theorem 1.

1.3 Preliminaries

For a graph $G = (V, E)$ we let $n = |V(G)|$, i.e. the number of vertices in G. Given an instance of EM and a perfect matching (abbreviated PM) M, we define edge weights as follows: blue edges get weight 0, matching red edges get weight -1 and non-matching red edges get weight $+1$. For G' a subgraph of G, we define $R(G')$ (resp. $B(G')$) to be the set of red (resp. blue) edges in G', $r(G') := |R(G')|$ and $w(G')$ to be the sum of the weights of edges in G'.

2 Exact Matching on Bounded Independence Number Graphs

To prove Theorem 1 we develop an algorithm that relies on a 2 phase process. The first phase is an algorithm that outputs a PM M with $|k - r(M)|$ bounded (by a function of α), i.e. with a number of red edges that only differs from k by a function of α. This algorithm is also of independent interest since it provides a solution that is close to optimal (for small independence number) while its running time is polynomial and independent of the independence number.

Theorem 3. *Given a 'Yes' instance of EM, there exists a deterministic polynomial time algorithm that outputs a PM M with $k - 2 \cdot 4^{\alpha} \leq r(M) \leq k$.*

The second phase is an algorithm that outputs a solution matching with a running time that depends on the size of the smallest color class in a symmetric difference between a given matching and a solution matching. It is also of independent interest as it can be more generally useful for the study of other parameterizations of EM as well as other matching problems with color constraints.

Proposition 1. *Let M and M' be two PMs in G s.t. $|B(M \Delta M')| \leq L$ or $|R(M \Delta M')| \leq L$. Then there exists a deterministic algorithm running in time $n^{O(L)}$ such that given M it outputs a PM M'' with $r(M'') = r(M')$.*

[2]The *bipartite independence number* of a bipartite graph G equipped with a bipartition of its vertices is defined as the largest number β such that G contains a *balanced independent set* of size 2β, i.e., an independent set using exactly β vertices from both color classes.

For the correctness of this phase, we need to show that there exists a PM M^* with exactly k red edges, where $M \Delta M^*$ has a bounded number of edges of some color class. The main technical challenge is to show that for this to be the case it is sufficient to have $|k - r(M)|$ bounded (which is guaranteed by the first phase). In the rest of this section we give a quick overview of the proof, for more details we refer the reader to the full version of the paper [3].

2.1 Proof Overview

Skips. The main tool we use to reduce the size of a cycle C is something we call a skip (see Figure 1). At a high level, a skip is simply a pair of edges that creates a new cycle C' by replacing two paths of C. If those paths have total length more than 2 then $|C'| < |C|$. Now we also want to preserve the weight of the cycle so that the new target PM still has k red edges, so we look for skips that do not change the total weight (we call them 0 skips). It can happen, however, that even though no 0 skip exists, a collection of skips exists, that can be used independently, and their total weight change is zero (we call them 0 skip sets). Also observe that these skips can come from different cycles of $M \Delta M^*$ and still be used to reduced its total number of edges. So by taking $M \Delta M^*$ to be minimal, we are guaranteed that no such skip sets exist.

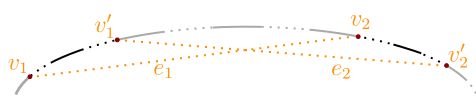

Figure 1: A skip formed by two non-matching edges e_1 and e_2 (in orange). Matching edges are represented by full lines and non-matching edges by dotted lines. The paths removed by the skip are depicted in black.

Skips from Paths. To find such skips, we rely on Ramsey theory to show that if we take a large enough (with respect to α) collection of disjoint paths from a cycle, starting and ending with non-matching edges, then they must form skips. Now if these paths had certain desired weights, then we could make sure that we get a 0 skip set as desired.

Paths from Edge Pairs. To prove the existence of paths of desired weight, we analyse the cycles in $M \Delta M^*$ by looking at their edge pairs, i.e. pairs of consecutive matching and non-matching edges. These edge pairs can have 3 configurations from which we can extract the paths. (1) Consecutive same sign pairs, (2) consecutive different sign pairs and (3) consecutive 0 pairs. We show that we can extract paths of the desired properties from all of these configurations, and the types of skips we get is dependent on the weights of the cycles and the sizes of their color classes.

Bounding the Cycle Weights. Next we show that if $M \Delta M^*$ is minimal, all of its cycles have bounded weight. This is mainly achieved by showing that cycles of large weight must have skips that reduce the weight. This changes the overall weight however, and must be compensated for either by skips on a cycle of the opposite weight, or by removing some of the cycles in $M \Delta M^*$.

Bounding one color class. With bounded weights, the number of cycles in $M \Delta M^*$ can also be bounded if their total weight is bounded. With these properties, we can show that if $M \Delta M^*$ has enough edges from both colors, then at least one of its cycles contains enough positive skips and one of its cycles contains enough negative skips, together forming a 0 skip set, i.e. it is not minimal. So choosing $M \Delta M^*$ minimal implies a bound on the size of one of its color classes.

3 Concluding remarks

In this paper we initiated the study of the parameterized complexity of EM by showing that it can be solved in deterministic polynomial time on graphs of bounded independence number and bipartite graphs of bounded bipartite independence number (i.e. we developed XP algorithms). This is an important step towards finding the right complexity class of the problem in general graphs as it generalizes the only previously known results on dense graph classes which were restricted to complete (bipartite) graphs.

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