Polynomial characterization of the set of r-splits of a graph

François Pitois — LIB, France, and LIRIS, UMR5205, France
Mohammed Haddad — Univ Lyon, UCBL, CNRS, LIRIS, UMR5205, F-69622 France
Hamida Seba — Univ Lyon, UCBL, CNRS, LIRIS, UMR5205, F-69622 France
Olivier Togni — Université de Bourgogne, LIB, France

Abstract

In this paper, we introduce r-splits, a generalization of graph splits. Splits are well-known graph cuts that allow representing a graph with a structure that can be described in linear space. We mainly prove that, analogously, r-splits can be described in polynomial space.

1 Introduction

Split-decomposition is a well-known graph decomposition [4] that decomposes a graph in linear time, using a structure called symmetric crossing family [1], which can be represented in linear space [3]. Hence, one entry point to decompose a graph via splits in linear time is the existence of a concise structure that represents every split. This paper aims at generalizing this concept. A split can be defined as a cut of rank at most 1. In a natural way, we introduce an r-split as a cut of rank at most r.

Definition 1. *Let $G = (V, E)$ be a graph. Let $(X, \overline{X})$ be a cut (i.e., a 2-partition of V). The rank of this cut, noted $\rho(X)$, is equal to the rank of the adjacency matrix of G where the rows are restricted to X and the columns are restricted to $\overline{X}$. The cut $(X, \overline{X})$ is an* r-split *if $\rho(X) \leq r$. Such a cut is said to be* trivial *when $\rho(X) = \min(|X|, |\overline{X}|)$.*

We focus on the set of non-trivial r-splits of a graph for a fixed r. Function ρ has several properties [2] which can be carried to r-splits. The most useful is *submodularity*, which is proved in [2] and stated as follows:

Lemma 1. *For all $X, Y \subseteq V$, we have $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.*

Oum also defined the notion of *r-rank connectivity*, as follows:

Definition 2. *[5] A graph G is* r-rank connected *if each k-split for $k < r$ is trivial.*

As a direct consequence, in an r-rank connected graph, if a set of vertices X has a cardinality satisfying $r \leq |X| \leq |V| - r$, then $\rho(X) \geq r$. Combined with submodularity, we get the following lemma:

Lemma 2. *If X and Y are two r-splits of an r-rank connected graph G, $|X \cap Y| \geq r$ and $|\overline{X \cup Y}| \geq r$, then $X \cap Y$ and $X \cup Y$ are both r-splits.*

Proof. Since X and Y are r-splits, we know that $\rho(X) \leq r$ and $\rho(Y) \leq r$. By submodularity, we deduce that $\rho(X \cup Y) + \rho(X \cap Y) \leq 2r$. The number of vertices of $X \cup Y$ and $X \cap Y$ is both between r and $|V| - r$. Hence, by r-rank connectivity of G, $\rho(X \cup Y) \geq r$ and $\rho(X \cap Y) \geq r$. All in all, $\rho(X \cup Y) = \rho(X \cap Y) = r$, which concludes the proof. □

This property is powerful as it organizes the set of all r-splits of a graph, as shown in the following theorem, which constitutes the main result of this paper:

Theorem 1. *Given an r-rank connected graph G with n vertices, and the set of all its r-splits, there exists a subset of r-splits of size $\mathcal{O}(n^{2r+1})$ that fully characterizes the whole set of r-splits.*

Next sections are dedicated to formalizing and proving this theorem.

2 Hypergraph of r-splits

Let $G = (V, E)$ be an r-rank connected graph with vertex set $V = [n]$. Let us denote by $H_r(G)$ the hypergraph whose set of vertices is the same as G, namely $V(H_r(G)) = [n]$, and whose set of hyperedges $\mathcal{E}$ is the set of all r-splits of G. From the previous section, we know that $H_r(G)$ satisfies the following properties: (1) if $A \in \mathcal{E}$, then $V \setminus A \in \mathcal{E}$; (2) for every set of vertices $X \subseteq V$, if $|X| \leq r$, then $X \in \mathcal{E}$; (3) if $A, B \in \mathcal{E}$, $|A \cap B| \geq r$ and $|\overline{A \cup B}| \geq r$, then $A \cap B \in \mathcal{E}$ and $A \cup B \in \mathcal{E}$. Therefore, we consider hypergraphs that satisfy these properties:

Definition 3. *Let $\mathbb{K}_r(n)$ be the class of hypergraphs with set of vertices $V = [n]$ and set of hyperedges $\mathcal{E}$ that satisfies:*

$\boldsymbol{K}_1$: *If $A \in \mathcal{E}$, then $V \setminus A \in \mathcal{E}$.*
$\boldsymbol{K}_2$: *For every set of vertices $X \subseteq V$, if $|X| \leq r$, then $X \in \mathcal{E}$.*
$\boldsymbol{K}_3$: *If $A, B \in \mathcal{E}$, $|A \cap B| \geq r$ and $|\overline{A \cup B}| \geq r$, then $A \cap B \in \mathcal{E}$ and $A \cup B \in \mathcal{E}$.*

We note that for each r-rank connected graph G of order n, the hypergraph $H_r(G)$ made of all r-splits of G belongs to the class $\mathbb{K}_r(n)$.

The class $\mathbb{K}_r(n)$ has the property of being a *closure system.* This means that: (1) the hypergraph with every possible hyperedge belongs to $\mathbb{K}_r(n)$; (2) if we take two hypergraphs $H_1, H_2 \in \mathbb{K}_r(n)$, then the intersection of those hypergraphs is also in $\mathbb{K}_r(n)$. We recall that the intersection of two hypergraphs H_1 and H_2 is the hypergraph whose vertex set is the same as H_1 and H_2 (namely, $[n]$), and whose hyperedge set is the intersection of the set of hyperedges of H_1 and H_2.

Lemma 3. *The class $\mathbb{K}_r(n)$ is a closure system.*

Proof. First, it is trivial that the hypergraph with all possible edges satisfies Definition 3, meaning that this hypergraph belong to $\mathbb{K}_r(n)$. Secondly, let $H_1, H_2 \in \mathbb{K}_r(n)$ and let us prove that $H_1 \cap H_2 \in \mathbb{K}_r(n)$. To this purpose, let A, B be hyperedges of $H_1 \cap H_2$ and let X be a subset of vertices of V, and let us prove that they satisfy properties $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3$ of Definition 3:

- For $\boldsymbol{K}_1$: As A is a hyperedge of $H_1 \cap H_2$, A is a hyperedge of H_1, and $V \setminus A$ is also a hyperedge of H_1 as $H_1 \in \mathbb{K}_r(n)$. For the same reason, $V \setminus A$ is a hyperedge of H_2. Hence, $V \setminus A$ is a hyperedge of $H_1 \cap H_2$.
- For $\boldsymbol{K}_2$: If $|X| \leq r$, then X is a hyperedge of H_1 as $H_1 \in \mathbb{K}_r(n)$, and X is a hyperedge of H_2 as $H_2 \in \mathbb{K}_r(n)$. Hence, X is a hyperedge of $H_1 \cap H_2$.
- For $\boldsymbol{K}_3$: If $|A \cap B| \geq r$ and $|\overline{A \cup B}| \geq r$, then $A \cap B$ and $A \cup B$ are hyperedges of H_1 as $H_1 \in \mathbb{K}_r(n)$. With the same argument, $A \cap B$ and $A \cup B$ are hyperedges of H_2. Hence, $A \cap B$ and $A \cup B$ are hyperedges of $H_1 \cap H_2$.

In conclusion, $H_1 \cap H_2$ fully satisfies Definition 3, proving that $H_1 \cap H_2 \in \mathbb{K}_r(n)$. $\square$

Having a closure system is convenient, as it induces a *closure operator*. In our case, the closure operator is defined as follows:

Definition 4. *Let H be a hypergraph with vertex set $V = [n]$. The* closure *of H in $\mathbb{K}_r(n)$, denoted $<H>_r$, is the hypergraph defined as the intersection of all hypergraphs that contain H and that belong to $\mathbb{K}_r(n)$. A hypergraph H satisfying $<H>_r = H$ is called a* closed hypergraph for $<\cdot>_r$.

In other words, A is a hyperedge of $<H>_r$ if and only if A is a hyperedge of every hypergraph that contains H and that belongs to $\mathbb{K}_r(n)$.

Just like any closure operator, $<\cdot>_r$ is *extensive* (for any hypergraph H, $H \subseteq <H>_r$), *monotone* (for any hypergraphs H, H', if $H \subseteq H'$, then $<H>_r \subseteq <H'>_r$), and *idempotent* (for any $H \in \mathbb{K}_r(n)$, we have $<H>_r = H$). We can now use $<\cdot>_r$ to formalize Theorem 1:

Formalization of Theorem 1. *Given an r-rank connected graph G with n vertices, there exists a hypergraph H with $\mathcal{O}(n^{2r+1})$ hyperedges such that $<H>_r = H_r(G)$.*

3 Proof of Theorem 1

To prove Theorem 1, we introduce the function Edg that takes two sets of vertices (V_+, V_-) of a hypergraph H, and that returns the smallest hyperedge (in term of inclusion) of that hypergraph that contains every vertex of V_+ and that avoids every vertex of V_-.

Lemma 4. *Let $H \in \mathbb{K}_r(G)$. Let V_+, V_- be two sets of vertices such that $V_+ \cap V_- = \emptyset$, $|V_+| \geq r$ and $|V_-| = r$. Let $\mathcal{A}$ be the set of all hyperedges A of H satisfying $V_+ \subseteq A \subseteq \overline{V_-}$. Then, the intersection of all hyperedges of $\mathcal{A}$ is also a hyperedge of H. We denote this hyperedge by* $\mathrm{Edg}_H(V_+|V_-)$.

Proof. First, let us check that this intersection is not empty, i.e., there exists a hyperedge A such that $V_+ \subseteq A \subseteq \overline{V_-}$. Let $A = \overline{V_-}$. We have $V_+ \subseteq A$ as $V_+ \cap V_- = \emptyset$. We claim that $\overline{V_-}$ is a hyperedge of H. Indeed, since $|V_-| \leq r$ and $H \in \mathbb{K}_r(n)$, by property $\boldsymbol{K}_2$, V_- is a hyperedge of H. Furthermore, by property $\boldsymbol{K}_1$, $\overline{V_-}$ is also a hyperedge of H.

Secondly, let us prove that if A and B are two hyperedges of $\mathcal{A}$ (i.e., they are hyperedges of H satisfying $V_+ \subseteq A \subseteq \overline{V_-}$ and $V_+ \subseteq B \subseteq \overline{V_-}$), then $A \cap B$ is in $\mathcal{A}$ as well. We know that $V_+ \subseteq A \cap B$, meaning that $|A \cap B| \geq |V_+| \geq r$; and $V_- \subseteq \overline{A \cup B}$, meaning that $|\overline{A \cup B}| \geq |V_-| = r$. By property $\boldsymbol{K}_3$, $A \cap B$ is a hyperedge of H. Furthermore, $A \cap B$ satisfies $V_+ \subseteq A \cap B \subseteq \overline{V_-}$. Hence, $A \cap B$ is a hyperedge of $\mathcal{A}$.

As $\mathcal{A}$ is made up of a finite number of hyperedges, and the intersection of any pair of hyperedges of $\mathcal{A}$ is also in $\mathcal{A}$, we deduce that the intersection of all hyperedges of $\mathcal{A}$ is a hyperedge of $\mathcal{A}$, and thus a hyperedge of H. $\square$

Before finally proving Theorem 1, let us first state a simple but useful lemma about a kind of closeness under union for hyperedges of a closed hypergraph (i.e., a hypergraph in $\mathbb{K}_r(n)$).

Lemma 5. *Let $H \in \mathbb{K}_r(n)$. Let V_- be a set of r vertices, and let $A_1, \ldots, A_k$ be k hyperedges of H. If every A_i satisfies $A_i \subseteq \overline{V_-}$, and if for every $1 \leq i < k$, we have $|A_i \cap A_{i+1}| \geq r$, then $\bigcup_{i=1}^{k} A_i$ is a hyperedge of H.*

Proof. We prove it by induction. For $k = 1$, the lemma is trivial. Suppose the lemma is true for k hyperedges and let us prove it for $k + 1$. Let $A' = A_1 \cup \ldots \cup A_k$. A' is a hyperedge of H by induction. It remains to prove that $A' \cup A_{k+1}$ is a hyperedge of H. Since $A_k \subseteq A'$, we have $A_k \cap A_{k+1} \subseteq A' \cap A_{k+1}$, meaning that $|A' \cap A_{k+1}| \geq |A_k \cap A_{k+1}| \geq r$. Besides, $A' \cup A_{k+1} \subseteq \overline{V_-}$, so $|\overline{A' \cup A_{k+1}}| \geq |V_-| \geq r$. Finally, by property $\boldsymbol{K}_3$, $A' \cup A_{k+1}$ is a hyperedge of H. $\square$

Now, given a hypergraph $H \in \mathbb{K}_r(n)$, let us build a hypergraph H' that has $\mathcal{O}(n^{2r+1})$ hyperedges, and such that $<H'>_r = H$.

Lemma 6. *Given $H \in \mathbb{K}_r(n)$, let H' be the sub-hypergraph of H whose hyperedge set is the set of all hyperedges $\mathrm{Edg}_H(V_+|V_-)$ for every $V_+, V_- \subseteq V = [n]$ such that the set of vertices V_+ and V_- are disjoint, $|V_+| = r+1$ and $|V_-| = r$. Then H' has $\mathcal{O}(n^{2r+1})$ hyperedges and $<H'>_r = H$.*

Proof. By construction, H' has a number of hyperedges at most equal to the number of hyperedges of the form $\mathrm{Edg}_H(V_+|V_-)$, with $|V_+| = r+1$ and $|V_-| = r$. This is equal to the number of ways of picking $r+1$ vertices from V to build V_+, multiplied by the number of ways of picking r vertices from V to build V_-, i.e.:

$$\binom{|V|}{r+1}\binom{|V|}{r} = \binom{n}{r+1}\binom{n}{r} = O(n^{2r+1}).$$

Now, let us prove that $<H'>_r = H$ by double inclusion.

Since $H' \subseteq H$ and since a closure operator is monotone, we have that $< H' >_r \subseteq < H >_r$. Since $H \in \mathbb{K}_r(n)$, and since a closure operator is idempotent, we have that $<H>_r = H$. Hence, $<H'>_r \subseteq H$.

Let A be a hyperedge of H. If $|A| \leq r$ or $|\overline{A}| \leq r$, A is a hyperedge of $<H'>_r$ by properties $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$. Now, suppose that $|A| > r$ and $|\overline{A}| > r$. Set V_- to be any subset of r vertices of $\overline{A}$, which is possible as $|\overline{A}| > r$. Let $k = |A|$, and let $A = \{u_1, \ldots, u_k\}$. For $1 \leq i \leq k-r$, let $V_i = \{u_i, \ldots, u_{i+r}\}$. Hence, each V_i is a set of vertices of size $r+1$, the union of all V_i's is A, and the intersection of V_i and V_{i+1} is of size r. Now, let $A_i = \mathrm{Edg}_{<H'>_r}(V_i|V_-)$, i.e., A_i is the smallest hyperedge of $<H'>_r$ that contains V_i and avoids V_-. By definition, $V_i \subseteq A_i \subseteq \overline{V}_-$, and A_i is a hyperedge of $<H'>_r$. Besides, $\bigcup_i V_i \subseteq \bigcup_i A_i$, i.e., A is included in the union of all A_i. Furthermore, by minimality of $\mathrm{Edg}_{<H'>_r}(V_i|V_-)$, $A_i \subseteq A$. Hence, the union of all A_i is exactly A. Using Lemma 5 on the hypergraph $<H'>_r \in \mathbb{K}_r(n)$, we have that A is a hyperedge of $<H'>_r$, concluding the proof that $H \subseteq <H'>_r$. $\square$

We can now prove Theorem 1. Let G be an r-rank connected graph with n vertices. Then $H_r(G) \in \mathbb{K}_r(n)$. Using Lemma 6, it exists H with $\mathcal{O}(n^{2r+1})$ hyperedges such that $<H>_r = H_r(G)$.

Acknowledgement: This work is supported by Agence Nationale de la Recherche (ANR-20-CE23-0002). We thank Michel Habib for his precious help. We also thank Mamadou M. Kanté for the hints that lead to the proof of Theorem 1. We thank the reviewers for their remarks and valuable feedback.

References

[1] W. Cunningham and J. Edmonds. A combinatorial decomposition theory. Canadian J. of Math, 1980.

[2] S. Oum and P. Seymour. Approximating clique-width and branch-width. Journal of Combinatorial Theory, Series B, 2006.

[3] B. Bui-Xuan, M. Habib and M. Rao. Tree-representation of set families and applications to combinatorial decompositions. European Journal of Combinatorics, 2012.

[4] E. Gioan and C. Paul Split decomposition and graph-labelled trees: characterizations and fully dynamic algorithms for totally decomposable graphs. Discrete Applied Mathematics, 2012.

[5] S. Oum. Rank connectivity and pivot-minors of graphs. Preprint arXiv:2011.03205v2, 2020.