Dominating and Independent Dominating Sets in Goldberg Snarks

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Abstract

A set S of vertices of a graph G is said to be a dominating set if every vertex of G either belongs to S or is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. An independent dominating set in a graph is a set that is both dominating and independent. The minimum cardinality of an independent dominating set $i(G)$ of G is the independent domination number of G. We show that, for every Goldberg Snark G_l, $l \geq 3$ and l odd, it holds that $\gamma(G_l) = i(G_l) = \lceil \frac{11l}{5} \rceil$.

1 Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d(v)$ the *degree* of $v \in V(G)$ and by $\delta(G)$ the *minimum degree* of G. A graph G is *k-regular* if $d(v) = k$ for every $v \in V(G)$. A graph is *cubic* if it is 3-regular. For $v \in V(G)$, the *closed neighbourhood* of v is $N[v] = \{u \in V(G) : uv \in E(G) \text{ and } u \neq v\} \cup \{v\}$. Extending these concepts to a set $S \subseteq V(G)$, we define $N[S] = \cup_{v \in S} N[v]$. A *dominating set* of G is a set $S \subseteq V(G)$ such that every $v \in V(G)$ is either in S or is adjacent to some vertex in S; we say that S *dominates* G and $u \in S$ *dominates* its adjacent vertices. We also say that a set $S' \subseteq S$ *dominates* a set $H \subseteq V(G)$ if $H \subseteq N[S']$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A set $S \subseteq V(G)$ is *independent* if its elements are pairwise nonadjacent. An *independent dominating set* of G is both dominating and independent. The *independent domination number* $i(G)$ of G is the minimum cardinality of an independent dominating set of G.

In 1979, Garey and Johnson [4] showed that determining both $\gamma(G)$ and $i(G)$ for arbitrary graphs are NP-hard problems and remain NP-hard even when restricted to cubic graphs [3, 7]. These results motivate the search for bounds for $\gamma(G)$ and $i(G)$ and, over the years, many results have been obtained considering classes of graphs. Let $n = |V(G)|$. Ore [8] proved that $\gamma(G) \leq \frac{n}{2}$ for graphs with $\delta(G) \geq 1$. Later on, Blank [2] proved that, except for seven graphs, every connected graph with $\delta(G) \geq 2$ has $\gamma(G) \leq \frac{2n}{5}$. Also, Reed [10] proved that $\gamma(G) \leq \frac{3n}{8}$ for connected graphs with $\delta(G) \geq 3$. All these bounds are tight. Reed [10] also conjectured that every connected cubic graph has $\gamma(G) \leq \lceil \frac{n}{3} \rceil$. However, Kostochka and Stodolsky [6] showed that this conjecture is false by constructing an infinite family of connected cubic graphs for which $\gamma(G) > \lceil \frac{n}{3} \rceil$. In fact, Reed's Conjecture remains false even for 2-connected cubic graphs [11]. Nevertheless, the search for classes of cubic graphs that verify or improve Reed's Conjecture is still a quite challenging problem.

Another interesting problem is establishing the relationship between $\gamma(G)$ and $i(G)$. Note that, by the definition, $\gamma(G) \leq i(G)$. Furthermore, deciding whether $\gamma(G) = i(G)$ is an NP-complete problem [1]. Although no necessary and sufficient conditions are known for the characterization of graphs having $\gamma(G) = i(G)$, there are some families of graphs that achieve this equality [5]. On the other hand, there exist infinite families of graphs for which the difference $i(G) - \gamma(G)$ is unbounded even when restricted to connected cubic graphs [12].

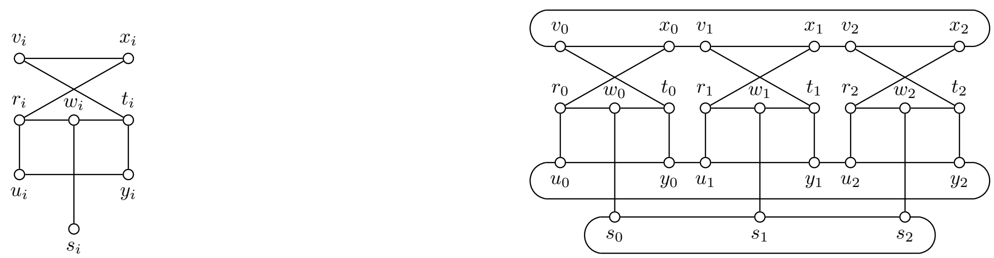
All these considerations motivate the search for cubic graphs with domination number bounded by the value conjectured by Reed, as well as the determination of the independent domination number of these graphs, so as to evaluate how far apart these two parameters are.

Snarks are connected cubic graphs without *bridges* (edges whose removal increases their number of connected components), which are not 3-edge-colourable, i.e., whose edges cannot be assigned three colours such that adjacent edges have distinct colours. Snarks are an important class of cubic graphs that played an essential role in the proof of the Four-Colour Theorem and that have been shown to be interesting when approaching other problems in Graph Theory ever since.

In this work, we determine the domination number and the independent domination number of an infinite family of snarks known as Goldberg Snarks. The domination problem seems unexplored for these graphs, with the known results focusing on variants of the classical problem. We show that the domination number and the independent domination number are equal for every graph in this family and remark that its domination number is less than the value conjectured by Reed.

2 Main Results

Let $\mathcal{G} = \{G_3, G_5, G_7, \ldots\}$ be the family of *Goldberg Snarks*. Each $G_l \in \mathcal{G}$ is formed by l copies of a block B with $V(B) = \{r, s, t, u, v, w, x, y\}$ and $E(B) = \{ru, rw, rx, sw, tv, tw, ty, uy, vx\}$; denote the i-th copy of B by B_i and attach index i to its vertices as in Figure 1(a). For each $l \geq 3$, l odd, graph G_l is built from $B_0, B_1, ..., B_{l-1}$ by adding edges $x_i v_{i+1}$, $y_i u_{i+1}$ and $s_i s_{i+1}$ for every $0 \leq i < l$, with indexes taken module l, as exemplified in Figure 1(b). In Theorem 1, we establish upper bounds for $\gamma(G_l)$ and $i(G)$.



(a) Block B_i.

(b) Graph G_3.

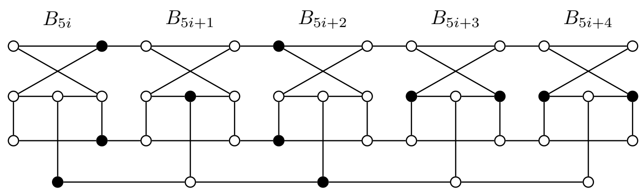
Figure 1: Block B_i and graph G_3, constructed from B_0, B_1 and B_2.

Theorem 1. *For Goldberg Snark G_l, $\gamma(G_l) \leq i(G_l) \leq \lceil \frac{11l}{5} \rceil$.*

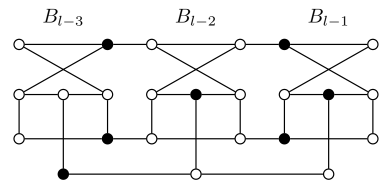
Proof (sketch). We construct an independent dominating set S for $G = G_l$, with $|S| \leq \lceil \frac{11l}{5} \rceil$ to prove that $\gamma(G) \leq i(G) \leq \lceil \frac{11l}{5} \rceil$. Let $l = 5t + r$, $t \geq 0$ and $r \in \{0, 1, 2, 3, 4\}$. Let $H_0, H_1, \ldots, H_t$ be subgraphs of G such that: $H_i = G[\cup_{j=0}^{4} V(B_{5i+j})]$ if $0 \leq i < t$; $H_t = G[\cup_{j=0}^{r-1} V(B_{l-r+j})]$ if $r \neq 0$. Note that $|V(H_i)| = 40$ for $0 \leq i < t$ and $|V(H_t)| = 8r$ if $r \neq 0$. Figure 2 exhibits subgraphs H_i and H_t. We construct an independent dominating set S for G where $S = \cup_{i=0}^{t} S_i$ such that, for $0 \leq i < t$,

$$S_i = \begin{cases} \{w_{5i}, x_{5i}, y_{5i}, w_{5i+1}, s_{5i+2}, u_{5i+2}, v_{5i+2}, r_{5i+3}, t_{5i+3}, r_{5i+4}, t_{5i+4}\} & \text{if } i = 0 \text{ and } r = 1; \\ \{s_{5i}, x_{5i}, y_{5i}, w_{5i+1}, s_{5i+2}, u_{5i+2}, v_{5i+2}, r_{5i+3}, t_{5i+3}, r_{5i+4}, t_{5i+4}\} & \text{otherwise; and} \end{cases}$$

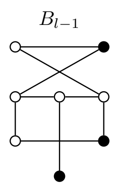
$$S_t = \begin{cases} \emptyset & \text{if } r = 0; \\ \{s_{l-1}, x_{l-1}, y_{l-1}\} & \text{if } r = 1; \\ \{s_{l-2}, x_{l-2}, y_{l-2}, r_{l-1}, t_{l-1}\} & \text{if } r = 2; \\ \{s_{l-3}, x_{l-3}, y_{l-3}, w_{l-2}, u_{l-1}, v_{l-1}, w_{l-1}\} & \text{if } r = 3; \\ \{s_{l-4}, x_{l-4}, y_{l-4}, w_{l-3}, s_{l-2}, u_{l-2}, v_{l-2}, r_{l-1}, t_{l-1}\} & \text{if } r = 4. \end{cases}$$



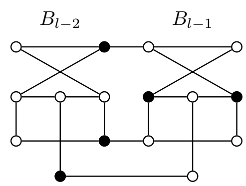
(a) Subgraph H_i, $i \neq 0$ or $r \neq 1$.



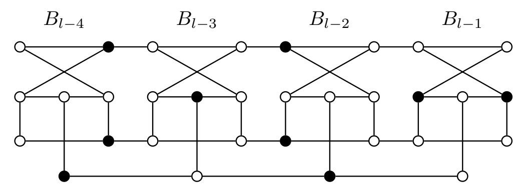
(b) Subgraph H_t, $r = 3$.



(c) Subgraph H_t, $r = 1$.



(d) Subgraph H_t, $r = 2$.



(e) Subgraph H_t, $r = 4$.

Figure 2: The dark vertices belong to S.

By construction, $|S_i| = 11$ when $0 \leq i < t$ and $|S_t| = 2r+1$ when $r \neq 0$. Then, $\gamma(G) \leq 11t+2r+1$ when $r \neq 0$ and $\gamma(G) \leq 11t$ when $r = 0$. In both cases, $\gamma(G) \leq \lceil \frac{11l}{5} \rceil$, which concludes the proof. $\square$

In Theorem 3, we show that the upper bounds previously established for $\gamma(G_l)$ and $i(G_l)$ are also lower bounds. The proof is based on Theorem 2 and Lemma 1. Theorem 2 shows that Theorem 1 is tight for G_l with $l \in \{3, 5, 7, 9, 11, 13\}$ and is crucial for Lemma 1, which is the key for Theorem 3. The complete proofs of these results can be found in A. A. Pereira Master's thesis [9].

Theorem 2. *For Goldberg Snark G_l, $l \in \{3, 5, 7, 9, 11, 13\}$, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$.* $\square$

Lemma 1. *For Goldberg Snark G_l, $l \geq 13$, $\gamma(G_{l-10}) \leq \gamma(G_l) - 22$.*

Proof (sketch). Let $H = G_l[\cup_{i=1}^{10} V(B_i)]$ and $\overline{H} = G_l[V(G_l) \setminus V(H)]$. Note that $\overline{H}$ is isomorphic to a subgraph of G_l. Let S be a minimum dominating set of G_l. Then, set $S_{\overline{H}} = S \setminus V(H)$ may be or may not be a dominating set of G_{l-10}. The core of the proof consists of building a dominating set S' for G_{l-10} from $S_{\overline{H}}$ such that $|S'| \leq |S| - 22$. Then, we show that $\gamma(G_{l-10}) \leq |S'| \leq |S| - 22 = \gamma(G_l) - 22$. In order to do this, we show $|S \cap V(H)| > 20$ and that $S' = S_{\overline{H}} \cup \{s_{11}, u_{11}, v_{11}\}$ is a dominating set of G_{l-10}. Thus, if $|S \cap V(H)| \geq 25$, $|S'| \leq |S| - 22$. Then, we prove the result for the remaining cases $|S \cap V(H)| \in \{21, 22, 23, 24\}$. $\square$

Theorem 3. *For Goldberg Snark G_l, $\gamma(G_l) = i(G_l) = \lceil \frac{11l}{5} \rceil$.*

Proof. By Theorem 1, $\gamma(G_l) \leq \lceil \frac{11l}{5} \rceil$ and, by Theorem 2, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$ for $l \in \{3, 5, 7, 9, 11, 13\}$. We prove that $\gamma(G_l) \geq \lceil \frac{11l}{5} \rceil$ for $l \geq 15$. Let $\mathcal{F} = \{G_l : \gamma(G_l) < \lceil \frac{11l}{5} \rceil\}$. Suppose $\mathcal{F} \neq \emptyset$. Let $G_l \in \mathcal{F}$

be the graph with minimum $|V(G_l)|$. Consider graph G_{l-10}. By Lemma 1, $\gamma(G_{l-10}) \leq \gamma(G_l) - 22 < \lceil \frac{11l}{5} \rceil - 22 < \lceil \frac{11(l-10)}{5} \rceil$. Hence, $G_{l-10} \in \mathcal{F}$. However, $|V(G_{l-10})| < |V(G_l)|$, which contradicts the choice of G_l. Therefore, $\mathcal{F} = \emptyset$. Then, $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$. In order to complete the proof, we show that $i(G_l) = \lceil \frac{11l}{5} \rceil$. Since $\gamma(G_l) = \lceil \frac{11l}{5} \rceil$ and $\gamma(G_l) \leq i(G_l)$, we conclude that $i(G_l) = \lceil \frac{11l}{5} \rceil$. $\square$

3 Concluding Remarks

In this work, we determined the domination and the independent domination numbers of Goldberg Snarks, which contribute to a deeper understanding of the domination problem in cubic graphs, in particular in view of the bound proposed by Reed [10]. Set S constructed in Theorem 1 is also an independent dominating set of *Twisted Goldberg Snarks*, TG_l, defined from G_l by replacing edges x_0v_1 and y_0u_1 by edges x_0u_1 and y_0v_1. Observe that $N[x_0] \cap N[y_0] \cap S = N[u_1] \cap N[v_1] \cap S = \{x_0, y_0\}$ regardless of the edges that link the pairs x_0, y_0 and u_1, v_1. Therefore, S is also independent dominating for TG_l and $\gamma(TG_l) \leq i(TG_l) \leq \lceil \frac{11l}{5} \rceil$. The complete proof of the lower bound for $\gamma(G_l)$ is extensive and very technical. We believe the same key ideas can be used to prove the lower bound for $\gamma(TG_l)$.

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