Making a tournament k-strong by adding new arcs

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Abstract

A **semicomplete digraph** is a digraph with no pair of non-adjacent vertices and a **tournament** is a semicomplete digraph with no cycle of length 2. A digraph is **k-strong** if it has $n \geq k+1$ vertices and every induced subdigraph on at least $n-k+1$ vertices is strongly connected.

We have proved that every semicomplete digraph on at least $k+1$ vertices can be made k-strong by adding at most $\binom{k+1}{2}$ new arcs. This confirms a conjecture of Bang-Jensen from 1994.

Combined with other work, our results also imply that every tournament on at least $3k-1$ vertices can be made k-strong by reversing at most $\binom{k+1}{2}$ arcs. This provides new support for the conjecture, due to Bang-Jensen, that we can make any tournament on at least $2k+1$ vertices k-strong by reversing at most $\binom{k+1}{2}$ arcs.

Keywords: Tournament; Semicomplete digraph; One-way pairs; Directed vertex-connectivity augmentation; Arc-reversal.

1 Introduction

The notation used below follows [1]. For a given digraph D on at least $k+1$ vertices we denote by $a_k(D)$ the minimum number of new arcs that must be added to D to obtain a k-strong digraph. For an arbitrary digraph on $n \geq k+1$ vertices which is not k-strong we may have to add nk new arcs to obtain a k-strong digraph. One example attaining this bound is the arc-less digraph $D_{n,0}$ on $n \geq k+1$ vertices which needs nk new arcs. We have $a_k(D_{n,0}) = nk$ as we can label the vertices of $D_{n,0}$ as $\{0, 1, 2, \ldots, n-1\}$ and add all arcs from vertex i to the vertices $i+1, i+2, \ldots i+k$ modulo n. This digraph, denoted C_n^k (also known as the k'th power of an n-cycle) is k-strong [1, Exercise 5.10]. Since any superdigraph of a k-strong digraph is again k-strong, this shows that $a_k(D) \leq nk$ for every digraph on $n \geq k+1$ vertices.

Let us first observe that $a_k(D)$ is bounded by some function of k for every tournament D on $n \geq k+1$ vertices. We will just sketch the argument here. We use the following two facts (see e.g. [2]):

(i) every tournament on at least $4k-2$ vertices has a vertex v with both in-degree and out-degree at least k

(ii) If $H = (V, A)$ is a k-strong digraph and we add a new vertex v along with an arc from v to each of k distinct vertices of V and from a set of k distinct vertices of V to v, then the resulting digraph is k-strong (see [1, Exercise 14.8])

These facts imply that for every tournament D on more than $4k-2$ vertices we have $a_k(D) \leq a_k(D')$ for some subtournament D' of D on at most $4k-2$ vertices. For such a tournament we clearly have that $a_k(D') \leq \binom{4k-2}{2}$ and hence $a_k(D')$ is bounded by a function of k.

The transitive tournament TT_n on n vertices has vertex set $\{1,2,\ldots,n\}$ and arc set $\{ij : 1 \leq i < j \leq n\}$. It is easy to check that $a_k(TT_n) \geq \binom{k+1}{2}$ as we need this many arcs just to get in- and out-degree at least k for every vertex. In fact we have equality above so it is a natural question how large $a_k(D)$ can become when D is a tournament.

Observations like this made the first author conjecture in 1994 that $a_k(D)$ is at most $\binom{k+1}{2}$ for every tournament D and hence also for every semicomplete digraph on at least $k+1$ vertices. The purpose of this talk is to give a detailed sketch of a short proof of this conjecture.

Theorem 1. *For every semicomplete digraph D on at least $k+1$ vertices and every positive integer k we have $a_k(D) \leq \binom{k+1}{2}$.*

2 The tool: one-way pairs

Frank and Jordán [5] solved the problem of characterizing $a_k(D)$ for a given digraph D and also gave a polynomial algorithm to find a set of arcs A' of minimum cardinality to add to a given digraph $D = (V, A)$ so that the resulting digraph $\hat{D} = (V, A \cup A')$ is k-strong. The tool they used is described below.

Let H, T be disjoint non-empty proper subsets of V. The ordered pair (H, T) is a **one-way pair** in $D = (V, A)$ if D has no arc with tail in T and head in H. This definition is due to Frank and Jordán [5] but we use a sligthly different notation here. For such a pair (H, T) we refer to H (T) as the **head** (**tail**) of the pair. Let $h(H, T) = |V - H - T|$. The **deficiency** $\eta(H, T)$ of a one-way pair (H, T) with respect to k-strong connectivity is defined as

$$\eta(H, T) = \max\{0, k - h(H, T)\}. \tag{1}$$

For instance, if $N^+[X] \neq V$ then the pair $(X, V - N^+[X])$ is a one-way pair with deficiency $\eta(X, V - N^+[X]) = \max\{0, k - |N^+(X)|\}$. Here $N^+[X]$ is the closed out-neighbourhood of the set $X \subset V$. One-way pairs are closely related to k-strong connectivity.

Lemma 1. *[5] A digraph $D = (V, A)$ is k-strong if and only if we have $h(H, T) \geq k$ for every one-way pair (H, T) in D.*

Two one-way pairs $(H_1, T_1), (H_2, T_2)$ are **independent** if either their heads or their tails are disjoint. By Lemma 1, in order to obtain a k-strong superdigraph of D, we must add enough new arcs to cover all one-way pairs with $\eta(H, T) > 0$: we must add at least $\eta(H, T)$ arcs from T to H). This is the reason why H (T) is called the head (tail) of the one-way pair (H, T). Clearly, if $(H_1, T_1), (H_2, T_2)$ are independent one-way pairs, then no new edge can decrease both $\eta(H_1, T_1)$ and $\eta(H_2, T_2)$. This implies that, if $\mathcal{F}$ is any family of pairwise independent one-way pairs in D, then we must add at least

$$\eta(\mathcal{F}) = \sum_{(H,T)\in\mathcal{F}} \eta(H, T) \tag{2}$$

new arcs to D in order to obtain a k-strong digraph. We call the number $\eta(\mathcal{F})$ the **deficiency** of $\mathcal{F}$.

The following theorem, due to Frank and Jordán, shows that the maximum deficiency over families of independent one-way pairs gives the right lower bound for the vertex-strong connectivity augmentation problem.

Theorem 2 (Frank and Jordán)**.** *[5] For every digraph D on at least $k+1$ vertices we have*

$$a_k(D) = \max\{\eta(\mathcal{F}) : \mathcal{F} \text{ is a family of independent one-way pairs in } D\}. \tag{3}$$

Frank and Jordán also gave a polynomial algorithm for finding an optimal augmentation (set of new arcs to add) for any given input digraph D.

Theorem 3. *[5] There exists a polynomial algorithm which, given a digraph $D = (V, A)$ and a natural number k, finds a minimum cardinality set F of new arcs to add to D so that the resulting digraph is k-strong.*

3 Short sketch of the proof of Theorem 1

We will use the notation (H, S, T) to denote a one-way pair. Here H, T are as above and $S = V - H - T$ so $\eta(H, S, T) = \max\{0, k - |S|\}$. For a one-way pair (H, S, T) with $|H| = 1$ ($|T| = 1$) we call H (T) a **singleton head** (**tail**) and say that (H, S, T) is a **singleton** one-way pair.

We consider a tournament D such that $a_k(D)$ is maximum and for all tournaments D' with $a_k(D') = a_k(D)$ we have $n = |V(D)| \leq |V(D')|$. We then prove that $n = k+1$ from which the theorem follows as we can make any tournament D on $k+1$ vertices k-strong by adding all the arcs of the converse of D and this has $\binom{k+1}{2}$ arcs. Here we used the easy fact that the complete digraph on $k+1$ vertices is k-strong.

Assume below that $n \geq k+2$ and let $\mathcal{F} = \{(H_1, S_1, T_1), \ldots, (H_p, S_p, T_p)\}$ be an independent family of one-way pairs of D achieving the value $a_k(D)$, that is, by Theorem 2, $\eta(\mathcal{F}) = a_k(D)$.

Claim : Every vertex of D is either a singleton head or a singleton tail of some one-way pair in $\mathcal{F}$

Proof of Claim: Suppose x is neither a singleton head nor a singleton tail. Then we consider the tournament $D' = D - x$ and the family $\mathcal{F}{-}x = \{(H'_1, S'_1, T'_1), \ldots, (H'_p, S'_p, T'_p)\}$ of one-ways pair in D', where $H'_i = H_i - x$, $S'_i = S_i - x$ and $T'_i = T_i - x$ (precisely one of H_i, S_i, T_i contains x). As x is neither a singleton head nor a singleton tail, each (H'_i, S'_i, T'_i) is a one-way pair in D' and $\eta(\mathcal{F}') \geq \eta(\mathcal{F})$ with strict in-equality if x belongs to at least one S_i, $i \in [p]$. This contradicts the choice of D. $\diamond$

By the choice of D, whenever we remove a vertex x from D the resulting tournament $D - x$ will have $a_k(D - x) < a_k(D)$.

Now the rest of the proof (not to be revealed here as the paper is under review) consists of a careful analysis of what happens to the deficiency of D when we remove a vertex x, in particular how removing x affects the deficiency of the family $\mathcal{F}$.

4 Reversing arcs to achieve high connectivity

For a given digraph D let $r_k(D)$ denote the minimum number of arcs one needs to reverse to obtain a k-strong reorientation of D. If no such reversal exists we set $r_k(D) = \infty$. Already for $k = 2$ it is an open problem whether $r_k(D)$ can be determined efficiently. For $k = 1$ the problem can be solved in polynomial time [4] (see also [1, Section 13.1]).

Since every tournament T on n vertices has a hamiltonian path $v_1v_2\dots v_n$ it is easy to see that $r_1(T) \leq 1$, because either v_nv_1 is an arc and T is already strong or v_1v_n is an arc and we can reverse that arc to obtain a strong tournament.

Using the fact that C_n^k is k-strong as well as (i) and (ii) as we did when we studied the function $a_k()$ it is easy to check that for every tournament T on at least $2k+1$ vertices we have $r_k(T) \leq \binom{4k-2}{2}/2$ so r_k is bounded by a function of k for every tournament T on at least $2k+1$ vertices.

Since adding a copy of an existing arc uv cannot increase the vertex connectivity it is easy to see that no optimal reversal will reverse an arc of a directed 2-cycle. This implies that we have $r_k(D) \geq a_k(D)$ for every digraph. Our earlier arguments for TT_n imply that when $n \geq 2k+1$ we have $r_k(TT_n) = \binom{k+1}{2}$. This made the first author pose the following stronger version of Theorem 1.

Conjecture 1 (Bang-Jensen, 1994)**.** *For every tournament T on at least $2k+1$ vertices we have $r_k(T) \leq \binom{k+1}{2}$.*

The proof of the following result also relies on a careful study of one-way pairs.

Theorem 4. *[2] For every semicomplete digraph on $n \geq 3k-1$ vertices we have $a_k(D) = r_k(D)$. There exists a semicomplete digraph D on $3k-2$ vertices for which we have $a_k(D) < r_k(D)$.*

Combining this with Theorem 1 we see that Conjecture 1 holds when $n \geq 3k-1$.

Corollary 1. *For every semicomplete digraph D on $n \geq 3k-1$ vertices we have $r_k(D) \leq \binom{k+1}{2}$.*

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