Factorially many maximum matchings close to the Erdős-Gallai bound

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Abstract

A classical result of Erdős and Gallai determines the maximum size $m(n,\nu)$ of a graph G of order n and matching number νn. We show that G has factorially many maximum matchings provided that its size is sufficiently close to $m(n,\nu)$.

We consider finite, simple, and undirected graphs. A *matching* in a graph G is a set of pairwise disjoint edges, and the *matching number* $\nu(G)$ of G is the largest size of a matching in G. For a matching M in G, let $V(M)$ be the set of vertices of G that are incident with an edge in M; the set $V(M)$ contains the vertices of G that are *saturated* by M.

A classical result of Erdős and Gallai, Theorem 4.1 in [4], states that a graph G of order n, size m, and matching number $\nu(G)$ such that $\nu(G)=\nu n$ for some $\nu\in\left[0,\frac{1}{2}\right]$ satisfies

$$m \quad\leq\quad m(n,\nu):=\begin{cases}\nu n(n-\nu n)+\binom{\nu n}{2} & \text{, if } \nu\leq\frac{2}{5}-\frac{3}{5n}\text{, and}\\ \binom{2\nu n+1}{2} & \text{, if } \frac{2}{5}-\frac{3}{5n}\leq\nu\leq\frac{1}{2}.\end{cases}\tag{1}$$

Furthermore, they showed that equality holds in (1) if and only if

(i) the complement $\overline{G}$ of G is $K_{n-\nu n}\cup\overline{K_{\nu n}}$ for $\nu\leq\frac{2}{5}-\frac{3}{5n}$, and

(ii) G is $K_{2\nu n+1}\cup\overline{K_{n-2\nu n-1}}$ for $\frac{2}{5}-\frac{3}{5n}\leq\nu\leq\frac{1}{2}$.

Recall that, for positive integers n and k with $k\leq n$, the *falling factorial* $n^{\underline{k}}$ is $n(n-1)\dots(n-k+1)$.

The starting point here was the observation that the two extremal graphs in (i) and (ii) have

$$(n-\nu n)^{\underline{\nu n}}\qquad\text{and}\qquad\frac{(2\nu n+1)!}{(\nu n)!2^{\nu n}}$$

maximum matchings, respectively. Estimating quite roughly, it follows that, for positive ν, the extremal graphs for (1) have between $\lceil 0.4n\rceil^{\underline{\lceil 0.5\nu n\rceil}}$ and $n^{\underline{\nu n}}$ maximum matchings. We show that G still has factorially many maximum matchings provided that $m(n,\nu)-m$ is sufficiently small. Since $m(n,\nu)=\Theta(\nu n^2)$, it is natural to bound $m(n,\nu)-m$ in terms of ν and n^2.

The following is our first main result; all proofs can be found in [1].

Theorem 1. *For every real ν with $\nu\in\left(0,\frac{1}{2}\right]$, the following holds: If G is a graph of order n, size m, and matching number νn such that $\left(\frac{\nu}{50}\right)^2 n\geq 1$ and $m\geq m(n,\nu)-\left(\frac{\nu}{50}\right)^2 n^2$, then G has at least $\lceil 0.1n\rceil^{\underline{\lceil 0.1\nu n\rceil}}$ maximum matchings.*

For the sake of simplicity, we did not try to optimize the constants that appear in this statement, which works over the full range $\left(0,\frac{1}{2}\right]$ of ν. Our purpose here is rather to illustrate the effect and present arguments and tools that allow to capture it. In particular, the exact dependence of the minimum number of maximum matchings on the difference $m(n,\nu)-m$ remains a natural yet challenging open problem.

Our second main result gives a better bound provided that ν is sufficiently small.

Theorem 2. *There are two functions $h_\nu : (0,1) \to \left(0, \frac{1}{2}\right]$ and $h_\delta : (0,1) \times \left(0, \frac{1}{2}\right] \to (0,1)$ with the following property: If $\epsilon \in (0,1)$, $\nu \in (0, h_\nu(\epsilon))$, and G is a graph of order n, size m, and matching number νn such that $h_\delta(\epsilon,\nu)n \geq 1$ and $m \geq m(n,\nu) - h_\delta(\epsilon,\nu)n^2$, then G has at least $\lceil (1-\epsilon)n \rceil^{\underline{\lceil (1-\epsilon)\nu n \rceil}}$ maximum matchings.*

Matchings in graphs are among the most well studied topics in graph theory [8], and we would like to mention only few related results. Computing the permanent of a matrix, and, hence, counting the perfect matchings of a given bipartite graph, is a well known #P-complete problem [10]. Van der Waerden's proved conjecture on the permanent of a doubly stochastic matrix [3, 6, 7, 9] allows to show that d-regular bipartite graphs have exponentially many perfect matchings for $d \geq 3$, and Brègman's [2] upper bound on the permanent allows to derive an exponential upper bound. Another famous related result, establishing a conjecture of Lovász and Plummer, is due to Esperet, Kardoš, King, Král, and Norine [5] who showed that cubic bridgeless graphs have exponentially many perfect matchings.

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