Long induced paths in graphs with bounded treewidth

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Abstract

Esperet, Lemoine and Maffray conjectured in 2017 that every k-degenerate graph that has a path on n vertices also has an induced path on at least $(\log n)^{\Omega_k(1)}$ vertices.

In this talk we prove that every graph of pathwidth less than k with an n-path has an induced path of order at least $\frac{1}{3}n^{1/k}$. This is an exponential improvement and a generalization of the previous polylogarithmic lower bounds for interval graphs of bounded clique number. This result is complemented with an upper bound. We use this result to prove the conjecture for graphs of treewidth less than k: their longest induced path has order at least $\frac{1}{4}(\log n)^{1/k}$.

1 Introduction

In this talk we study the relationship between the existence of a long path and of a long induced path in a graph. Observe for example that a biclique has a Hamiltonian path, but the longest induced path in a biclique has at most 3 vertices.

Let $\mathcal{G}$ be a class of graph, we define $L(\mathcal{G}, n)$ as the maximum integer ℓ such that for every graph $G \in \mathcal{G}$ with a path on n vertices, G has an induced path on ℓ vertices.

From the previous example, when $\mathcal{B}$ is the class of bicliques, for any $n \geq 3$, $L(\mathcal{B}, n) = 3$.

In 1982, Galvin, Rival and Sands [3] showed that as soon as the graph excludes a biclique as subgraph, the relationship between path and induced subgraph is much more interesting. They proved that when $\mathcal{G}_t$ is the family of graphs excluding $K_{t,t}$ as subgraph, then $L(\mathcal{G}_t, n)$ is unbounded. However, the lower bound that they provide (relying on the infinite Ramsey's theorem for 4-uples) increases extremely slowly and can essentially only be used as an existential theorem.

Note that if $\mathcal{G}'$ is a subclass of $\mathcal{G}$, then $L(\mathcal{G}, n) \leq L(\mathcal{G}', n)$, thus their result motivates the search of more accurate bounds for some subclasses of graphs excluding a $K_{t,t}$.

An interesting subclass is the class of k-degenerate graphs (*i.e.* graphs where every subgraph has a vertex of degree at most k), which are graphs excluding $K_{k+1,k+1}$ as subgraph. Nešetřil and Ossona de Mendez [4] proved in 2012 that if G is k-degenerate and has a path of order n then G has an induced path of order at least $\frac{\log \log n}{\log(k+1)}$.

On the negative side, Arocha and Valencia [1] proved in 2000 that there are outerplanar graphs (which form a subclass of 2-degenerate graphs) with a path of size n but the longest induced path is at most logarithmic in n. This motivates the search of a polylogarithmic bound for classes of degenerate graphs. This was conjectured in 2017 by Esperet, Lemoine and Maffray [2].

Conjecture 1 ([2, Conjecture 1.1])**.** *For every integer k, there is a constant d such that every k-degenerate graph that has a path of order n also has an induced path of order at least* $(\log n)^d$.

This conjecture is open even when $k = 2$. In the same paper, they showed a lower bound in $\Omega(\sqrt{\log n})$ for planar graphs and more generally for graphs embeddable in a surface of bounded genus, which have bounded degeneracy.

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Besides graphs of bounded genus, a prominent type of degenerate graphs consists of graphs of bounded treewidth: a graph of treewidth k is k-degenerate.

If G has treewidth k and any addition of an edge between two vertices of G yields a graph of treewidth larger than k, then G is called a k-tree. Esperet et al. obtained the lower bound $\frac{\log n}{k \log k}$ on the size of the longest induced path of k-trees with a n-path. They also found an upper bound of $(k+1)(\log n)^{2/(k-1)}$ for graphs with treewidth less than k.

Another class of graphs with bounded treewidth considered in their paper is the class of interval graphs with bounded pathwidth: if G is an interval graph with pathwidth less than k and has a path of order n, then G has an induced path of order $\Omega((\log n)^{1/(k-1)^2})$.

2 Contribution

Our first result is an exponential improvement of the bound above, as well as a generalization to all graphs of pathwidth less than k.

Theorem 1. *For every $k \in \mathbb{N}$, if G is a graph of pathwidth less than k that has a path of order n, then G has an induced path of order at least $\frac{1}{3} n^{1/k}$.*

Its proof consists in a simple win/win strategy. We identify an induced path whose removal decreases the pathwidth of G. Then either this path is at least as long as the bound promised by the statement and we are done, or its removal decreases the pathwidth without decreasing the number of vertices much, and we conclude by applying the induction hypothesis.

This result is complemented by the following upper bound that even holds for interval graphs, showing that the exponential dependency in $1/k$ in our lower bound above is unavoidable.

Theorem 2. *For every $k, n \in \mathbb{N}$ with $2 \leq k \leq n$, there exists an interval graph $G_{n,k}$ with a path of order at least n and clique number at most k, such that every induced path of $G_{n,k}$ has order at most $n^{2/k} + 1$.*

We then show that in a graph of bounded treewidth that has a large path, there is always (as a contraction) a graph of bounded pathwidth that has a long path. This statement is used to obtain the following polylogarithmic bound for graphs of bounded treewidth.

Theorem 3. *For every $k \in \mathbb{N}$, if G is a graph of treewidth less than k that has a path of order n, then G has an induced path of order at least $\frac{1}{4}(\log n)^{1/k}$.*

As mentioned above, Esperet et al. constructed chordal graphs of treewidth k that have a path of order n and where no induced path has order more than $(k+1)(\log n)^{\frac{2}{k-1}}$. Therefore neither the logarithmic dependency in n nor the exponential dependency in $1/k$ could be improved in our lower bound above.

The full article describing this research is available at `https://arxiv.org/abs/2201.03880` and gives polylogarithmic bounds for more general classes of graphs including topological-minor-free graphs.

References

[1] J. Arocha and P. Valencia. Long induced paths in 3-connected planar graphs. Discussiones Mathematicae Graph Theory, 20(1):105–107, 2000.

[2] L. Esperet, L. Lemoine and F. Maffray. Long induced paths in graphs. European Journal of Combinatorics, 62:1–14, 2017.

[3] F. Galvin, I. Rival and B. Sands. A Ramsey-type theorem for traceable graphs. Journal of Combinatorial Theory, Series B, 33(1):7–16, 1982.

[4] J. Nešetřil and P. Ossona de Mendez Sparsity: graphs, structures, and algorithms,volume 28. Springer Science & Business Media, 2012.