A bound on the dissociation number

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Abstract

The dissociation number $\mathrm{diss}(G)$ of a graph G is the maximum order of a set of vertices of G inducing a subgraph that is of maximum degree at most 1. Computing the dissociation number of a given graph is algorithmically hard even when restricted to subcubic bipartite graphs. For a graph G with n vertices, m edges, k components, and c_1 induced cycles of length 1 modulo 3, we show $\mathrm{diss}(G) \geq n - \frac{1}{3}\Big(m + k + c_1\Big)$. Furthermore, we characterize the extremal graphs in which every two cycles are vertex-disjoint.

Keywords: Dissociation set, dissociation number, cactus

1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology. A set D of vertices of a graph G is a *dissociation set* in G if the subgraph $G[D]$ of G induced by D has maximum degree at most 1. The *dissociation number* $\mathrm{diss}(G)$ of G is the maximum order of a dissociation set in G. The dissociation number is algorithmically hard even when restricted, for instance, to subcubic bipartite graphs [3, 9, 12]. Fast exact algorithms [8], (randomized) approximation algorithms [7, 8], and fixed parameter tractability [11] have been studied for this parameter or its dual, the 3-*path (vertex) cover* number. Several lower bounds on the dissociation number were proposed: If G is a graph of order n and size m, then

$$\mathrm{diss}(G) \quad \geq \quad \begin{cases} \frac{n}{\lceil \frac{\Delta+1}{2} \rceil} & \text{, if } G \text{ has maximum degree } \Delta \text{ [4],} \\ \frac{4}{3} \sum\limits_{u \in V(G)} \frac{1}{d_G(u)+1} & \text{, if } G \text{ has no isolated vertex [4],} \\ \sum\limits_{u \in V(G)} \frac{1}{d_G(u)+1} + \sum\limits_{uv \in E(G)} \binom{|N_G[u] \cup N_G[v]|}{2}^{-1} & \text{, [6],} \\ \frac{n}{2} & \text{, if } G \text{ is outerplanar [4]} \\ \frac{2n}{3} & \text{, if } G \text{ is a tree [4],} \\ \frac{2n}{k+2} - \frac{m}{(k+1)(k+2)} & \text{, if } k = \lceil \frac{m}{n} \rceil - 1 \text{ [5], and} \\ \frac{2n}{3} - \frac{m}{6} & \text{, [4].} \end{cases} \tag{1}$$

The results in the present papers were inspired by bounds in (1).

Our main result is the following.

Theorem 1. *If G is a graph with n vertices, m edges, k components, and c_1 induced cycles of length 1 modulo 3, then*

$$\mathrm{diss}(G) \quad \geq \quad n - \frac{1}{3}\Big(m + k + c_1\Big). \tag{2}$$

Theorem 1 generalizes the lower bound $2n/3$ for trees of order n in (1), strengthens the general lower bound $\frac{2n}{3} - \frac{m}{6}$ in (1) for many graphs, and almost implies the lower bound $n/2$ for subcubic graphs of order n, which follows from the first bound in (1). In the proof of Theorem 1, graphs in which all cycles are pairwise vertex-disjoint play an essential role. We call such graphs *cycle-disjoint*; their components are restricted cactus graphs, where a *cactus* is a connected graph in which every block is either a K_2 or a cycle. As a step towards the understanding of all extremal graphs for Theorem 1, we consider the extremal cycle-disjoint graphs in more detail. We propose three extension operations (O_1), (O_2), and (O_3) applicable to a given graph G', attaching a P_3 or a cycle of length not 0 modulo 3 by a bridge to G', illustrated in Figure 1. It is easy to see that applying one of these operations to a graph that satisfies (2) with equality yields a graph that satisfies (2) with equality. Since P_3 and the cycles of lengths not 0 modulo 3 satisfy (2) with equality, this already allows to construct quite a rich family of extremal graphs, yet not all of them.

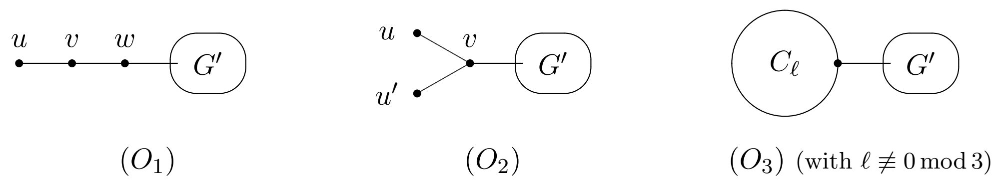

Figure 1: Operations constructing an extremal graph from a smaller extremal graph G'.

The two operations (O_1) and (O_2) are sufficient for the constructive characterization of all trees T of order n with $\mathrm{diss}(T) = 2n/3$, that is, of all trees that are extremal for the bound from [4] stated in (1). Let $\mathcal{T}$ be the set of all trees that arise from P_3 by repeated applications of the two operations (O_1) and (O_2), attaching a new P_3 by a bridge to trees in $\mathcal{T}$.

Theorem 2. *For a tree T of order n, the following statements are equivalent.*

(a) $\mathrm{diss}(T) = \frac{2n}{3}$.

(b) $T \in \mathcal{T}$.

(c) $n \equiv 0 \bmod 3$, *and, for every vertex y of T, at most two components of $T - y$ have order not* 0 *modulo* 3.

Next to the three simple operations illustrated in Figure 1, we introduce one slightly more complicated operation involving so-called *((very) good) spiked cycles*: For positive integers ℓ and k with $\ell \geq \max\{3, k\}$, and indices $i_1, \ldots, i_k \in [\ell]$ with $i_1 < i_2 < \ldots < i_k$, a *spiked cycle C^* with k spikes at* $\{i_1, \ldots, i_k\}$ arises from the cycle $C : u_1u_2 \ldots u_\ell u_1$ of length ℓ by attaching a new endvertex v_{i_j} to u_{i_j} for every $j \in [k]$. The spiked cycle C^* is *good* if either $k = 1$ and $\ell \equiv 1 \bmod 3$ or $k \geq 2$,

- $i_{j+1} - i_j \equiv 2 \bmod 3$ for every $j \in [k-1]$, and
- $\ell + i_1 - i_k \equiv 1 \bmod 3$,

that is, the k paths in C^* between vertices of degree 3 whose internal vertices have degree 2, have lengths $2, \ldots, 2$, and 1 modulo 3. The spiked cycle C^* is *very good* if it is good and

- $\ell \not\equiv 1 \bmod 3$,

that is, in particular, $k \geq 2$. See Figure 2 for an illustration.

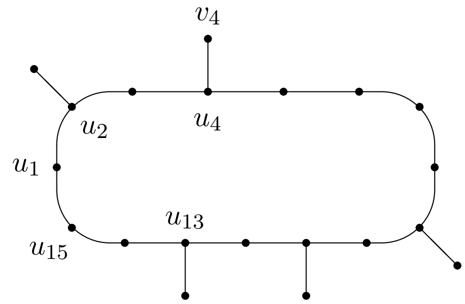

Figure 2: A very good spiked cycle with $\ell = 15$ and $k = 5$ spikes at $\{i_1, \ldots, i_k\} = \{2, 4, 9, 11, 13\}$. Note that removing v_4 results in a spiked cycle that is not good.

Let $\mathcal{C}$ be the set of all graphs that arise from the graphs in

$$\mathcal{C}_0 = \Big\{P_3\Big\} \cup \Big\{C_\ell : \ell \in \mathbb{N},\, \ell \geq 3, \text{ and } \ell \not\equiv 0 \bmod 3\Big\} \cup \Big\{C^* : C^* \text{ is a very good spiked cycle}\Big\}$$

by repeated applications of the three operation (O_1), (O_2), and (O_3), as well as the fourth operation (O_4) of forming the disjoint union of some graph G' in $\mathcal{C}$ with a very good spiked cycle C^*, and adding a bridge between $V(G')$ and $V(C^*)$.

Lemma 1. *All graphs in $\mathcal{C}$ satisfy (2) with equality. Furthermore, for every vertex u of every graph G in $\mathcal{C}$, the graph G has a maximum dissociation set not containing u.*

As our final result, we show that $\mathcal{C}$ contains all connected cycle-disjoint extremal graphs for Theorem 1. Figure 3 shows two extremal graphs that are not cycle-disjoint.

Theorem 3. *A connected cycle-disjoint graph satisfies (2) with equality if and only if it belongs to $\mathcal{C}$.*

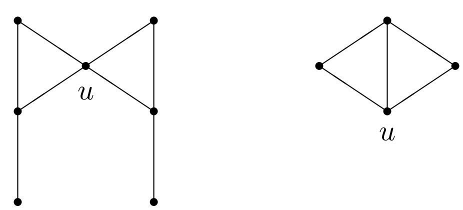

Figure 3: Two graphs G that satisfy (2) with equality. Note that the removal of the vertex u, which lies on two cycles, yields $d_G(u) - 2$ components.

Within our results, the value c_1 can be replaced by the maximum number of pairwise vertex-disjoint cycles of length 1 modulo 3. It remains to elucidate the structure of all extremal graphs for Theorem 1.

All proofs are provided in the arxiv version of this paper. [2]

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