Non-empty intersection of longest paths in H-free graphs

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Abstract

We make progress toward a characterization of the graphs H such that every connected H-free graph has a longest path transversal of size 1. In particular, we show that the graphs H on at most 4 vertices satisfying this property are exactly the linear forests. We also show that if the order of a connected graph G is large relative to its connectivity $\kappa(G)$ and $\alpha(G) \leq \kappa(G)+2$, then each vertex of maximum degree forms a longest path transversal of size 1.

1 Introduction

It is a classic result in graph theory that every two longest paths in a connected graph share at least one vertex. Gallai [4] asked whether in fact all longest paths in a connected graph share at least one vertex. This was answered in the negative by Walther [9], who provided a counterexample with 25 vertices. A counterexample with 12 vertices was later constructed by Walther and Voss [10] and, independently, by Zamfirescu [12] (see Figure 1).

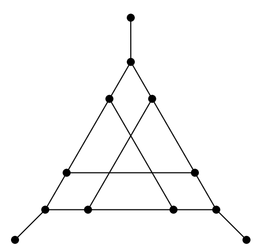

Figure 1: The graph G_0: A 12-vertex graph with no Gallai vertex.

A *Gallai set* (or *longest path transversal*) in a graph G is a set of vertices S intersecting every longest path in G. The *Gallai number* of G, denoted by $\mathrm{lpt}(G)$, is the minimum size of a Gallai set and a *Gallai family* is a family of graphs $\mathcal{G}$ such that $\mathrm{lpt}(G) = 1$ for each connected graph $G \in \mathcal{G}$. A vertex v in G is a *Gallai vertex* if $\{v\}$ is a Gallai set and a graph is *Gallai* if it has a Gallai vertex. The counterexamples mentioned above consist of connected graphs G for which $\mathrm{lpt}(G) = 2$. In fact, there are examples of connected graphs G for which $\mathrm{lpt}(G) = 3$ [12] and Walther [9] and Zamfirescu [11] asked if the Gallai number of connected graphs is bounded. In a companion paper [7] we addressed this fifty-year-old question. Improving on [8], we showed that connected graphs admit sublinear longest path transversals. The gap between our upper bound and the constant lower bound 3 remains large.

In this paper we focus on another natural variant of Gallai's question: Which classes of graphs form Gallai families? It is well known that a family of pairwise intersecting subtrees of a tree has non-empty intersection; in particular, trees form a Gallai family. Several other Gallai families have been identified: for example, $2K_2$-free graphs [6] and P_4-sparse graphs [1].

Let $\mathrm{Free}(H)$ be the class of H-free graphs. A *monogenic class* of graphs has the form $\mathrm{Free}(H)$, for some graph H. In this paper we aim at characterizing monogenic Gallai families. In Section 2, we make progress by showing that if $\mathrm{Free}(H)$ is a Gallai family, then H is a linear forest, and this suffices when $|V(H)| \leq 4$. In the spirit of [6], we in fact prove something more general: if H is a linear forest on at most 4 vertices and G is a connected H-free graph, then all maximum degree vertices in G are Gallai. We also show that if G is a connected graph with independence number $\alpha(G) \leq 4$ (i.e., G is $5P_1$-free), then G is Gallai.

A celebrated result of Chvátal and Erdős [2] asserts that a graph G has a Hamiltonian cycle when $|V(G)| \geq 3$ and $\alpha(G) \leq \kappa(G)$, and that G has a Hamiltonian path when $\alpha(G) \leq \kappa(G)+1$. It follows that every vertex in G is Gallai when $\alpha(G) \leq \kappa(G)+1$. In Section 3, we show that if a connected graph G is large relative to its connectivity $\kappa(G)$ and $\alpha(G) \leq \kappa(G)+2$, then each vertex of maximum degree is a Gallai vertex. Our result has the following immediate consequence: if a regular graph G is large relative to its connectivity and $\alpha(G) \leq \kappa(G)+2$, then G contains a Hamiltonian path.

2 Monogenic Gallai families

In this section we make progress toward a classification of monogenic Gallai families. We first show that a necessary condition for a monogenic family $\mathrm{Free}(H)$ to be Gallai is that H is a linear forest on at most 9 vertices, where a *linear forest* is a forest in which every component is a path. Let G_0 be the graph in Figure 1 with $\mathrm{lpt}(G_0)=2$ [10, 12]. In the following, we say that a graph H is a *fixer* if $\mathrm{Free}(H)$ is a Gallai family; that is, forbidding H "fixes" the answer to Gallai's question.

Proposition 1. *If H is a fixer, then H is a linear forest on at most 9 vertices.*

Remark 2. Gao and Shan [5] asked whether all longest paths in a connected claw-free graph have a non-empty intersection. Proposition 1 answers this question in the negative.

For $|V(H)| \leq 4$, we show that H is a fixer if and only if H is a linear forest. Necessity follows from Proposition 1. For sufficiency, we show that every 4-vertex linear forest is a fixer. The linear forests of order 4 are P_4, P_3+P_1, $2P_2$, P_2+2P_1, and $4P_1$. Cerioli and Lima [1] showed that P_4-sparse graphs, a superclass of P_4-free graphs, form a Gallai family, whereas Golan and Shan [6] showed that $2P_2$-free graphs form a Gallai family. In other words, P_4 and $2P_2$ are fixers. In the following, we address the remaining cases: P_3+P_1, P_2+2P_1, and $4P_1$.

Theorem 3. *If G is a connected (P_3+P_1)-free graph, then every vertex of degree at least $\Delta(G)-1$ is a Gallai vertex.*

The degree assumption in Theorem 3 is best possible. Indeed, the complete bipartite graph $K_{t,t+2}$ is (P_3+P_1)-free, has maximum degree $t+2$, and the vertices of degree t are not Gallai.

Proposition 4. *If G is a connected (P_2+2P_1)-free graph, then every vertex of maximum degree is a Gallai vertex.*

Vertices of degree $\Delta(G)-1$ in a (P_2+2P_1)-free graph G need not be Gallai. Indeed, consider the graph G obtained from $K_{t,t+2}$ by removing a matching saturating the part of size t. G is (P_2+2P_1)-free and $\Delta(G)=t+1$. The longest paths in G omit one vertex, and the Gallai vertices are those in the smaller part. Two of the non-Gallai vertices in the larger part have degree t, which equals $\Delta(G)-1$.

Theorem 5. *Let $k \in \{1,2\}$. If G is k-connected and $\alpha(G) \leq k+2$, then every longest path in G contains every vertex of degree at least $\Delta(G)-(2-k)$.*

Corollary 6. *If G is a connected graph with $\alpha(G) \leq 3$ and $\Delta(G)-\delta(G) \leq 1$, or if G is a 2-connected regular graph with $\alpha(G) \leq 4$, then G has a Hamiltonian path.*

Corollary 7. *The graph $4P_1$ is a fixer.*

To show that every connected graph G with $\alpha(G) \leq 4$ has a Gallai vertex, we distinguish two cases. If G is 2-connected, the result already follows from Theorem 5. When G has cut-vertices, we exploit the block-cutpoint structure of G.

Theorem 8. *Let G be a connected graph. If $\alpha(G) \leq 4$, then G has a Gallai vertex. Equivalently, $5P_1$ is a fixer.*

Note that there are connected $5P_1$-free graphs in which no vertex of maximum degree is Gallai (see Example 11). This is in contrast to the case of fixers F of order at most 4, where the vertices of maximum degree in a connected F-free graph are all Gallai (Golan and Shan [6] show this for $F = 2P_2$, we show it for $F \in \{P_3 + P_1, P_2 + 2P_1, 4P_1\}$, and the case $F = P_4$ is an easy exercise).

The graph G_0 from Figure 1 shows that there is a connected graph G such that G has no Gallai vertex and $\alpha(G) = 6$. The case $\alpha(G) \leq 5$ remains open.

Conjecture 9. *If $\alpha(G) \leq 5$ and G is connected, then G has a Gallai vertex.*

When G is 3-connected, $\alpha(G) \leq 5$, and G is sufficiently large, Theorem 10 below shows that G has a Gallai vertex. Outside of a finite number of cases when $\kappa(G) \geq 3$, resolving Conjecture 9 reduces to the cases that $\kappa(G) = 1$ and $\kappa(G) = 2$. Although it is reasonable to expect that the case $\kappa(G) = 1$ may be treated by analyzing the block structure of G, it is less clear how to handle the case $\kappa(G) = 2$.

3 A Chvátal–Erdős type result

We show that if $\alpha(G) \leq \kappa(G) + 2$ and G is sufficiently large in terms of $\kappa(G)$, then the maximum degree vertices in G are Gallai.

Theorem 10. *For each positive integer k, there exists an integer n_0 such that if G is an n-vertex k-connected graph with $\alpha(G) \leq k + 2$ and $n \geq n_0$, then each vertex of maximum degree is Gallai.*

Example 11. The assumption $\alpha(G) \leq \kappa(G) + 2$ in Theorem 10 is best possible. Let G be the graph obtained from the star $K_{1,k+2}$ with leaves $\{x_1, \ldots, x_{k+2}\}$ by replacing the center vertex with a k-clique S and replacing each leaf vertex x_i with a t-clique X_i containing a set of k distinguished vertices Y_i that are joined to S. Since $V(G)$ can be covered by $k+3$ cliques, we have $\alpha(G) \leq k+3$. Also, we have $\kappa(G) = k$ since S is a cutset of size k and when $R \subseteq V(G)$ and $|R| < k$, the graph $G - R$ contains at least one vertex in each of $S, Y_1, \ldots, Y_{k+2}$, implying that $G - R$ is connected.

We claim that the set of Gallai vertices in G is S. Since $|S| = k$ and $G - S$ is the disjoint union of $k + 2$ copies of K_t, it follows that every path in G has at most $|V(G)| - t$ vertices. Paths in G that achieve this bound contain S and all but one of $X_1, \ldots, X_{k+2}$, implying that $u \in V(G)$ is Gallai if and only if $u \in S$. By construction, each vertex in S has degree $k(k+2) + (k-1)$. Hence, when t is sufficiently large, the set of vertices in G of maximum degree is $Y_1 \cup \cdots \cup Y_{k+2}$, and none of these is Gallai.

Although maximum degree vertices are not Gallai, our construction still has Gallai vertices. It is natural to ask whether every graph with sufficiently high connectivity has a Gallai vertex [11, 13]. As noted in Section 1, there are k-connected graphs having no Gallai vertices when $k \leq 3$. The question remains open for $k \geq 4$.

The complete bipartite graphs $K_{s,s+2}$ show that the condition $\alpha(G) \leq \kappa(G) + 1$ cannot in general be relaxed to $\alpha(G) \leq \kappa(G) + 2$ while still guaranteeing existence of Hamiltonian paths [2]. However, Theorem 10 immediately implies that this is possible for sufficiently large regular graphs.

Corollary 12. *For each positive integer k, there exists n_0 such that every k-connected regular graph G with $\alpha(G) \leq k+2$ and $n \geq n_0$ vertices has a Hamiltonian path.*

We do not know whether the condition $\alpha(G) \leq k+2$ in Corollary 12 is best possible. The following construction from [3] shows that it cannot be relaxed to $\alpha(G) \leq k+5$.

Example 13. Let $k \geq 6$ be even. Let G_1 be K_{k+1} minus an edge and let G_2 be K_{k+1} minus a matching on $k-4$ vertices. Let G be the graph obtained from two copies of G_1 and one copy of G_2 by adding a new vertex adjacent to all k vertices of degree $k-1$. We have that G is a 1-connected regular graph with $\alpha(G) = 6$ and no Hamiltonian path.

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