Improved square coloring of planar graphs[1][2]

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Abstract

Square coloring is a variant of graph coloring where vertices within distance 2 must receive different colors. When considering planar graphs, the most famous conjecture about this coloring (Wegner, 1977) states that $\frac{3}{2}\Delta + 1$ colors are sufficient to square color every planar graph of maximum degree Δ. This conjecture has been proven asymptotically for graphs with large maximum degree. We consider here planar graphs with small maximum degree and show that $2\Delta + 7$ colors are sufficient, which improves the best known bounds when $6 \leqslant \Delta \leqslant 31$.

1 Introduction

In graph theory, graph coloring is among the most studied problems. The history of graph coloring begins in the 19th century with the coloration of maps and the question asking whether four colors are sufficient to color differently regions sharing a same border. This question can be rephrased as whether every planar graph can be properly colored with four colors, which was proved by Appel and Haken in 1976 [3]. The proof of this result is known for being one of the first major computer-assisted proof.

Many other types of colorings were studied in the last decades, in particular in the case of planar graphs. We are interested here in the so-called *square coloring*, where two vertices must receive distinct colors if they are at distance at most 2. Given a graph G, the *square chromatic number*, denoted by $\chi_2(G)$, is the minimum number of colors to color vertices at distance at most 2 differently. The name "square coloring" comes from the fact that $\chi_2(G)$ can also be defined as the chromatic number of the square G^2 of G, *i.e.* the graph obtained from G by adding edges between vertices at distance 2.

Given a graph G, it is not hard to see that $\chi_2(G) \geq \Delta(G) + 1$ where $\Delta(G)$ (or Δ when G is clear from the context) is the maximum degree of G. Indeed, all the neighbors of a vertex v are at distance at most 2 in G and then must receive distinct colors. On the other hand, we have $\chi_2(G) \leq \Delta(G)^2 + 1$ since every vertex is adjacent to at most $\Delta(G)^2$ vertices in G^2 and in particular G^2 can be colored greedily with $\Delta(G)^2 + 1$ colors. One can prove that in general graphs this bound is tight for a finite number of graphs called the Moore graphs [7], and asymptotically tight for the infinite family of incidence graphs of projective planes (which require $\Delta^2 - \Delta + 1$ colors).

In this paper, we focus on the particular case of planar graphs, that are graphs that can be embedded on the plane without edge-crossings. Using degeneracy arguments, one can show a linear bound for planar graphs, namely $\chi_2 \leq 9\Delta$ [8]. However, even if linear, this bound seems far from

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[2]Reference to the full text: [5]

tight, since the graphs from Figure 1 satisfy $\chi_2(G) = \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$, which is the highest known value of χ_2.

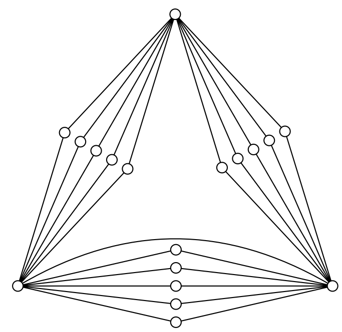

Figure 1: Wegner's construction

A famous conjecture from Wegner [14] states that this should be asymptotically the right bound.

Conjecture 1 (Wegner [14])**.** *Every planar graph G with maximum degree Δ satisfies:*

$$\chi_2(G) \leq \begin{cases} 7 & \textit{if } \Delta = 3, \\ \Delta + 5 & \textit{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1 & \textit{if } \Delta \geq 8. \end{cases}$$

Despite receiving considerable attention, Wegner's conjecture is still open today, except for the case of subcubic graphs solved by Thomassen [12]. As shown independently in [6] and [2], the conjecture asymptotically holds: $\chi_2(G) = 3/2\Delta + o(\Delta)$ when $\Delta \to \infty$. Many results of the form $c\Delta + O(1)$ (where c is a constant) were also found, culminating with $c = 5/3$ from [11].

However, since the constants hidden in these proofs are large (and these results only hold for large enough Δ), the picture is far from being complete, especially for small values of Δ. An extensive line of work consists in improving the function when Δ is small. Some of the existing results are summarized in Table 1.

Authors	Restriction	Result
Thomassen [12]	$\Delta \leq 3$	$\chi_2(G) \leq 7$
Jonas [8]	$\Delta \geq 7$	$\chi_2(G) \leq 8\Delta - 22$
Wong [15]	$\Delta \geq 7$	$\chi_2(G) \leq 3\Delta + 5$
Madaras and Marcinova [10]	$\Delta \geq 12$	$\chi_2(G) \leq 2\Delta + 18$
Borodin et al. [4]	$\Delta \leq 20$ $21 \leq \Delta \leq 46$ $\Delta \geq 47$	$\chi_2(G) \leq 59$ $\chi_2(G) \leq \Delta + 39$ $\chi_2(G) \leq \lceil \frac{9\Delta}{5} \rceil + 1$
Van den Heuvel and McGuinness [13]	$\Delta \geq 5$	$\chi_2(G) \leq 9\Delta - 19$ $\chi_2(G) \leq 2\Delta + 25$
Agnarsson and Halldorsson [1]	$\Delta \geq 749$	$\chi_2(G) \leq \lfloor \frac{9\Delta}{5} \rfloor + 1$
Molloy and Salavatipour [11]	$\Delta \geq 249$	$\chi_2(G) \leq \lceil \frac{5\Delta}{3} \rceil + 25$ $\chi_2(G) \leq \lceil \frac{5\Delta}{3} \rceil + 78$
Zhu and Bu [16]	$\Delta \leq 5$ $\Delta \geq 6$	$\chi_2(G) \leq 20$ $\chi_2(G) \leq 5\Delta - 7$
Zhu and Bu [17]	$\Delta = 4$	$\chi_2(G) \leq 13$
Krzyzinski et al. [9]	$\Delta \geq 6$	$\chi_2(G) \leq 3\Delta + 4$

A graph summarizing all these results is depicted in Figure 2.

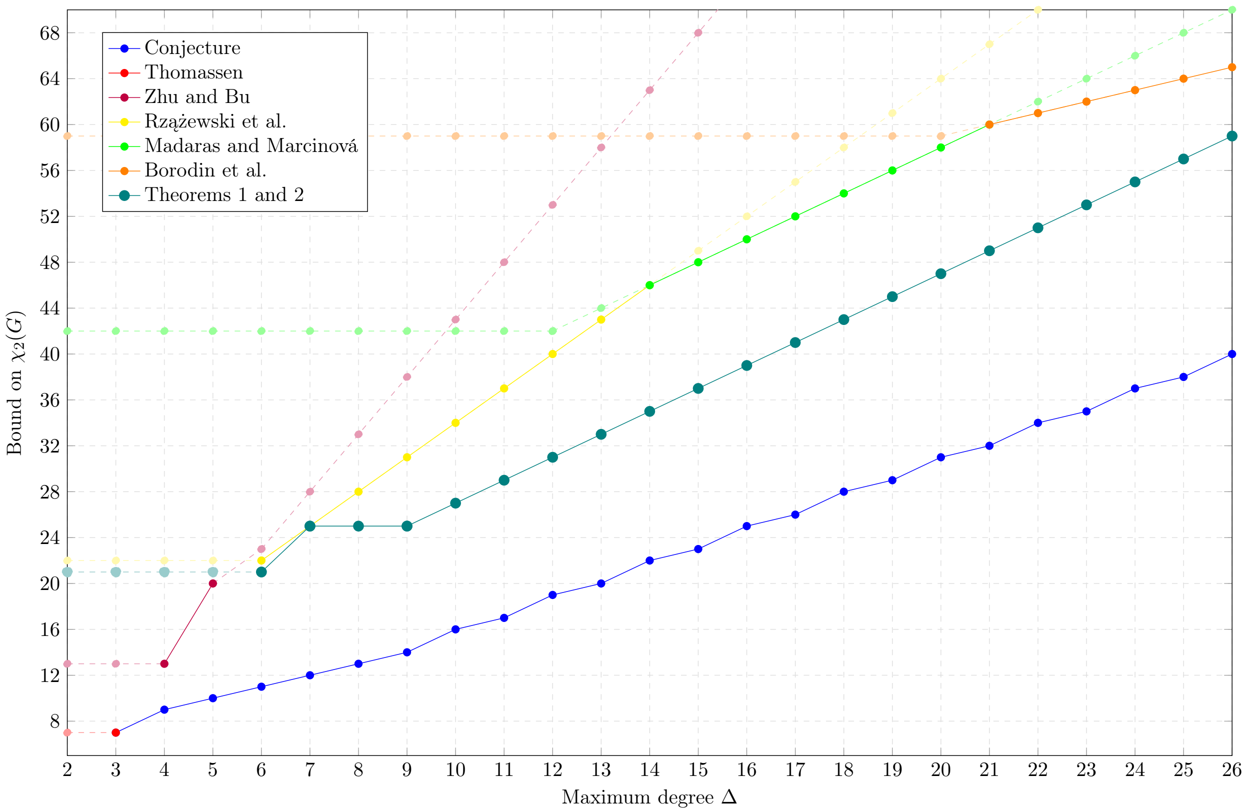

Figure 2: Bounds on $\chi_2(G)$ depending on Δ for $3 \leq \Delta \leq 26$

2 Our results

We follow this line of research by improving the best known bound on $\chi_2(G)$ for every planar graph whose maximum degree is between 6 and 31. More formally, we prove the following.

Theorem 1. *Let G be a planar graph of maximal degree $\Delta \geq 9$. Then, $\chi_2(G) \leq 2\Delta + 7$.*

For $\Delta \leqslant 6$, the $2\Delta + 7$ bound is 19. Reusing our analysis, we can easily prove a slightly worse bound, which still improves the best current bound of [9].

Theorem 2. *Let G be a planar graph of maximal degree $\Delta \leq 6$. Then, $\chi_2(G) \leq 21$.*

The two theorems are proved using the discharging method. The idea of this method is to assume that there is a counter-example G to one of these theorems, and to choose it as a minimal one. Then we look for a contradiction by double counting some suitable quantity. To this end, we put charges on faces and vertices such that the total weight is negative (by Euler's formula). We then define some "discharging rules", *i.e.* ways to transfer the charges between the faces and vertices. On the other hand we prove that G does not contain some subgraphs (called configurations). The classical way to forbid a configuration is, assuming G contains it, to build a counter-example smaller than G for the theorem (which contradicts the minimality of G). Finally, we use this structural information to show every vertex and face has a non-negative weight after applying the rules. This yields a contradiction and proves that G cannot exist.

We use a not so standard order, by choosing a counter-example with minimum number of vertices and maximum number of edges. Then we provide a set of 4 rules and 7 configurations to prove Theorem 1. Using the same rules and adding two configurations, we also obtain Theorem 2.

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