On three domination-based identification problems in block graphs[1]

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Abstract

The problems of determining minimum identifying or (open) locating-dominating codes are special search problems that are challenging from both theoretical and computational viewpoints. In these problems, one selects a dominating set C from the vertex set $V(G)$ of a graph G such that the vertices of a chosen subset of $V(G)$ (e.g. $V(G) \setminus C$ or $V(G)$ itself) are uniquely determined by their neighborhoods in C. Hence, a typical line of attack for these problems is to determine tight bounds for the minimum codes in special graphs. In this work, we do the same for block graphs (i.e. diamond-free chordal graphs). We present for all three codes tight lower and upper bounds, and examples of block graphs which attain these bounds. Our upper bounds are in terms of the number of maximal cliques $n_Q(G)$, the order $|V(G)|$ and other structural properties of a block graph G. As for the lower bounds, we prove them to be linear in terms of (i) $n_Q(G)$, and (ii) the order of G.

1 Introduction

For a graph $G = (V(G), E(G))$ that models a facility, detection devices can be placed at its vertices to locate an intruder (like a fire or a thief). Depending on the features of the detection devices, different types of dominating sets can be used to determine the optimal distribution of the detection devices in G. In the following, we study three problems arising in this context which have all been actively studied during the last decades, see the bibliography maintained by Lobstein [7].

Let $G = (V(G), E(G))$ be a graph. The (open) neighborhood of a vertex u is the set $N(u)$ of all vertices of G adjacent to u, and $N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u. A subset $C \subseteq V(G)$ is an *identifying code* [6] (for short: ID-code) of G if $N[u] \cap C \neq \emptyset$ for all $u \in V$ (domination); and $N[u] \cap C \neq N[v] \cap C$ for all $u, v \in V(G)$ (separation) [refer Figure 1 (a)]. A graph G admits an identifying code if and only if G is true-twin-free (i.e., there is no pair of distinct vertices $u, v \in V$ with $N[u] = N[v]$). A subset $C \subseteq V(G)$ is a *locating-dominating code* [8, 9] (for short: LD-code) if $N[u] \cap C \neq \emptyset$ for all $u \in V$ (domination); and $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V \setminus C$ (open-separation) [refer Figure 1 (b)]. A subset $C \subseteq V(G)$ is an *open locating-dominating code* [10] (for short: OLD-code) of G if $N(u) \cap C \neq \emptyset$ for all $u \in V$ (open-domination); and $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V$ (open-separation) [refer Figure 1 (c)]. A graph G admits an open-locating-dominating code if and only if G has no isolated vertices and is false-twin-free (i.e., there is no pair of distinct vertices $u, v \in V(G)$ such that $uv \notin E(G)$ and that $N(u) = N(v)$). The *identifying code number* $\gamma^{ID}(G)$, *locating-dominating number* $\gamma^{LD}(G)$ and the *open locating-dominating number* $\gamma^{OLD}(G)$ of a graph G are the minimum cardinalities of an ID-code, an LD-code and an OLD-code, respectively, of G. If $A \triangle B$ denotes the symmetric difference between any two sets A and B, for a set $C \subset V(G)$ and $u, v \in V(G)$, any $w \in (N(u) \cap C) \triangle (N(v) \cap C)$ (resp. $(N[u] \cap C) \triangle (N[v] \cap C)$) is said to open-separate (resp. closed-separate) u and v in C.

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A lot of research has been done to determine these codes of special graphs (see [3, 4, 6, 8, 9, 10]). In this paper, we consider the family of block graphs. A *block graph* is a graph in which every maximal 2-connected subgraph induces a clique; or precisely, a diamond-free chordal graph. Linear-time algorithms to compute all three domination numbers in block graphs are presented in [1]. In this paper, we complement this result by determining tight lower and upper bounds for all three domination numbers in block graphs. We give bounds using both the number of vertices – as it has been done for several classes of graphs – but also using the number $n_Q(G)$ of maximal cliques of G, that is relevant to block graphs. In doing so, we also prove the conjecture posed in [2] that the identifying code number of block graphs can be bounded from above by $n_Q(G)$. We also address similar questions for LD- and OLD-codes. For a block graph G, let $\mathcal{K}$ be the set of all maximal cliques of G. Any $x \in V(G)$ such that $\{x\} = K \cap K'$, for two distinct $K, K' \in \mathcal{K}$, is called an *articulation vertex* of both K and K'. Let $art(K)$ be the set of all articulation vertices of $K \in \mathcal{K}$.

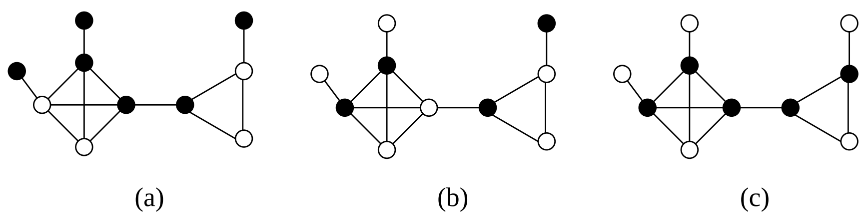

Figure 1: Block graphs whose black vertices form a minimum (a) ID-, (b) LD- and (c) OLD-code.

2 Upper and lower bounds

Theorem 1. *Let G be a true-twin-free block graph. Then $\gamma^{ID}(G) \leq n_Q(G)$.*

Sketch of the proof. The proof is by contradiction assuming G to be a block graph of minimum cardinality to have the property $\gamma^{ID}(G) > n_Q(G)$. Since G does not contain true twins, there exists in G a vertex, say x (in a leaf-clique), of degree 1. We then take the graph $G' = G - x$ and notice that $n_Q(G') = n_Q(G) - 1$. By the minimality of G therefore, $\gamma^{ID}(G') \leq n_Q(G) - 1$. If C' is an ID-code of G', by some graph theoretic analysis and case studies, we are then able to show that adding just one appropriate vertex to C' gives an ID-code, say C, for G. However, this is a contradiction since $|C| \leq n_Q(G)$ which contradicts the assumption that $\gamma^{ID}(G) > n_Q(G)$. $\square$

The bound in Theorem 1 can be refined for LD-codes as follows:

Theorem 2. *Let G be a block graph, $n'_Q(G)$ be the number of maximal cliques with no true twins and $\mathcal{S} = \{S \subset K \in \mathcal{K} : S$ is a maximal set of pairwise true twins$\}$. Then, $\gamma^{LD}(G) \leq n'_Q(G) + \sum_{S \in \mathcal{S}}(|S| - 1)$.*

Sketch of the proof. Let $K_0 \in \mathcal{K}$. Define a function $f : \mathcal{K} \to \mathbb{Z}$ by: $f(K_0) = 0$, and for any other $K \in \mathcal{K}$, define $f(K) = i$ if $K \cap K' \neq \emptyset$ for some $K' \in \mathcal{K}$ such that $f(K') = i - 1$. Then, let $art^-(K) = K \cap K'$. Next, we define a set $C \subset V(G)$ by the rules that, for all $K \in \mathcal{K}$ not containing any true twins, pick any one vertex from $K \setminus art^-(K)$ in C; and, for all $K \in \mathcal{K}$ containing true twins, pick any $|K \setminus art(K)| - 1$ vertices from $K \setminus art(K)$ in C. On counting, we find that $|C| = n'_Q(G) + \sum_{S \in \mathcal{S}}(|S| - 1)$. Next, we notice that for every $K \in \mathcal{K}$, $|K \cap C| \geq 1$. Therefore,

C is a dominating set of G. Further, for any $u, v \in V(G) \setminus C$, we have $u \in K$ and $v \in K'$ for a distinct pair of $K, K' \in \mathcal{K}$ and there exist $v_K \in K \cap C$ and $v_{K'} \in K' \cap C$ such that $v_K \neq v_{K'}$. This completes the proof, as either one of v_K and $v_{K'}$ open-separates u and v in C. □

For a false-twin-free graph G, $\gamma^{OLD}(G) \leq |V(G)| - 1$ unless G is a half-graph (a special kind of bipartite graph) [5]. We extend the result for block graphs, noting that the only half-graphs which are block graphs are P_2 and P_4:

Theorem 3. *Let G be a connected, false-twin-free block graph which is neither a P_2, nor a P_4. Let $m_Q(G)$ be the number of maximal cliques K in G such that $|K \setminus art(K)| \geq 1$ and $|art(K)| \geq 2$. Then $\gamma^{OLD}(G) \leq |V(G)| - m_Q(G) - 1$.*

Sketch of the proof. Take a clique $K_0 \in \mathcal{K}$ such that $|K_0| = \min\{|K| : K \in \mathcal{K}$ such that $|art(K)| = 1\}$ and define $C \subset V(G)$ by the following rules: For all $K \in \mathcal{K} \setminus \{K_0\}$ such that $|art(K)| = 1$, pick all vertices of K in C; and for all $K \in \{K \in \mathcal{K} : |art(K)| > 1\} \cup \{K_0\}$, let $art(K) \subset C$ and pick any $\max\{|K \setminus art(K)| - 1, 0\}$ vertices of $K \setminus art(K)$ in C. First of all, every vertex in K_0 is dominated by $art(K_0) \subset C$; and for each other maximal clique K in G, $|K \cap C| \geq 2$. This makes C an open-dominating set of G. Then, by case analysis, one shows that C is also an open-separating set of G thus making it an OLD-code of G. Moreover, observing that, for K_0 and for all other maximal cliques K for which $|K \setminus art(K)| \geq 1$ and $|art(K)| \geq 2$, exactly one vertex is left out from it in C, this gives the count for the size of C and the upper bound for $\gamma^{OLD}(G)$ in the theorem. □

The upper bounds in Theorems 1, 2 and 3 are tight, as they are attained, respectively, by:
(1) for ID-codes: stars and thin headless spiders [3];
(2) for LD-codes: graphs obtained by taking a complete graph K of any size and attaching another complete graph of any size to each vertex of K; and
(3) for OLD-codes: graphs constructed by taking k (≥ 2) triangles with vertices a_i, b_i, c_i ($1 \leq i \leq k$), attaching a degree 1 vertex d_i to each a_i and identifying all the b_is to a single vertex.

Theorem 4. *Let G be a block graph. Then $\gamma^{ID}(G)$, $\gamma^{OLD}(G) \geq \frac{|V(G)|}{3} + 1$ and $\gamma^{LD}(G) \geq \frac{|V(G)|+1}{3}$.*

Sketch of the proof. As an example, we give the sketch of the proof only for $\gamma^{OLD}(G) \geq \frac{|V(G)|}{3} + 1$. The other two cases of $\gamma^{ID}(G)$ and $\gamma^{LD}(G)$ follow similarly. Let C be an OLD-code for G and k be the number of connected components of G. We partition $V(G)$ into V_1, V_2, V_3 and V_4 where, $V_1 = C$; $V_2 = \{v \in V(G) : |N(v) \cap C| = 1\}$; V_3 contains vertices that have neighbors in at least two different connected components of C; and $V_4 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Then, through a series of arguments, one shows that $|V_2| \leq |C| - n_1(G[C])$, $|V_3| \leq k - 1$ and $|V_4| \leq |C| - 3k + n_1(G[C])$, where $n_i(H)$ denotes the number of degree i vertices in the subgraph H of G. Noting that $k \geq 1$, this gives the result by $n = |C| + |V_2| + |V_3| + |V_4| \leq |C| + |C| - n_1(G[C]) + k - 1 + |C| - 3k + n_1(G[C]) = 3|C| - 2k - 1$. □

Extremal cases where these bounds are attained can be constructed as follows: Consider the graph with one path $u_1, ..., u_k$ (the vertices in the code C) and attach further vertices:
(1) for an ID-code: attach a single vertex to each u_i and vertices to the pairs u_i, u_{i+1} for $1 < i < k-1$;
(2) for an OLD-code: attach a single vertex to u_1, u_k and each u_i for $2 < i < k-1$ and vertices to all the pairs u_i, u_{i+1}; and
(3) for an LD-code: attach a single vertex to each u_i and vertices to all the pairs u_i, u_{i+1}.

Theorem 5. *Let G be a block graph. Then $\gamma^{ID}(G)$, $\gamma^{OLD}(G) \geq \frac{3(n_Q(G)+2)}{7}$, $\gamma^{LD}(G) \geq \frac{n_Q(G)+2}{3}$ and $\gamma^{OLD}(G) \geq \frac{n_Q(G)+3}{2}$.*

Sketch of the proof. We give the sketch of the proof for the case $\gamma^{OLD}(G) \geq \frac{n_Q(G)+3}{2}$ and the other cases follow similarly. We use the same definitions of k, V_1, V_2, V_3, V_4 and $n_i(H)$ as in the sketch of the proof for Theorem 4. We see that there are four types of maximal cliques: (1) Maximal cliques that are maximal cliques of C (of size at least 2) or a maximal clique of C with one vertex of V_4. We show that there are at most $n_Q(G[C]) - n_0(G[C]) \leq |C| - 3k + n_1(G[C])$ of them; (2) Maximal cliques of size 2 between V_2 and its unique neighbor in the code. Again, we show that there are at most $|V_2| \leq |C| - n_1(G[C])$ such cliques; (3) Maximal cliques formed with a vertex of V_3 and some vertices of C; and (4) Maximal cliques that are included in $V(G) \setminus C$. We prove that the number of maximal cliques of the types (3) and (4) is at most $2k-2$. Thus, noting that $k \geq 1$ and putting together the upper bounds for all the four types of maximal cliques, $n_Q(G) \leq n_Q(G[C]) - n_0(G[C]) + |V_2| + 2k - 2 \leq |C| - 3k + n_1(G[C]) + |C| - n_1(G[C]) + 2k - 2 = 2|C| - k - 2$. □

Note that for trees, since we have $n_Q(G) = |E(G)| = |V(G)| - 1$, these bounds are equivalent to the known lower bounds using the number of vertices (see [4] for ID-codes, [8] for LD-codes and [10] for OLD-codes). In particular, there are infinite families of trees reaching the three bounds.

3 Concluding remarks

We provided tight upper and lower bounds for all three dominating codes for block graphs by studying their structural properties. Perhaps, similar ideas also work for graph classes with a similar structure, e.g. cacti or block-cacti (graphs in which every maximal 2-connected subgraph is either a complete graph or a cycle). Other sub-classes of chordal graphs could also be of interest.

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