Ramsey numbers of Boolean lattices

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Abstract

The *poset Ramsey number* $R(Q_m, Q_n)$ is the smallest integer N such that any blue-red coloring of the elements of the Boolean lattice Q_N has a blue induced copy of Q_m or a red induced copy of Q_n. The *weak poset Ramsey number* $R_w(Q_m, Q_n)$ is defined analogously, with weak copies instead of induced copies.

Axenovich and Walzer showed that $n+2 \leq R(Q_2, Q_n) \leq 2n+2$. Recently, Lu and Thompson improved the upper bound to $\frac{5}{3}n + 2$. We solve this problem asymptotically by showing that $R(Q_2, Q_n) = n + O(n/\log n)$. Recent work of Axenovich and Winter implies that the $n/\log n$ term is required.

In the diagonal case, Cox and Stolee proved $R_w(Q_n, Q_n) \geq 2n + 1$ using a probabilistic construction. In the induced case, Bohman and Peng showed $R(Q_n, Q_n) \geq 2n + 1$ using an explicit construction. Improving these results, we show that $R_w(Q_m, Q_n) \geq n + m + 1$ for all $m \geq 2$ and large n by giving an explicit construction.

1 Introduction

Background and definitions. The classical Ramsey theorem asserts that for any m and n, there is an integer N such that every blue-red edge coloring of the complete graph on N vertices contains a blue clique on m vertices or a red clique on n vertices. Determining the smallest such integer N, known as the Ramsey number is a central problem in combinatorics. More generally, for any two graphs G and H, the Ramsey number is the smallest integer N such that every blue-red edge coloring of the complete graph on N vertices contains a red copy of G or a blue copy of H. Several natural variations of these problems such as multicolor Ramsey numbers, and hypergraph Ramsey numbers are major subjects of ongoing research. For further examples, we refer the reader to the surveys [6, 9].

In this paper, we will study poset Ramsey numbers. A *partially ordered set* (or a *poset* for short) is a set with an accompanying relation $\leq$ which is transitive, reflexive, and antisymmetric. A *Boolean lattice* of dimension n, denoted by Q_n, is the power set of $[n] = \{1, 2, \ldots, n\}$ equipped with the inclusion relation. If $(P, \leq)$ and $(Q, \leq')$ are posets, then an injection $f : P \to Q$ is *order-preserving* if $f(x) \leq' f(y)$ whenever $x \leq y$; we say that $f(P)$ is a *weak copy* of P in Q and that P is a *weak subposet* of Q. An injection $f : P \to Q$ is an *order-embedding* if $f(x) \leq' f(y)$ if and only if $x \leq y$; we say that $f(P)$ is an *induced copy* of P in Q and that P is an *induced subposet* of Q.

The *(induced) poset Ramsey number* $R(Q_m, Q_n)$ is defined to be the smallest integer N such that every blue-red coloring of the elements of the Boolean lattice Q_N contains an induced copy of Q_m whose elements are blue or an induced copy of Q_n whose elements are red. Similarly, the *weak poset Ramsey number* $R_w(Q_m, Q_n)$ is defined to be the smallest integer N such that every blue-red coloring of the elements of the Boolean lattice Q_N contains a weak copy of Q_m whose elements are blue or a weak copy of Q_n whose elements are red. (For convenience, we will call a copy Q_m all of whose elements are blue is called a blue Q_m, and a copy of Q_n all of whose elements are red is called a red Q_n.) It is easy to see that $R(Q_m, Q_n) \geq R_w(Q_m, Q_n)$. Recently, variants of this problem, such as rainbow poset Ramsey numbers have been studied in [4, 5, 7].

Induced poset Ramsey numbers. For the diagonal poset Ramsey number $R(Q_n, Q_n)$, Axenovich and Walzer [1] showed that $2n \le R(Q_n, Q_n) \le n^2 + 2n$. Walzer [10] improved the upper bound to $R(Q_n, Q_n) \le n^2 + 1$. Recently, Lu and Thompson [8] further improved it to $R(Q_n, Q_n) \le n^2 - n + 2$. On the other hand, Cox and Stolee [7] showed that for $n \ge 13$, $R_w(Q_n, Q_n) \ge 2n + 1$, which implies that $R(Q_n, Q_n) \ge 2n + 1$.

More generally, Axenovich and Walzer [1] showed that $n + m \le R(Q_m, Q_n) \le mn + n + m$ for any integers $n, m \ge 1$. Lu and Thompson [8] improved this bound by showing that $R(Q_m, Q_n) \le (m - 2 + \frac{9m-9}{(2m-3)(m+1)})n + m + 3$ for all $n \ge m \ge 4$. See [1, 7, 8, 10] for several other interesting results.

For the off-diagonal poset Ramsey number $R(Q_2, Q_n)$, Axenovich and Walzer [1] showed that $n+2 \le R(Q_2, Q_n) \le 2n+2$. Recently, Lu and Thompson [8] improved the upper bound by proving that $R(Q_2, Q_n) \le \frac{5}{3}n + 2$. In this paper, we determine $R(Q_2, Q_n)$ asymptotically by proving the following theorem.

Theorem 1. *For every $c > 2$, there exists an integer n_0 such that for all $n \ge n_0$, we have*

$$R(Q_2, Q_n) \le n + c\frac{n}{\log_2 n}.$$

Combining Theorem 1 with the lower bound $R(Q_2, Q_n) \ge n + 2$, we obtain that $R(Q_2, Q_n)$ is asymptotically equal to n. It follows from our proof that for all $n \ge 2$, we have $R(Q_2, Q_n) \le n + 6.14\frac{n}{\log_2 n}$. Recently, it has been shown by Axenovich and Winter [2] that a second order term of the form $n/\log_2(n)$ is required. Moreover, they gave an alternate proof of the upper bound.

Weak poset Ramsey numbers. A chain of length k is a poset of k distinct, pairwise comparable elements and is denoted by C_k. Cox and Stolee [7] showed that $R_w(C_k, Q_n) = n+k-1$; since Q_m is a weak subposet of C_{2^m}, this implies that $R_w(Q_m, Q_n) \le n + 2^m - 1$. The lower bound $R_w(Q_m, Q_n) \ge m+n$ is obtained by a simple "layered" coloring of Q_{m+n-1} considered by Axenovich and Walzer [1], which is described as follows. The collection of all subsets of $[N]$ of a given size k is called a *layer*. A coloring of Q_N is *layered* if for every layer, all sets on that layer have the same color. A layered coloring of Q_{m+n-1} with m blue layers and n red layers does not contain a (weak) blue copy of Q_m or a (weak) red copy of Q_n. Therefore, $R_w(Q_m, Q_n) \ge m + n$ (which implies $R(Q_m, Q_n) \ge m + n$). Despite the work of several researchers, so far this lower bound on $R_w(Q_m, Q_n)$ has not been improved except in the diagonal case: Cox and Stolee [7] showed that $R_w(Q_n, Q_n) \ge 2n + 1$ for $n \ge 13$ using a probabilistic construction. Recently, in the induced case, Bohman and Peng [3] gave an explicit construction showing the bound $R(Q_n, Q_n) \ge 2n + 1$. Note that these constructions showing $R(Q_n, Q_n) \ge 2n + 1$ cannot be layered.

We give an explicit construction which yields a lower bound on $R_w(Q_m, Q_n)$ for all m and $n \ge 68$, thereby generalizing the results of Bohman and Peng to the weak poset case, and additionally extending their results and those of Cox and Stolee to the off-diagonal case.

Theorem 2. *For any $m \ge 2$ and $n \ge 68$, we have*

$$R_w(Q_m, Q_n) \ge m + n + 1.$$

Note that Theorem 2 shows that $R_w(Q_2, Q_n) = n + 3$ since $R_w(Q_2, Q_n) \le n + 2^2 - 1 = n + 3$ by the upper bound mentioned earlier.

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