Matching Theory and Barnette's Conjecture

Maximilian Gorsky — Technische Universität Berlin, Germany
Raphael Steiner — ETH Zürich, Switzerland
Sebastian Wiederrecht — LIRMM, University of Montpellier, France

Abstract

Barnette's Conjecture claims that all cubic, 3-connected, planar, bipartite graphs are Hamiltonian. Using a matching-theoretic perspective, we can relax the requirement of planarity to give the equivalent conjecture that all cubic, 3-connected, Pfaffian, bipartite graphs are Hamiltonian.

A graph, other than the path of length three, is a brace if it is bipartite and any two disjoint edges are part of a perfect matching. We observe that Barnette's Conjecture can be reduced to cubic, planar braces. We show a similar reduction to braces for cubic, 3-connected, bipartite graphs regarding four stronger versions of Hamiltonicity. Note that in these cases we do not need planarity. As a practical application of these results, we provide some supplements to a generation procedure for cubic, 3-connected, planar, bipartite graphs discovered by Holton et al. in 1985. These allow us to check whether a graph we generated is a brace.

1 Introduction

In 1884, Tait conjectured that every cubic, 3-connected, planar graph is Hamiltonian [Tai84], which Tutte disproved in 1946 [Tut46]. Subsequently, Tutte conjectured that the result might hold if the graph was cubic, 3-connected, and bipartite [Tut71], which was disproven by Horton [Hor82]. Since all known counterexamples to Tait's conjecture are non-bipartite and all known counterexamples to Tutte's conjecture are non-planar, Barnette's conjecture from 1969 seems like a sensible compromise. Let $\mathcal{B}$ be the class of all cubic, 3-connected, planar, bipartite graphs.

Conjecture 1 (Barnette [Bar69])**.** *All graphs in $\mathcal{B}$ are Hamiltonian.*

This conjecture has received considerable attention and is known to be true for some subclasses of $\mathcal{B}$, but an approach to the problem in general continues to be elusive. Though the recent resolution of the related conjecture that all cubic, 3-connected, planar graphs with faces of size at most six are Hamiltonian, proven by Kardoš [Kar20], provides some hope. Significant effort has also been devoted to finding strengthenings of Conjecture 1; that is statements which also concern $\mathcal{B}$ but demand more than a simple Hamiltonian cycle (for an overview see [Her05]).

In a graph G, we call a set of mutually disjoint edges a *matching* and we call it a *perfect matching* if the union of the edges in the set contains all vertices of G. A connected graph G with $|V(G)| \geq 2k+2$ is called *k-extendable* if for every matching $F \subseteq E(G)$ with $|F| \leq k$ there exists a perfect matching M with $F \subseteq M$. If G is isomorphic to C_4 or a bipartite, 2-extendable graph, it is called a *brace*. The following can be derived from a combination of a result by Kelmans [Kel86] and a result by Holton and Plummer [HP91].

Lemma 1. *Conjecture 1 holds if and only if every cubic, planar brace is Hamiltonian.*

2 Pfaffian graphs, tight cuts, and Hamiltonicity

Lemma 1 leads us to an extension of Barnette's Conjecture, which despite Tutte's Conjecture on the matter being false, is stated for a class of cubic, 3-connected, non-planar, bipartite graphs. In a graph G, a cycle C is called M-conformal, for a perfect matching M of G, if half of the edges of C are contained in M. A graph is then G is called *Pfaffian* if it has an orientation of its edges such that for every perfect matching M of G and every M-conformal cycle C, the number of edges in C oriented in the same way is odd, for either direction of traversal. Pfaffian graphs have a myriad of connections to other problems, both in mathematics and outside of it (see [McC04] for an excellent overview), and they are a central object of study in Matching Theory.

Theorem 1. *Conjecture 1 holds if and only if all cubic, 3-connected, Pfaffian, bipartite graphs are Hamiltonian.*

At the core of the proof of this result lie a few lemmas concerning the interaction of Hamiltonian cycles and a certain type of cut central to structural Matching Theory. We call a connected graph G *matching covered* if it is 1-extendable. A cut $\partial(X)$ in a matching covered graph G is called *tight* if for any perfect matching M of G, we have $|\partial(X) \cap M| = 1$. We call a tight cut *non-trivial* if each side of the cut contains at least two vertices. If we contract one of the sides of a tight cut into a single vertex, we call the resulting graph a *tight cut contraction*. By repeatedly searching out non-trivial tight cuts and repeating this contraction operation on the resulting tight cut contractions, we arrive at a list of graphs which do not have non-trivial tight cuts. Such graphs are called bricks if they are non-bipartite and braces otherwise. This procedure is known as the *tight cut decomposition* due to Lovász [Lov87].

In the general setting of undirected graphs, we note that any Hamiltonian cycle in a graph G with an even number of vertices can be split into two perfect matchings. Thus, to use the tight cut decomposition, one simply needs to restrict themselves to the subgraph induced by those edges that are contained in a perfect matching. This graph is known as the *cover graph* of G. The following general result is not hard to prove, but has gone unstated so far.

Theorem 2. *The bricks and braces of the cover graph of a Hamiltonian graph with an even number of vertices are Hamiltonian.*

The reverse direction is not true in general, as can be seen by performing a tight cut decomposition on the smallest cubic, 2-connected, planar, bipartite graph not containing a Hamiltonian cycle found by Asano et al. [ASEH82]. Nonetheless, motivated by the explanatory power the reverse direction promises, much of our work focuses on finding graphs classes and stronger Hamiltonicity properties for which it holds. One promising candidate are k-regular, connected, bipartite graphs. A classic result by Kőnig [Kön16] implies that they are matching covered, providing a good basis for their study using Matching Theory.

We present the next result, which is also used in the proof of Theorem 1, that illustrates that, given the right assumptions, a reverse version of Theorem 2 can be recovered. We call a graph G with at least k vertices P_k-*Hamiltonian* if any path of length $k-1$ in G is contained in some Hamiltonian cycle of G. Furthermore, we say that a graph G has the H^--property, respectively the H^{+-}-property, if for any edge in G there exists a Hamiltonian cycle in G which avoids said edge, respectively for any two distinct edges e and f there exists a Hamiltonian cycle containing e and avoiding f.

Theorem 3. *Let G be a cubic, 3-connected, bipartite graph.*

1. *If the braces of G are P_4-Hamiltonian, then G is P_4-Hamiltonian.*

2. *G has the H^--property if and only if the braces of G have the H^--property.*

3. G is P_3-Hamiltonian if and only if the braces of G are P_3-Hamiltonian.

4. G has the H^{+-}-property if and only if the braces of G have the H^{+-}-property.

Furthermore, in the study of Pfaffian braces, another operation has proved to be very useful, as it allows one to construct all Pfaffian braces (except for one) from planar braces (see [McC04] and [RST99]). Let G_1, G_2, G_3 be three bipartite graphs such that their pairwise intersection is a cycle C of length four and we have $V(G_i) \setminus V(C) \neq \emptyset$ for all $i \in \{1, 2, 3\}$. Further, let $S \subseteq E(C)$ be some subset of the edges of C. A *trisum of G_1, G_2, G_3 at C* is a graph $(\bigcup_{i=1}^{3} G_i) - S$. We call a trisum *cubic* if $S = E(C)$. Note that the cubic trisum of three cubic, bipartite graphs is itself again cubic. In this context, we provide the following result, again used in the proof of Theorem 1.

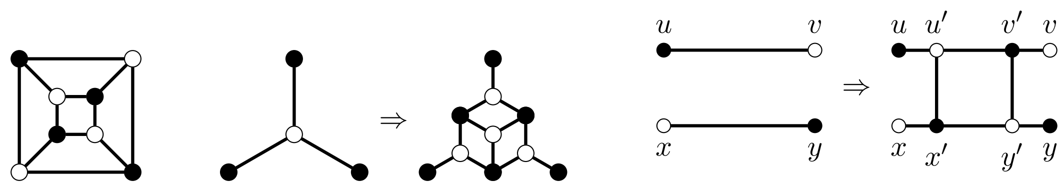
Theorem 4. *Let G_1, G_2, and G_3 be cubic, Pfaffian braces and let G be the result of a cubic trisum.*

1. If G_1, G_2, and G_3 are P_4-Hamiltonian, then G is P_4-Hamiltonian.

2. If G is Hamiltonian, then G_1, G_2, and G_3 are Hamiltonian.

3 Checking Hamiltonicity for Barnette's graphs more efficiently

The first large computational effort with regards to Conjecture 1 was undertaken by Holton et al. [HMM85] in 1985, which also introduced an elegant generation method for the graphs in $\mathcal{B}$ that is still used in the most recent computational efforts (see [BGM21]). In particular Holton et al. show that all graphs in $\mathcal{B}$ can be generated from the cube using the two operations depicted in Figure 1.



(a) The cube. (b) The cube-expansion. (c) The C_4-expansion.

Figure 1: The cube expansion can be used independent of the colour of the central vertex. The C_4-expansion must be used on two edges inside a facial cycle C such that the removal of the two edges from C results in two odd paths.

In [HMM85], Conjecture 1 is confirmed for graphs in $\mathcal{B}$ with up to 64 vertices and [BGM21] improves this to 90 vertices using the same method. A key part of their argument requires checking whether a given graph in $\mathcal{B}$ has the H^{+-}-property. Afterwards a number of reductions allow one to reduce larger graphs to graphs with the H^{+-}-property to verify their Hamiltonicity. Since this has to be done for all graphs in $\mathcal{B}$ up to certain number of vertices and the H^{+-}-property in particular implies Hamiltonicity, which on its own is already NP-complete, this is quite computationally expensive.

We provide supplements to the generation procedure from [HMM85] that allow us to determine the tight cut structure of each newly generated graph. These additions can be computed in linear time for each newly generated graph G and take up at most $3/7|V(G)|$ additional space per generated graph. Using this modified procedure, we can recognize whether a generated graph is a brace, since braces have no non-trivial tight cuts. This significantly narrows the number of graphs for which we have to check the H^{+-}-property, thanks to the fourth point of Theorem 3.

References

[ASEH82] Takao Asano, Nobuji Saito, Geoffrey Exoo, and Frank Harary. The smallest 2-connected cubic bipartite planar nonhamiltonian graph. *Discrete Mathematics*, 38(1):1–6, 1982.

[Bar69] David W Barnette. Conjecture 5. *Recent Progress in Combinatorics (W. T. Tutte, ed.)*, 343, 1969.

[BGM21] Gunnar Brinkmann, Jan Goedgebeur, and Brendan McKay. The Minimality of the Georges-Kelmans Graph. Sep 2021. URL: http://dx.doi.org/10.1090/mcom/3701, doi:10.1090/mcom/3701.

[Her05] Alexander Hertel. A Survey & Strengthening of Barnette's Conjecture. Technical report, University of Toronto, 2005. URL: http://www.cs.toronto.edu/~ahertel/WebPageFiles/Papers/StrengtheningBarnette'sConjecture10.pdf.

[HMM85] Derek A Holton, Bennet Manvel, and Brendan D McKay. Hamiltonian Cycles in Cubic 3-Connected Bipartite Planar Graphs. *Journal of Combinatorial Theory, Series B*, 38:279–297, 1985.

[Hor82] Joseph D Horton. On Two-Factors of Bipartite Regular Graphs. *Discrete Mathematics*, 41(1):35–41, 1982.

[HP91] Derek A Holton and Micheal D Plummer. Matching Extension and Connectivity in Graphs ii. *Graph Theory, Combinatorics, and Applications*, 2:651–665, 1991.

[Kar20] František Kardoš. A Computer-Assisted Proof of the Barnette–Goodey Conjecture: Not Only Fullerene Graphs Are Hamiltonian. *SIAM Journal on Discrete Mathematics*, 34(1):62–100, 2020.

[Kel86] Alexander K Kelmans. Konstruktsii Kubicheskih Dvudolnyh 3-Svyaznyh Bez Gamiltonovyh Tsiklov. *Sb. Tr. VNII Sistem. Issled*, 10:64–72, 1986.

[Kön16] Dénes König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Mathematische Annalen*, 77(4):453–465, 1916.

[Lov87] László Lovász. Matching Structure and the Matching Lattice. *Journal of Combinatorial Theory, Series B*, 43(2):187–222, 1987.

[McC04] William McCuaig. Pólya's Permanent Problem. *Electronic Journal of Combinatorics*, pages R79–R79, 2004.

[RST99] Neil Robertson, Paul D Seymour, and Robin Thomas. Permanents, Pfaffian Orientations, and Even Directed Circuits. *Annals of Mathematics*, pages 929–975, 1999.

[Tai84] Peter G Tait. Listing's Topologie. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 17(103):30–46, 1884.

[Tut46] William T Tutte. On Hamiltonian Circuits. *Journal of the London Mathematical Society*, 1(2):98–101, 1946.

[Tut71] William T Tutte. On the 2-Factors of Bicubic Graphs. *Discrete Mathematics*, 1(2):203–208, 1971.