Contact graphs of boxes with unidirectional contacts

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Abstract

This paper is devoted to the study of particular classes of geometrically defined intersection classes of graphs. Those were previously studied in [6], where the authors shown that these graphs have arbitrary large chromatic number, while being triangle free. We give several structural properties of these graphs, and we raise several questions. These graphs have the particular feature to serve as a counter-example for a conjecture in [2].

1 Introduction

A lot of graph classes studied in the literature are defined by a geometric models, where vertices are represented by geometric objects (e.g. intervals on a line, disks in the plane, chords inscribed in a circle...) and the adjacency of two vertices is determined according to the relation between the corresponding objects. A large amount of graph classes consider the intersection relation (e.g. interval graphs, disk graphs or circle graphs). However, some other relations might be considered such as the containment, the overlap or also the contact between objects. In this note, we consider a new class defined by a contact model. More precisely, we consider the class of graphs defined by contact of axis parallel boxes in $\mathbb{R}^d$ where the contact occurs on $(d-1)$-dimensional box in only one direction (CBU).

The motivation for this class of graph originates from an article of Magnant and Martin [7] where a wireless channel assignment is considered for a building with rectangular rooms. We start here a study on the structural and algorithmic properties of this class.

Formally, given a basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$, we consider interior disjoint d-dimensional axis-parallel boxes. Furthermore, two such boxes are only allowed to intersect on a $(d-1)$-dimensional box orthogonal to e_1. This class of graphs is denoted by *d-CBU*, for *Contact* graphs of d-dimensional *Boxes* with *Unidirectional* contacts. We denote *CBU* the union of d-CBU for all d. Note that 1-CBU correspond to the forests of paths, and the major property is that, *for every $d \geq 1$, d-CBU graphs are triangle-free.*

This can be shown by orienting the edges according to vector e_1 and labeling each arc with the coordinate of the corresponding $(d-1)$-hyperplane, one obtains an acyclic orientation such that for every vertex, all the outgoing arcs have the same label, all the ingoing arcs have the same label, and the label of ingoing arcs is smaller than the label of outgoing arcs. We call such a labeling of the arcs an *homogeneous arc labeling* (See an example in Figure 1).

2 Boxicity

The *boxicity* $box(G)$ of a graph G, is the minimum dimension d such that G admits an intersection representation with axis-aligned boxes. Of course, the graphs in d-CBU have boxicity at most d. The converse cannot hold for the graphs containing a triangle, as those are not in CBU. However, some relations hold for triangle-free graphs.

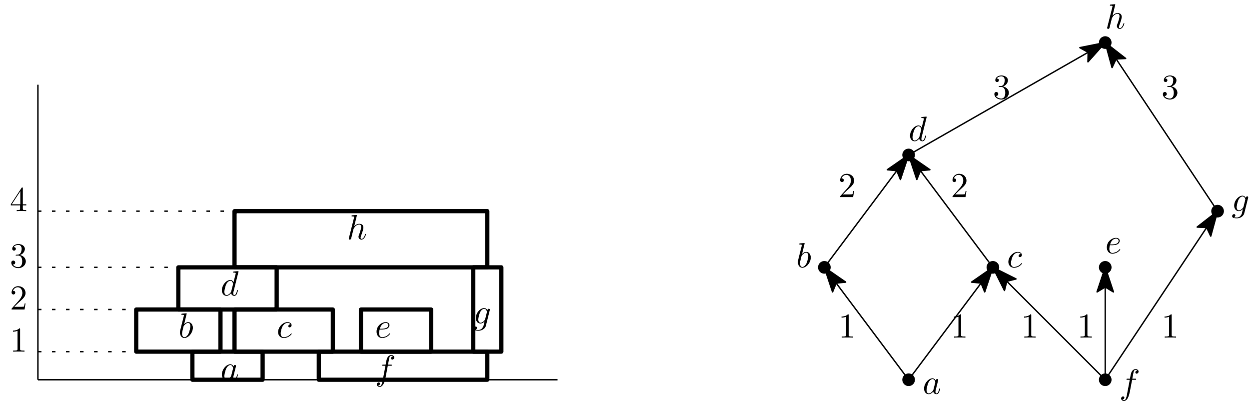

Figure 1: An example of 2-CBU graph and its associated acyclic orientation

Theorem 1. *Every bipartite graphs of boxicity b belongs to $(b+1)$-CBU.*

Theorem 1 does not extend to triangle-free graphs (see Lemma 4). Nevertheless, subdividing the edges enables to consider every triangle-free graph.

Theorem 2. *For every triangle-free graphs G of boxicity b, the 1-subdivision of G belongs to $(b+1)$-CBU.*

3 Planar graphs

While planar graphs have boxicity 3 [11], many subclasses of planar graphs are known to have boxicity at most 2. This is the case for 4-connected planar graphs [10], and their subgraphs. The subgraphs of 4-connected graphs include every triangle-free planar graph. By Theorem 1, we thus have the following.

Corollary 3. *Every bipartite planar graph belongs to 3-CBU.*

This property does not generalize, in a strong sense, to triangle-free planar graphs.

Lemma 4. *There exists a girth 5 planar graph that is not CBU.*

However, several subclasses of planar graphs belong to 2-CBU.

Theorem 5. *Every triangle-free outerplanar graph is 2-CBU.*

Theorem 6. *There are series-parallel graphs of girth 6 that are not 2-CBU.*

Problem 7. *Is there a girth g such that every series parallel graph of girth at least g belongs to 2-CBU ?*

We know that for planar graphs, such property does not hold.

Theorem 8. *For any $g \geq 3$, there are planar graphs of girth g that do not belong to 2-CBU.*

4 Structural properties of CBU

Theorem 9. *For every $d \geq 1$, the class of d-CBU graphs is strictly contained in the class of $(d+1)$-CBU graphs.*

It is clear that CBU is hereditary (i.e. closed under induced subgraphs) but actually it is also closed under subgraphs.

Theorem 10. *For any subgraph H of G, $G \in$ CBU implies that $H \in$ CBU.*

As every complete bipartite graph belongs to 3-CBU.

Corollary 11. *Every bipartite graph belongs to CBU.*

Problem 12. *Is there a g such that any graph G of girth at least g is in CBU ?*

Finally, we provide a characterization of CBU graphs in terms of homogenous arc labeling.

Theorem 13. *A graph G belongs to CBU if and only if it admits an (acyclic) orientation with an homogeneous arc labeling.*

5 Recognition

Theorem 14. *It is NP-complete to decide whether a graph belongs to d-CBU for any integer $d \geq 3$.*

Problem 15. *Is it Polynomial/NP-complete to recognize 2-CBU graphs ?*

It is proven in [1] that it is not possible to approximate the boxicity of bipartite graph within a $O(n^{0.5-\varepsilon})$-factor unless $NP = ZPP$.

Theorem 16. *Unless $NP = ZPP$, for a bipartite graph G, one cannot approximate, within a $O(n^{0.5-\varepsilon})$-factor, the minimum value d such that G is d-CBU.*

Problem 17. *Is it polynomial to decide whether a graph G belongs to CBU ?*

We have seen that some triangle-free planar graphs are not in CBU, thus :

Problem 18. *Is it polynomial to decide whether a planar graph G belongs to CBU ?*

6 Chromatic Number and Independent Sets

While 2-CBU graphs have chromatic number at most 3 (by Grotchtch's theorem), 3-CBU graphs have unbounded chromatic number.

Theorem 19 (Magnant and Martin [6])**.** *For any $\chi \geq 1$, there exist a graph in 3-CBU, with chromatic number χ.*

Note that as these graphs are triangle-free, they also have unbounded co-chromatic number. However, these graphs have large independent sets.

Theorem 20. *For any graph $G \in$ CBU, $\alpha(G) \geq |V(G)|/4$.*

This contradicts Conjecture 1 of [2].

References

[1] Abhijin Adiga, Diptendu Bhowmick, and L. Sunil Chandran. The hardness of approximating the boxicity, cubicity and threshold dimension of a graph. *Discret. Appl. Math.*, 158(16):1719–1726, 2010.

[2] A. Atminas, V. Lozin, and V. Zamaraev. Linear ramsey numbers. In *Proc. of IWOCA 2018*, pages 26–38. Springer, 2018.

[3] H. de Fraysseix, P. Ossona de Mendez, and J. Pach. Representation of planar graphs by segments. *Intuit. Geom. (Szeged, 1991), Colloq. Math. Soc. János Bolyai*, 63:109–117, 1994.

[4] Hubert de Fraysseix, Patrice Ossona de Mendez, and Pierre Rosensthiel. On triangle contact graphs. *Probability and Computing*, 3, 1994.

[5] Pavel Irzhavsky. On 2-subdivision graphs we have the following algorithms and reductions. www.graphclasses.org.

[6] C. Magnant and D. M. Martin. Coloring rectangular blocks in 3-space. *Discussiones Mathematicae Graph Theory*, 31:161–170, 2011.

[7] Colton Magnant and Daniel L. Martin. Coloring rectangular blocks in 3-space. *Discussiones Mathematicae Graphe Theory*, 31, 2011.

[8] Svatopluk Poljak. A note on stable sets and colorings of graphs. *Commentationes Mathematicae Universitatis Carolinae*, 15(2):307 – 309, 1974.

[9] Fred S. Roberts. On the boxicity and cubicity of a graph. In William T. Tutte, editor, *Recent Progress in Combinatorics*, pages 301–310.

[10] C. Thomassen. Plane representations of graphs. *Progress in graph theory (Bondy and Murty, eds.)*, pages 336–342, 1984.

[11] C. Thomassen. Interval representations of planar graphs. *J. Combin. Theory Ser. B*, 40, pages 9–20, 1986.