Minimally tough graphs in special classes

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Abstract

Let t be a real number. A graph is called t-tough if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. We investigate the minimum degree and the recognizability of minimally t-tough graphs in the classes of chordal graphs, split graphs, claw-free graphs, and $2K_2$-free graphs.

1 Introduction

All graphs considered here are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number and $\kappa(G)$ denote the connectivity number of the graph G. (Using $\omega(G)$ to denote the number of components may be confusing, however, most of the literature on toughness uses this notation.)

The notion of toughness was introduced by Chvátal in [9].

Definition 1. *Let t be a real number. A graph G is called* t-tough *if*

$$\omega(G-S) \leq \frac{|S|}{t}$$

for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e. for any $S \subseteq V(G)$ with $\omega(G-S) > 1$). The toughness *of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.*

Definition 2. *A graph G is said to be* minimally t-tough *if $\tau(G) = t$ and $\tau(G-e) < t$ for all $e \in E(G)$.*

It follows directly from the definition that every t-tough noncomplete graph is $2t$-connected, implying $\kappa(G) \geq 2\tau(G)$ for noncomplete graphs. Therefore, the minimum degree of any t-tough noncomplete graph is at least $\lceil 2t \rceil$.

The following conjecture is motivated by a theorem of Mader [10], which states that every minimally k-connected graph has a vertex of degree k.

Conjecture 1 (Kriesell [8])**.** *Every minimally 1-tough graph has a vertex of degree 2.*

This conjecture can be naturally generalized.

Conjecture 2 (Generalized Kriesell's Conjecture)**.** *Every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$.*

In [4] we proved that every minimally 1-tough graph has a vertex of degree at most $n/3 + 1$.

In general, determining whether a graph is minimally t-tough is hard, in [5] we proved that it is DP-complete for any positive rational number t. DP-complete problems are believed to be even harder than NP-complete ones. For more details about the complexity class DP, see [1].

Proving the generalized Kriesell's conjecture seems to be difficult, so the main motivation is to study the conjecture in some special graph classes. It turns out that in the classes of chordal graphs, split graphs, claw-free graph and for some values of t we can prove the conjecture by giving a characterization of the special graph class, which easily implies the affirmative answer to the conjecture. In the case of $2K_2$-free graphs, the conjecture is still open, but we can at least show that these graphs can be recognized in polynomial time, which may give a chance to find a characterization later.

2 Chordal graphs

Definition 3. *A graph is* chordal *if it does not contain an induced cycle of length at least* 4.

Theorem 1. *Let $t \leq 1$ be a positive rational number.*

- *If $t \leq 1/2$, then all minimally t-tough, chordal graphs can be obtained from a tree of maximum degree $\Delta \geq 3$ by removing some (or all) of its vertices with degree 3 whose neighbors have degree Δ, and joining these neighbors with triangle.*
- *If $1/2 < t \leq 1$, then there exist no minimally t-tough, chordal graphs.*

This implies that Conjecture 2 holds for chordal graphs if $t \leq 1$, but the conjecture is open for larger t values.

3 Split graphs

Definition 4. *A graph is a* split *graph if its vertices can be partitioned into a clique and an independent set.*

The toughness of split graphs can be computed in polynomial time. First, it was shown for $t = 1$ in [2], then in [3] for all positive rational number t.

Theorem 2 ([3])**.** *For any rational number $t > 0$, the class of t-tough, split graphs can be recognized in polynomial time.*

Such an algorithm turns out to be helpful in giving a complete characterization of minimally t-tough, split graphs.

Theorem 3. *For any rational number $t > 1/2$ there exist no minimally t-tough, split graphs.*

Theorem 4. *Let $t \leq 1/2$ be an arbitrary positive rational number and G a minimally t-tough, split graph partitioned into a clique C and an independent set I. Then there exists an integer $b \geq 2$ for which $t = 1/b$, and $|C| \leq 3$. Moreover,*

- *either G is a tree with at most two internal vertices and with $\Delta(G) = b$,*

- *or* $|C| = 3$, *every vertex in* I *has degree* 1 *and every vertex in* C *has degree* $b + 1$.

Corollary 1. *Let* t *be a positive rational number. If* G *is a minimally* t*-tough, split graph, then* G *has a vertex of degree* $\lceil 2t \rceil$.

So Conjecture 2 holds for split graphs.

4 Claw-free graphs

In [7] it was shown that the toughness of any claw-free graph is either an integer or half of an integer. In this section we deal with minimally 1-tough and minimally 1/2-tough, claw-free graphs.

Definition 5. *The graph* $K_{1,3}$ *is called a* claw. *A graph is said to be* claw-free *if it does not contain a claw as an induced subgraph.*

Theorem 5 ([7])**.** *If* G *is a noncomplete claw-free graph, then* $2\tau(G) = \kappa(G)$.

Corollary 2. *For any rational number* $t > 0$ *the class of* t*-tough, claw-free graphs can be recognized in polynomial time.*

We have a characterization of of t-tough, claw-free graphs if $t \leq 1$.

Theorem 6 ([4])**.** *The class of minimally* 1*-tough, claw-free graphs consists of the cycles of length at least* 4.

Theorem 7. *The class of minimally* $1/2$*-tough, claw-free graphs are exactly those graphs that can be built up in the following way.*

- *Take a tree* T *with maximum degree at most 3 where the set of vertices of degree 1 and 3 together form an independent set.*
- *Now delete every vertex of degree 3, but connect its 3 neighbors with a triangle.*

Since the toughness of any claw-free graph is either an integer or half of an integer, in the class of claw-free graphs Conjecture 2 is true for all rational number $0 < t \leq 1$, but it is open for larger t values.

5 $2K_2$-free graphs

For split graphs and claw-free graphs it was useful to be able to compute the toughness in polynomial time. In this section we prove that for any positive rational number t the class of minimally t-tough, $2K_2$-free graphs can be recognized in polynomial time. This might be helpful in obtains results connected to Conjecture 2.

Definition 6. *A graph is said to be* $2K_2$*-free if it does not contain an independent pair of edges as an induced subgraph.*

Theorem 8 ([6])**.** *For any rational number* $t > 0$ *the class of* t*-tough,* $2K_2$*-free graphs can be recognized in polynomial time.*

Theorem 9. *For any positive rational number* t *the class of minimally* t*-tough,* $2K_2$*-free graphs can be recognized in polynomial time.*

Conjecture 2 is open for every positive rational t.

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