Introducing lop-kernels: a framework for kernelization lower bounds

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Abstract

We introduce a simple general framework to obtain kernelization lower bounds for a certain type of kernels for optimization problems, which we call lop-*kernels*. Informally, this type of kernels is required to preserve large optimal solutions in the reduced instance, and captures the vast majority of existing kernels in the literature. As a consequence of this framework, we show that the trivial quadratic kernel for Maximum Minimal Vertex Cover (MMVC) is essentially optimal, answering a question of Boria et al. [Discret. Appl. Math. 2015], and that the known cubic kernel for Maximum Minimal Feedback Vertex Set is also essentially optimal. We present further applications for Tree Deletion Set and for Maximum Independent Set on K_t-free graphs.[1].

1 Context

A *vertex cover* in a graph G is a subset of vertices containing at least one endpoint of every edge. In the associated optimization problem, called Minimum Vertex Cover, the objective is to find, given an input graph G, a vertex cover in G of minimum size. This problem has been one of the leitmotifs of the area of parameterized complexity [6, 8], serving as a test bed for many of the most fundamental techniques. An instance of a *parameterized problem* is of the form (x, k), where x is the total input (typically, a graph) and k is a positive integer called the *parameter*. The crucial notion is that of *fixed-parameter tractable* algorithm, FPT for short, which is an algorithm deciding whether (x, k) is a positive instance in time $f(k) \cdot |x|^{\mathcal{O}(1)}$, where f is a computable function depending only on k. In the parameterized Vertex Cover problem, we are given a graph G and an integer parameter k, and the objective is to decide whether G contains a vertex cover of size at most k. One of the main fields within parameterized complexity is *kernelization* [13], where the objective is to decide whether an instance (x, k) of a parameterized problem can be transformed in polynomial time into an equivalent instance (x', k') whose total size is bounded by a function of k; the reduced instance is called a *kernel*, and finding kernels of small size, typically polynomial or even linear in k in the best case, is one of the most active areas of parameterized complexity.

Considering the "max-min" version of minimization problems, that is, maximizing the size of a *minimal* solution of the corresponding problem, is a natural approach that has been applied to several problems such as Dominating Set [3, 10] (whose "max-min" version is called Upper Domination), Feedback Vertex Set [9], or Hitting Set [7, 2]. The initial motivation of this article is the "max-min" version of Minimum Vertex Cover, called Maximum Minimal Vertex Cover, or just MMVC for short.

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2 Previous work on kernelization of MMVC

Fernau [12] presented FPT algorithms for MMVC as well as some results about its kernelization parameterized by the solution size k. It is easy to note, as observed in [12], that the problem admits a kernel with at most k^2 vertices. Boria et al. [5] initiated a study of the complexity of MMVC and presented a number of results, in particular a polynomial-time approximation algorithm with ratio $n^{1/2}$ on n-vertex graphs, and showed that, unless $\mathsf{P} = \mathsf{NP}$, no polynomial-time approximation algorithm with ratio $\mathcal{O}(n^{1/2-\varepsilon})$ exists for any $\varepsilon > 0$. The authors asked explicitly whether kernels of size $o(k^2)$ exist for MMVC parameterized by k.

3 Our results and techniques

The starting motivation of this article is the kernelization of the MMVC problem. This initial motivation has resulted in a general framework that can be applied to a broad class of optimization problems in order to derive kernelization lower bounds. Namely, motivated by the question of Boria et al. [5] about the existence of subquadratic kernels for MMVC, we introduce a generic framework to obtain kernelization lower bounds for a "certain type" of kernels (called lop-kernels) for parameterized maximization or minimization problems (in particular, for MMVC), based on a hypothesis that guarantees an inapproximability result, typically $\mathsf{P} \neq \mathsf{NP}$. The following definitions of lop-rule and lop-kernel are specialized for vertex-maximization problem (where the input is a graph, and the output is a subset of vertices), and we refer the reader to the full version for the more general definitions.

Definition 1. *A* large optimal preserving *reduction rule, or* lop-rule *for short, for a vertex-maximization problem* Π*, is a polynomial-time algorithm* R *that, given a pair* (G, k)*, where* G *is a graph and* k *is a positive integer, computes another pair* (G', k') *with* $0 \leq k' \leq k$ *such that*

1. *if* (G, k) *is a* no*-instance of* Π*, then* (G', k') *is a* no*-instance of* Π*, and*
2. *if* (G, k) *is a* yes*-instance of* Π*, then* $opt(G') \geq opt(G) - (k - k')$*, implying that* (G', k') *is a* yes*-instance of* Π*.*

Definition 2. *Let* Π *be a vertex-maximization problem and let* $s : \mathbb{N} \to \mathbb{N}$ *be a computable function. A* lop-kernel of size s *for* Π *parameterized by the solution size is a polynomial-time algorithm that takes as input an instance* (G, k)*, produces a reduced instance* (G', k') *by applying a (possibly empty) sequence of* lop*-rules to* (G, k)*, and either*

- *determines that* (G', k') *is a* yes*-instance or a* no*-instance, or*
- *outputs* (G', k') *with* $|V(G')| \leq s(k)$*.*

Even if this type of kernels may seem restrictive, in particular we are not aware of any "non-artificial" kernel for a maximization problem, such as those that have become nowadays standard [13], which is *not* a lop-kernel. We do have such an example for a minimization problem, as discussed later.

The core result of our approach is the following result, showing that a lop-kernel yields a polynomial-time approximation algorithm whose ratio depends on the size of the kernel. Thus, using known inapproximability results allows to obtain the desired lower bound on lop-kernel size.

Theorem 1. *Let Π be a vertex-maximization problem whose decision version is in* NP*.*

1. *For every real $c > 1$, if Π admits a* lop*-kernel with $\mathcal{O}(k^c)$ vertices, then it admits a polynomial-time value-approximation algorithm with ratio $\mathcal{O}(n^{\frac{c-1}{c}})$ on n-vertex graphs.*
2. *For every real $\beta \geq 1$, if Π admits a* lop*-kernel with βk vertices, then for any real $\varepsilon > 0$, it admits a polynomial-time value-approximation algorithm with ratio $(\beta + 1 + \varepsilon)$.*

4 Applications of our framework

Combining the previous property with the $\mathcal{O}(n^{\frac{1}{2}-\varepsilon})$-inapproximability result for MMVC by Boria et al. [5] immediately rules out the existence of a lop-kernel for MMVC with $\mathcal{O}(k^{2-\varepsilon})$ vertices for any $\varepsilon > 0$, unless P = NP. Thus, while it does not completely rule out the existence of subquadratic kernels for MMVC, it tells that, if such a kernel exists, it should consist of "non-standard" reduction rules.

Interestingly, our framework has consequences beyond the MMVC problem. One of them concerns the Maximum Minimal Feedback Vertex Set (MMFVS) problem, defined in the natural way. Dublois et al. [9] recently provided a cubic kernel for MMFVS parameterized by the solution size, and proved that the problem does not admit an $\mathcal{O}(n^{\frac{2}{3}-\varepsilon})$-approximation algorithm for any $\varepsilon > 0$, unless P = NP. Again, our framework directly implies that the cubic kernel of Dublois et al. [9] is "essentially" optimal.

Another application of our results deals with the Tree Deletion Set problem. In this case, the fact that this problem does not admit a polynomial-time $\mathcal{O}(n^{1-\varepsilon})$-approximation for any $\varepsilon > 0$ unless P $\neq$ NP [17] implies using our framework that Tree Deletion Set parameterized by the solution size does not admit a polynomial lop-kernel, unless P = NP. However, Tree Deletion Set *does* admit a polynomial kernel with $\mathcal{O}(k^4)$ vertices [14]. Therefore, this polynomial kernel cannot be a lop-kernel, and so far it constitutes the only non-artificial example of non-lop-kernel that we are aware of.

Our last application concerns the Maximum Independent Set problem restricted to K_t-free graphs. In particular, we show that a lop-kernel with $\mathcal{O}(k^{t-1-\varepsilon})$ vertices for Maximum Independent Set on K_t-free graphs would improve the best known approximation ratio $n^{\frac{t-2}{t-1}}$ that follows from Ramsey's theorem [16]. Finally, generalizing a conjecture of Bonnet et al. [4], we conjecture that for every fixed graph H, the Maximum Independent Set problem restricted to H-free graphs admits a polynomial lop-kernel.

5 Other results on the kernelization of MMVC

Coming back to the MMVC problem parameterized by the solution size, given the above negative result on general graphs, we identify graph classes where MMVC is still NP-hard and admits a subquadratic kernel. In particular, we deal with graph classes defined by excluding an *induced* subgraph H that satisfies the *Erdős-Hajnal property* [11], that is, for which there exists a constant $\delta > 0$ such that every H-free graph on n vertices contains either a clique or an independent set of size n^δ. In particular, we present a kernel for MMVC with $\mathcal{O}(k^{7/4})$ vertices on the well-studied class of bull-free graphs, with $\mathcal{O}(k^{\frac{2t-3}{t-1}})$ vertices on K_t-free graphs graphs for every $t \geq 3$, and with $\mathcal{O}(k^{5/3})$ vertices on paw-free graphs. To the best of our knowledge, this is the first time that the Erdős-Hajnal property is used to obtain polynomial kernels (we would like to note that it was used by Kratsch et al. [15] to obtain kernelization lower bounds).

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