Using Edge Contractions and Vertex Deletions to Reduce the Independence Number and the Clique Number

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Abstract

For a graph G and two integers k and d, can we apply a graph operation at most k times to reduce a given graph parameter π by at least d? We show that this problem is NP-hard when the parameter is independence number and the graph operation is vertex deletion or edge contraction, even for $d = 1$ and when restricted to chordal graphs. We also give a polynomial time algorithm for bipartite graphs when the operation is edge contraction, the parameter is independence number and d is fixed. Further, we complete the complexity dichotomy on H-free graphs when the parameter is the clique number and the operation is edge contraction by showing that this problem is NP-hard in (C_3+P_1)-free graphs even for fixed $d = 1$. This answers open questions stated in [Diner et al., Theoretical Computer Science, 746, p. 49-72 (2012)].

1 Introduction

Blocker problems are a type of graph modification problems which are characterised by a graph parameter π and a set $\mathcal{O}$ of graph modification operations (for example vertex deletion or edge contraction). The problem consists in determining, for a given graph G and a threshold d, the shortest sequence of operations from $\mathcal{O}$ which transforms G into a graph G' such that $\pi(G') \leq \pi(G) - d$.

Several graph parameters have been studied in this context and in this paper we are going to continue the research that has been done for the independence number and the clique number. The set of graph operations we consider will always consist of a single operation, either *vertex deletion* or *edge contraction*. Given a graph G, we denote by $G-U$ the graph from which a subset of vertices $U \subseteq V(G)$ has been deleted. Given an edge $uv \in E(G)$, contracting the edge uv means deleting the vertices u and v and replacing them with a single new vertex which is adjacent to every neighbour of u or v. We denote by G/S the graph in which every edge from an edge set $S \subseteq E(G)$ has been contracted. We consider the following two problems, where $d \geq 1$ is a fixed integer.

d-Deletion Blocker(π)

Instance: A graph G and an integer k.

Question: Is there a set $U \subseteq V(G)$, $|U| \leq k$, such that $\pi(G-U) \leq \pi(G) - d$?

d-Contraction Blocker(π)

Instance: A graph G and an integer k.

Question: Is there a set $S \subseteq E(G)$, $|S| \leq k$, such that $\pi(G/S) \leq \pi(G) - d$?

When d is not fixed but part of the input, the problems are called Deletion Blocker(π) and Contraction Blocker(π), respectively.

Table 1: The table of complexities for some graph classes. Here, P means solvable in polynomial time, whereas NP-h and NP-c mean NP-hard and NP-complete, respectively. All cases are solved or referenced in [1]. We marked the cases for which we determine the complexity when d is fixed in **bold**.

Class	CONTRACTION BLOCKER(π)		DELETION BLOCKER(π)	
	$\pi = \alpha$	$\pi = \omega$	$\pi = \alpha$	$\pi = \omega$
Tree	P	P	P	P
Bipartite	**NP-h**;	P	P	P
Cobipartite	$d = 1$: NP-c	NP-c; d fixed: P	P	P
Cograph	P	P	P	P
Split	NP-c; d fixed: P	NP-c; d fixed: P	NP-c; d fixed: P	NP-c; d fixed: P
Chordal	**NP-c**	$d = 1$: NP-c	**NP-c**	$d = 1$: NP-c
Perfect	$d = 1$: NP-h	$d = 1$: NP-h	**NP-c**	$d = 1$: NP-c

When $\pi = \alpha$ or $\pi = \omega$, both problems above are NP-hard on general graphs [1], so it is natural to ask if these problems remain NP-hard when the input is restricted to a special graph class.

The authors of [1] establish the complexities given in Table 1. In the table there are some entries which give the complexity of a problem for the case when d is in the input but not when d is fixed (marked bold). In this paper, we determine the complexity of all these open cases.

A *monogenic* graph class is characterised by a single forbidden induced subgraph H. For a given graph parameter π, it is interesting to establish a *complexity dichotomy for monogenic graphs*, that is, to determine the complexity of (d-)DELETION BLOCKER(π) or (d-)CONTRACTION BLOCKER(π) in H-free graphs, for every graph H. For example, such a dichotomy has been established for DELETION BLOCKER(π) for all $\pi \in \{\alpha, \omega, \chi\}$ and CONTRACTION BLOCKER(π) for $\pi \in \{\alpha, \chi\}$ (all [1]), CONTRACTION BLOCKER(γ_{t2}) (for $d = k = 1$, [4]), CONTRACTION BLOCKER(γ_t) (for $d = k = 1$, [2]) and CONTRACTION BLOCKER(γ) (for $d = k = 1$, [3]). In [1], the computational complexity of CONTRACTION BLOCKER(ω) in H-free graphs has been determined for every H except $H = C_3 + P_1$. We show that this case is NP-hard even when $d = 1$ and complete hence the dichotomy.

2 An Alternative Notation for Edge Contraction

Before we go on to present our results, we first describe a useful tool that helps analyzing paths and distances in graphs in which edges have been contracted. Let G be a graph in which we want to contract a set of edges $S \subseteq E(G)$. Observe that any two vertices in G between which there is a path consisting of edges in S correspond to the same vertex in G/S. To generalize this idea, let $G\big|_S$ be the graph with $V(G\big|_S) = V(G)$ and $E(G\big|_S) = S$ (note that there might be isolated vertices). Observe that any two vertices in $G\big|_S$ which lie in the same connected component do correspond to the same vertex in G/S. We can thus see G/S as the graph whose vertices are the connected components in $G\big|_S$ and in which two vertices are adjacent if the vertex sets of their corresponding connected components have distance one in G. We use this alternative notion in the proofs of our results which are concerned with edge contraction. When analyzing independent sets, this means that an independent set corresponds to a set of connected components of $G|_S$ which pairwise have distance at least two in G.

3 Hardness proofs

In this section, we determine which of the open cases in Table 1 are NP-hard or NP-complete. We further complete the dichotomy of H-free graphs for 1-Contraction Blocker(ω).

The following two theorems and the corollary are shown by a polynomial reduction from Vertex Cover. Interestingly, 1-Deletion Blocker(α) and 1-Contraction Blocker(α) are equivalent on the graph which is constructed in the proof of Theorem 1 and thus the same construction is used to show NP-hardness of 1-Deletion Blocker(α) in chordal graphs. The analysis, however, is quite different.

Theorem 1. 1-Contraction Blocker(α) *is* NP-*complete in chordal graphs.*

Theorem 2. 1-Deletion Blocker(α) *is* NP-*complete in chordal graphs.*

Corollary 1. 1-Deletion Blocker(α) *is* NP-*complete in perfect graphs.*

Finally, we finish this section by answering a question asked in [1]. Indeed, Theorem 4 settles the missing case of [1, Theorem 24] and completes the complexity dichotomy for H-free graphs, which is as follows. Here, the *paw* is the graph given in Figure 1.

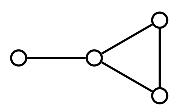

Figure 1: The paw

Theorem 3. *Let H be a graph. If H is an induced subgraph of P_4 or of the paw, then* Contraction Blocker(ω) *is polynomial-time solvable for H-free graphs, otherwise it is NP-hard or co-NP-hard for H-free graphs.*

It is known that Vertex Cover is NP-hard [5] in C_3-free graphs. Adding a universal vertex to a C_3-free graph yields a $(C_3 + P_1)$-free graph. The proof of the following theorem is achieved by showing that adding a universal vertex to a C_3-free instance of Vertex Cover yields an equivalent instance of 1-Contraction Blocker(ω).

Theorem 4. *The decision problem* 1-Contraction Blocker(ω) *is NP-hard in $(C_3 + P_1)$-free graphs.*

4 Algorithms

In this section, we settle the final remaining case highlighted in Table 1 by giving a polynomial-time algorithm for d-Contraction Blocker(α) in bipartite graphs for any fixed d. Observe that d-Contraction Blocker(α) is always a No-instance for all graphs whose independence number is at most d, since contracting edges cannot reduce the independence number to zero. Further, for all graphs whose number of vertices is bounded by a constant, we can solve d-Contraction Blocker(α) in constant time by brute-forcing over all possible sets of edges. As a structural result, we show that in any graph whose number of vertices and independence number is "large enough", it always suffices to contract only $2d+1$ edges in order to reduce the independence number by d.

Theorem 5. *Let G be a connected, bipartite graph with $|V(G)| \geq 2d+2$ and $\alpha(G) \geq d+1$, where $d \geq 1$ is an integer. Then $(G, 2d+1)$ is a* Yes*-instance of d-*Contraction Blocker(α)*.*

It remains to settle the instances in which only a "small" number of edges is contracted. Since the number of such small edge sets is also small, it is tempting to contract each of them and see if their contraction reduces the independence number enough. However, we are then confronted with the problem that contracting edges in a bipartite graph does not necessarily yield a bipartite graph and thus we do not know how to compute the independence number of the resulting graph. We solve this problem in the following theorem by giving an algorithm which determines the independence number of bipartite graphs in which a small number of edges has been contracted. It makes use of the fact that the part of the graph which is affected by the contraction is small and can be tackled by a brute-force algorithm, while the remainder of the graph is still bipartite and its independence number can be computed in polynomial time.

Theorem 6. d-Contraction Blocker(α) *is solvable in polynomial time in bipartite graphs.*

References

[1] Diner, Ö.Y., Paulusma, D., Picouleau, C., Ries, B.: Contraction and deletion blockers for perfect graphs and h-free graphs. Theoretical Computer Science **746**, 49–72 (2018). https://doi.org/10.1016/j.tcs.2018.06.023

[2] Galby, E., Mann, F., Ries, B.: Blocking total dominating sets via edge contractions. Theoretical Computer Science **877**, 18–35 (2021). https://doi.org/10.1016/j.tcs.2021.03.028

[3] Galby, E., Mann, F., Ries, B.: Reducing the domination number of (p3+kp2)-free graphs via one edge contraction. Discrete Applied Mathematics **305**, 205–210 (2021). https://doi.org/10.1016/j.dam.2021.09.009

[4] Galby, E., Lima, P.T., Mann, F., Ries, B.: Using edge contractions to reduce the semitotal domination number (2021)

[5] Poljak, S.: A note on stable sets and colorings of graphs. Commentationes Mathematicae Universitatis Carolinae **15**(2), 307–309 (1974)

[6] C. Bazgan, S. Toubaline, Z. Tuza: The most vital nodes with respect to independent set and vertex cover. Discrete Appl. Math., 159 (2011), pp. 1933-1946