The list-packing number

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Abstract

List-colouring is an influential and classic topic in graph theory. We initiate the study of a natural strengthening of this problem, where instead of one list-colouring, we seek many in parallel. Given the assignment of a list $L(v)$ of k colours to each vertex $v \in V(G)$, we study the existence of k pairwise-disjoint proper colourings of G using colours from these lists. We refer to this as a *list-packing* and we define the *list-packing number* $\chi_\ell^\star(G)$ as the smallest k for which every list-assignment of G admits a list-packing. We prove several results that (asymptotically) match the best-known bounds for the list chromatic number, among which: $\chi_\ell^\star(G) \le n$ with equality if and only if G is the complete graph on n vertices, $\chi_\ell^\star(G) \le (1+o(1))\log_2(n)$ if G is bipartite on n vertices, and $\chi_\ell^\star(G) \le (1+o(1))\Delta/\log(\Delta)$ if G is bipartite with maximum degree Δ. We conjecture that the last statement also holds for triangle-free graphs. Our main open question is whether $\chi_\ell^\star(G)$ can be bounded by a constant times the list chromatic number.

1 Introduction

We are interested in a strengthening of *list-colouring* [8], where we seek to find not just one list-colouring, but many disjoint list-colourings that moreover induce a partition of each list. We ask how large the lists need to be to guarantee such a partition. It was Alon, Fellows and Hare [4] who first hinted at this direction, right at the end of their paper, but ours [7] is the first work to embrace this suggestion.

Given a graph G, a *list-assignment* L of G is a mapping $L : V(G) \to \mathbb{N}$ that assigns a list $L(v)$ of available colours to each vertex $v \in V(G)$. Given a positive integer k, a *k-list-assignment* is a list-assignment for which each list has cardinality k. A *proper L-colouring* is a colouring $c : V(G) \to \mathbb{N}$ such that every $v \in V(G)$ receives a colour $c(v)$ from its list $L(v)$, and adjacent vertices receive distinct colours. The *list chromatic number* $\chi_\ell(G)$ is the least k such that G admits a proper L-colouring for *every* k-list-assignment L of G. By definition, $\chi(G) \le \chi_\ell(G)$ for every graph G. On the other hand, $\chi_\ell(G)$ cannot be upper bounded by a function of $\chi(G)$ alone.

We formulate the question above concretely within the framework of list-colouring. Given a list-assignment L of G, an *L-packing of G of size k* is a collection of k mutually disjoint L-colourings $c_1, \dots, c_k$ of G, that is, $c_i(v) \ne c_j(v)$ for any $i \ne j$ and any $v \in V(G)$. We say that an L-packing is proper if each of the disjoint L-colourings is proper. We define the *list (chromatic) packing number* $\chi_\ell^\star(G)$ of G as the least k such that G admits a proper L-packing of size k for any k-list-assignment L of G. Note that $\chi_\ell^\star(G)$ is necessarily at least $\chi_\ell(G)$.

Once the lists of a graph are of size much larger than $\chi_\ell(G)$, it is clear that there also must exist many disjoint list-colourings: just iteratively extract list-colourings until the lists have size less than $\chi_\ell(G)$. However, this greedy procedure leaves small remainder lists from which one may not be able to extract any further list-colouring, thus preventing the formation of a partition. Therefore the reader may not find it obvious that $\chi_\ell^\star$ is actually well-defined (i.e. finite) for every graph. It turns out that it is. This follows for instance from our following result, which is immediate in the context of list-colouring, but much less so for list-packing.

Theorem 1. *$\chi_\ell^\star(G) \le n$ for any graph G on n vertices. Equality holds if and only if G is K_n, the complete graph on n vertices.*

A natural potential strengthening of Theorem 1 remains wide open: in our full paper [7], we also study *correspondence colouring*, which in turn is a well-studied generalization of list-colouring that allows for forbidding custom pairs of colours along each pair of adjacent vertices. Seeking a partition into correspondence colourings analogously leads to the definition of the *correspondence packing number* $\chi_c^\star(G)$ of a graph G. One always has $\chi_\ell^\star(G) \le \chi_c^\star(G)$. An elegant conjecture going back to constructions of Catlin (see [11]), originally formulated in terms of disjoint independent transversals in sparse n-partite graphs, can be reformulated as: $\chi_c^\star(G) \le 2\lceil \frac{n}{2} \rceil$ for every n-vertex graph G. It is sufficient to prove this for K_n. However, the best known upper bound is $1.78n$, for n large enough, due to Yuster [11].

Usually, one considers the easy bound $\chi_\ell(G) \le n$ as a corollary of a more refined statement about greedy colouring. A graph is *d-degenerate* if there is an ordering of its vertices such that each vertex has at most d neighbours preceding it in the order. By colouring the vertices greedily along this order, it is easy to see that $\chi_\ell(G) \le d+1$ for every d-degenerate graph G. Leveraging Hall's theorem and iterative constructions, we demonstrate that this bound only partially survives in the packing context.

Theorem 2. *For any d-degenerate graph G,*

$$\chi_\ell^\star(G) \le \chi_c^\star(G) \le 2d.$$

Conversely, for every integer $d \ge 2$, there exists a d-degenerate graph G with $\chi_c^\star(G) = 2d$ and $\chi_\ell^\star(G) \ge d+2$.

While this completely settles the behaviour of $\chi_c^\star(G)$ with respect to degeneracy, we still do not know the right answer for $\chi_\ell^\star(G)$. Turning to the maximum degree $\Delta(G)$ of G, in a follow-up paper (work in progress), we explore the possibility that $\chi_c^\star(G) \le 2\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ and we prove this for $\Delta(G) \le 4$. In a slightly different direction (in the current paper) we obtain:

Theorem 3. *$\chi_\ell^\star(G) \le 1 + \Delta(G) + \chi_\ell(G)$ and $\chi_c^\star(G) \le 1 + \Delta(G) + \chi_c(G)$, for any graph G.*

Another important way to bound list chromatic number is to estimate it in terms of the chromatic number. For bipartite graphs G on n vertices, Erdős, Rubin and Taylor [8] already showed that $\chi_\ell(G) \le \log_2(n) + 1$, and this is optimal up to a factor $1 + o(1)$ as $n \to \infty$. More generally, by a result of Alon [1], $\chi_\ell(G)$ is within a factor $\log n$ of $\chi(G)$ for every graph G on n vertices. Through an analysis of random binary matrices, we show that asymptotically these bounds also hold in the list-packing setting.

Theorem 4. *For graphs G on n vertices we have, as $n \to \infty$,*

$$\chi_\ell^\star(G) \le \begin{cases} (1+o(1))\log_2 n & \textit{if } G \textit{ bipartite}, \\ (1+o(1))\chi_f(G)\log n & \textit{if } \chi_f(G) \textit{ uniformly bounded as } n \to \infty, \\ (5+o(1))\chi_f(G)\log n & \textit{in general.} \end{cases}$$

Here we present a further strengthening in terms of the *fractional chromatic number* $\chi_f(G)$ of G, which is the linear program relaxation of the chromatic number and as such a lower bound on $\chi(G)$. In the case that $\chi_f(G)$ is bounded, Theorem 4 is asymptotically sharp for the complete multipartite graphs (see [8]). For each graph G, one can write $\chi_f(G) = \frac{a}{b}$ for some integers a, b. Our proof proceeds by considering many independent random $[a]$-colourings of the union of the lists, which we use (via some optimal fractional colouring of G) to associate a random binary matrix with each vertex, in such a way that there is a list-packing if none of these matrices has permanent 0. We then find good estimates for the probability that a random binary matrix has permanent 0:

Theorem 5. *Let $0 \leq p < 1$ be a real number. Let A be a random $k \times k$-matrix with negatively correlated Bernoulli$(1-p)$ distributed entries. Then*

$$\textit{Permanent}(A) = 0 \textit{ with probability } 2kp^k(1+o(1)) \textit{ as } k \to \infty.$$

Returning to list-colouring bipartite graphs, Alon and Krivelevich [5] conjectured something more refined in terms of maximum degree, in particular, that for some $C > 0$ we have $\chi_\ell(G) \leq C \log \Delta(G)$ for any bipartite G. Since its formulation there has been surprisingly little progress on this essential problem. Already then, it was known that for some $C > 0$ $\chi_\ell(G) \leq C\Delta/\log \Delta(G)$ for any bipartite G, a statement which is a corollary of the seminal result of Johansson for triangle-free graphs [9]. Recent related efforts have only affected the asymptotic leading constant C, bringing it down to 1; see [10, 3]. Our work matches these recent efforts, but for a much stronger structural parameter.

Theorem 6. $\chi_\ell^\star(G) \leq (1+o(1))\Delta/\log\Delta$ *for any bipartite graph G with $\Delta(G) \leq \Delta$, as $\Delta \to \infty$.*

Theorem 7. $\chi_c^\star(G) \leq (1+o(1))\Delta/\log\Delta$ *for any bipartite graph G with $\Delta(G) \leq \Delta$, as $\Delta \to \infty$.*

In stark contrast to Theorem 6, our Theorem 7 is best possible up to a factor two, due to an elegant probabilistic lower bound for $\chi_c(G)$ by Bernshteyn [6]. We conjecture that Theorem 7 also holds for triangle-free graphs, thus generalizing Johanssons result. However, despite recent relatively short proofs of the latter, none of them seem to be easily adaptable to list-packing. For instance, these proofs often rely on a 'finishing blow', where a random partial list-colouring can be completed greedily by drawing colours from suitably structured smaller lists. In the list-packing setting this does not seem enough, as we require access to the full lists to construct a partition. Already a bound strictly smaller than Δ would be welcome progress.

Even for bipartite graphs, our proof of Theorem 7 is rather involved. We draw on the 'coupon collector intuition', which we know can be combined with the Lovász local lemma to obtain the analogous bound on $\chi_c(G)$ (see [3]). Unfortunately, this approach does not immediately work for Theorem 7 as the requisite negative correlation property becomes false in the packing setting; however, we managed to circumvent this obstacle via a suitable result on transversals in a large sum of independent uniformly random permutation matrices, using significant further probability estimates.

2 Concluding remarks

We have set the stage for a natural fusion between two classic notions in (extremal) graph theory: packing and colouring. Our programme would see significant progress if one could prove the following list-packing conjecture:

Conjecture 1. *There exists $C > 0$ such that $\chi_\ell^\star(G) \le C \cdot \chi_\ell(G)$ for any graph G.*

It could even be the case that, for each $\epsilon > 0$ there is some χ_0 such that $\chi_\ell^\star(G) \le (1+\epsilon) \cdot \chi_\ell(G)$ for all G with $\chi_\ell(G) \ge \chi_0$. A resolution of Conjecture 1, either affirmatively or negatively, would be very interesting. Together with the result proved by Alon [2] that for some constant $C > 0$, $\chi_\ell(G) \ge C \log d$ for all graphs G that are d-degenerate but not $(d-1)$-degenerate, note that Theorem 2 implies that $\chi_\ell^\star(G)$ is bounded by an exponential function of $\chi_\ell(G)$, which serves as modest support for Conjecture 1. Regardless of its status in general, Conjecture 1 may be specialised to various fundamental graph classes to push for further progress, e.g.:

- *Planar graphs.* We do not yet have any constructions to rule out the possibility that $\chi_\ell^\star(G) \le 5$ for all planar G. Theorem 2 implies that it is at most 10. What is the optimal value?
- *Line graphs.* Based on the List Colouring Conjecture, we surmise for every $\epsilon > 0$ that $\chi_\ell^\star(G) \le (1+\epsilon)\omega$ for every line graph G with clique number $\omega \ge \omega_0$. Due to its connection to Latin squares, here even the case where G is the line graph of the complete bipartite graph $K_{\omega,\omega}$ is enticing to narrow in on.
- *Random graphs.* Does it hold that $\chi_\ell^\star(G_{n,1/2}) \le (1+o(1))n/(2\log_2 n)$ a.a.s.?

Finally, we reiterate the open problems of determining the right bound for $\chi_\ell^\star(G)$ for d-degenerate graphs G (see Theorem 2), and extending Theorem 7 from bipartite to triangle-free graphs.

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