On Strict Brambles[1]

Emmanouil Lardas — Department of Mathematics, NKUA, Athens, Greece
Evangelos Protopapas — LIRMM, Univ Montpellier, CNRS, Montpellier, France
Dimitrios M. Thilikos — LIRMM, Univ Montpellier, CNRS, Montpellier, France
Dimitris Zoros — Department of Mathematics, NKUA, Athens, Greece

Abstract

A *strict bramble* of a graph G is a collection of pairwise-intersecting connected subgraphs of G, whose *order* is the minimum size of a set of vertices intersecting all its sets. The *strict bramble number* of G, denoted by $\mathsf{sbn}(G)$, is the maximum order of a strict bramble in G. sbn can be seen as a way to extend the notion of acyclicity, as for (non-empty) acyclic graphs every strict bramble has order 1. We initiate the study of this parameter by providing three alternative definitions, each revealing different structural characteristics. The first asserts that $\mathsf{sbn}(G)$ is the minimum k for which G is a minor of the lexicographic product of a tree and a clique on k vertices (known as the *lexicographic tree product number*). The second is in terms of a new variant of a tree decomposition called *lenient tree decomposition*. We show that $\mathsf{sbn}(G)$ is the minimum k for which there is a lenient tree decomposition of G of width at most k. The third is in terms of extremal graphs. For this we define for each k the concept of a *k-domino-tree* and we prove that every edge-maximal graph of $\mathsf{sbn} \leq k$ is a k-domino-tree. We also identify the graphs that constitute the minor-obstruction set of the class of graphs with $\mathsf{sbn} \leq 2$. We complete our results by proving that deciding whether $\mathsf{sbn}(G) \leq k$ is an NP-complete problem.

Introduction. A well-known definition of acyclicity is the following: a non-empty graph G is acyclic if for every collection of pairwise intersecting subtrees of G there is some vertex appearing in every subtree. In this paper we deal with a natural parametric extension of acyclicity, that is, the minimum k such that for every collection of pairwise intersecting subtrees of G there is a set of k vertices intersecting all of them. To our knowledge, this graph parameter[2] was studied for the first time by Kozawa, Otachi and Yamazaki in [1] with the name *PI number* (where PI stands for "Pairwise Intersecting") and was used in order to derive lower bounds for the treewidth of several classes of product graphs. The same parameter was recently introduced by Aidun, Dean, Morrison, Yu, and Yuan in [2] with the name *strict bramble number* and is the term that we adopt in this paper. The strict bramble number was used in [2] in order to study the relation of treewidth and the gonality on particular classes of graphs.

Strict brambles. Given a graph G, a collection $\mathcal{B}$ of vertex sets of G and some vertex set X, we say that X *covers* $\mathcal{B}$ if every set in $\mathcal{B}$ has some vertex in common with X. We say that a vertex set S is *connected* if the subgraph of G induced by S is connected. A *strict bramble* of a graph G is a collection $\mathcal{B}$ of vertex sets of G such that:

(1) every set in $\mathcal{B}$ is connected;
(2) every two sets in $\mathcal{B}$ have some vertex in common.

The *order* of a strict bramble $\mathcal{B}$ of G is the minimum size of a set that covers $\mathcal{B}$ and is denoted by $\mathsf{order}(\mathcal{B})$. The *strict bramble number* of G, denoted by $\mathsf{sbn}(G)$, is the maximum order of a strict bramble of G.

[1] Full version preprint available at `arXiv:2201.05783`
[2] We use the term *graph parameter* for every function mapping graphs to non-negative integers.

Brambles. Given two vertex sets S and S' of a graph G we say that S and S' *touch* in G if either they have some vertex in common or there is an edge with one endpoint in S and the other in S'. If we relax the definition of strict bramble by substituting (2) with:

(2′) every two sets in $\mathcal{B}$ are touching,

then we obtain the (classic) notion of *bramble* and the parameter *bramble number* ($\mathsf{bn}(G)$), introduced by Seymour and Thomas in [3][3]. The study of brambles attracted a lot of attention because of the main result in [3], that is a min-max theorem asserting that for every graph G, the treewidth of G ($\mathsf{tw}(G)$), is one less than its bramble number. As already observed in [1] (using the results of [4]), for every graph G, it holds that $\mathsf{sbn}(G) \leq \mathsf{bn}(G) \leq 2 \cdot \mathsf{sbn}(G)$ which, in turn, implies that:

$$\mathsf{sbn}(G) - 1 \leq \mathsf{tw}(G) \leq 2 \cdot \mathsf{sbn}(G) - 1. \tag{1}$$

Treewidth, the min-max analogue of brambles, is one of the most important graph parameters. Reinvented by Robertson and Seymour in [5] (see [6, 7] for earlier appearances), treewidth served as a cornerstone parameter of the Graph Minors series of Robertson and Seymour and is omnipresent in a wide range of topics in combinatorics and graph algorithms [8].

In this paper we initiate the study of the strict bramble number, mainly motivated by the fact that, so far, no min-max analogue, parallel to treewidth, is known for this graph parameter. In this direction, we provide three alternative definitions of the strict bramble number, each revealing different characteristics of this parameter. We continue with a brief introduction of these definitions.

Lexicographic tree product. Let G, H be a pair of graphs. The *lexicographic product* of G and H, denoted by $G \cdot H$, is the graph whose vertex set is the Cartesian product of the vertex sets of G and H and where the vertex (u, v) is adjacent with the vertex (w, z) in $G \cdot H$ if and only if either u is adjacent with w in G or it holds that $u = w$ and v is adjacent with z in H. The *lexicographic tree product number* of G is defined by Harvey and Wood in [9] as:

$$\mathsf{ltp}(G) \quad = \quad \min\{k \in \mathbb{N} \mid \text{there is a tree } T \text{ such that } G \text{ is a minor of } T \cdot K_k\}.$$

Our first contribution is to show that the lexicographic tree product number and the strict bramble number are the same parameter. Unsurprisingly it was already proved in [9] that (1), holds if we replace sbn with ltp.

Lenient tree decompositions. Let G be a graph, T a tree and let χ be a function mapping vertices of T to vertex sets of G. We say that two vertices t, t' of T are *close* in T if either they are identical or they are adjacent. The pair (T, χ) is a *lenient tree decomposition* of G if it satisfies the following three conditions:

(C1) $\bigcup_{t \in V(T)} \chi(t)$ is the vertex set of G;
(C2) for every edge e of G, there are two close vertices t, t' of T such that, $e \subseteq \chi(t) \cup \chi(t')$;
(C3) for every vertex x of G, the set $\{t \mid x \in \chi(t)\}$ is connected in T.

We define the *width* of (T, χ), as the maximum size of a set $\chi(t)$, for vertices t of T. Our second characterization of the strict bramble number is that, for every graph G, $\mathsf{sbn}(G)$ is equal to the minimum width of a lenient tree decomposition of G. In that way, lenient tree decompositions can serve as the analogue of tree decompositions for the case of strict brambles. Notice that the definition of a standard tree decomposition follows from the above definition if we substitute "close" by "identical".

[3] We wish to stress that in [3] the term "*screen*" was used, instead of the term "bramble".

k-domino-trees. Given a non-negative integer k, a *k-tree* is recursively defined as follows: a graph G is a *k-tree* if it is either isomorphic to K_r, for some $r \leq k$, or it contains a vertex v of degree k in G whose neighborhood induces a clique in G and whose removal from G yields a k-tree. It is known that among all the graphs with treewidth at most k, those that are edge-maximal (that is, after the addition of any edge they have treewidth more than k) are precisely the k-trees. This implies that the treewidth of a graph can be defined as the minimum k for which G is a spanning subgraph of a k-tree. Is there an analogous definition for the strict bramble number? What are the edge-extremal graphs of strict bramble number at most k?

Our third characterization is obtained by answering the above questions. For this, we introduce the concept of a k-domino-tree.

Given a graph G, we denote by $\mathsf{cc}(G)$ the set of all connected components of G. Let $S \subseteq V(G)$. We define the *connectivity-degree* of S as $\mathsf{cdeg}_G(S) = |\mathsf{cc}(G - S)|$, i.e. the number of connected components of $G-S$. Let G be a chordal graph. We call a maximal clique of G, *external* (respectively *internal*), if its vertex set contains at most one (respectively at least two) minimal separator(s) of G. We say that all external maximal cliques containing the same minimal separator S, form an *external family of* S, and we denote it by $\mathcal{K}_G(S)$. For each $K \in \mathcal{K}_G(S)$, we define its *valiancy* to be $\mathsf{val}(K) = |V(K) \setminus S|$, i.e. the number of private vertices of K. We call a minimal separator S, *external* (respectively *internal*), if $\mathcal{K}_G(S) \neq \emptyset$ (respectively $\mathcal{K}_G(S) = \emptyset$).

Let $k \in \mathbb{N}$. A graph G is a *k-domino-tree* if it is either K_r for some $r \leq k$, or it satisfies the following properties:

i. G is chordal;
ii. Every minimal separator of G has size k;
iii. Every maximal clique of G has size in $[k+1, 2k]$;
iv. The vertex set of every maximal clique of G contains at most two minimal separators;
v. The vertex set of every maximal clique of G that contains exactly two minimal separators S, S' is equal to $S \cup S'$;
vi. Every internal minimal separator of G of connectivity-degree two, is not contained in the union of two other minimal separators;
vii. For every external minimal separator S of connectivity-degree two, the union of the vertex sets of the maximal cliques that contain S, has size greater than $2k$;
viii. For every external minimal separator S, with $|\mathcal{K}_G(S)| > 1$, for any different pair $K, K' \in \mathcal{K}_S$, $\mathsf{val}(K) + \mathsf{val}(K') > k$.

A graph G is a *partial k-domino-tree* if it is a spanning subgraph of a k-domino-tree.

We prove that $\mathsf{sbn}(G)$ is equal to the minimum k for which G is a partial k-domino-tree. Moreover, we prove that the edge-extremal graphs of strict bramble number at most k are precisely the k-domino-trees. Interestingly, k-domino-trees enjoy a more elaborate structure than the one of k-trees. While all k-trees on n vertices have the same number of edges, the same does not hold for the k-domino-trees on n vertices as the number of edges may vary considerably.

Results. We prove all aforementioned equivalencies by proving the following theorem.

Theorem 1. *Let G be a graph and $k \in \mathbb{N}$. The following statements are equivalent:*

1. *There is a tree T such that G is a minor of $T \cdot K_k$.*
2. *G has a lenient tree decomposition of width at most k.*
3. *G has no strict bramble of order greater than k.*

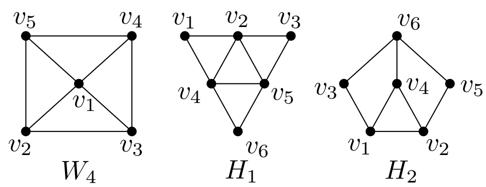

Figure 1: The graphs in $\mathcal{Z}$.

4. G is a partial k-domino-tree.

We identify the exact structure of all edge maximal graphs of $\mathsf{sbn}(G) \leq k$ by proving that the edge-maximal graphs of $\mathsf{sbn}(G) \leq k$ are exactly the k-domino-trees.

Theorem 2. *Let G be a graph with $\mathsf{sbn}(G) \leq k$. G is an edge-maximal graph if and only if, G is a k-domino-tree.*

Moreover, we prove that the obstruction set of the class of graphs of strict bramble number at most two are precisely the graphs in Figure 1.

Theorem 3. *The obstruction set of the graphs G with $\mathsf{sbn}(G) \leq 2$, consists of the graphs in $\mathcal{Z}$.*

To complete our results we prove that, given a graph G and an integer $k \in \mathbb{N}$, deciding whether $\mathsf{sbn}(G) \leq k$ is NP-complete. To that end we reduce our problem to deciding whether $\mathsf{tw}(G) \leq k$.

Theorem 4. *There exists a polynomially computable function that, given a graph G and an integer $k \in \mathbb{N}$, outputs a graph H such that, $\mathsf{tw}(G) \leq k - 1$ if and only if, $\mathsf{sbn}(H) \leq k$.*

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