On the Cospectrality of Hermitian Adjacency Matrices of a Mixed Graph

Mohammad Abudayah, German Jordanian University, Jordan
Omar Alomarit, German Jordanian University, Jordan.

Abstract

A digraph $\mathbb{X}$ where pothways oriented edges considered as non-oriented edges (digons) is called mixed graphs. Let α be a unit complex number, then the $\alpha-$Hermitian adjacency matrix of a mixed graph $\mathbb{X}$ is defined to be the matrix $H_\alpha = [h_{rs}]$ where $h_{rs} = \alpha$ if rs is an arc in $\mathbb{X}$, $h_{rs} = \overline{\alpha}$ if sr is an arc in $\mathbb{X}$, $h_{rs} = 1$ if sr is a digon in $\mathbb{X}$ and $h_{rs} = 0$ otherwise. In this paper we study the cospectrality of the Hermitian adjacency matrices of a mixed graph.

1 Introduction

A digraph $\mathbb{X} = (V(\mathbb{X}), E_0(\mathbb{X}), E_1(\mathbb{X}))$ is a set of vertices together with set of bothways oriented arcs E_0 and set of one way oriented edges E_1. A digraph $\mathbb{X}$ where pothways oriented edges (the elements of the set $E_0(\mathbb{X})$) considered as non-oriented edges is called mixed graphs; such edges are called digons. The underlying graph of the mixed graph $\mathbb{X}$, denoted by $\Gamma(\mathbb{X})$, is the graph all elements of E_1 considered as digons.
Algebraic graph theory is the study of graphs and digraphs with respect to some graph matrix and its spectrum, where the adjacency matrix intensively studied. For undirected graphs researchers focused on two kinds of adjacency matrices, the traditional adjacency matrix and the Laplacian adjacency matrix. On the other hand for directed graphs (digraphs) the traditional adjacency matrix was very challenging to deal with. Recently, many researchers have proposed other Hermitian adjacency matrices of mixed graphs, the most interesting Hermitian adjacency matrix of mixed graphs was proposed by Guo and Mohar in [1] as follows: For a mixed graph $\mathbb{X}$ with n vertices, the $i-$Hermitian adjacency matrix of $\mathbb{X}$ is an $n \times n$ matrix $H_i(\mathbb{X}) = [h_{uv}]$, where

$$h_{uv} = \begin{cases} 1 & \text{if } uv \in E_0(\mathbb{X}), \\ i & \text{if } uv \in E_1(\mathbb{X}), \\ -i & \text{if } vu \in E_1(\mathbb{X}), \\ 0 & \text{otherwise.} \end{cases}$$

Authors in [1] proved many interesting properties of H_i spectrum. Mohar in [2] extended the previously proposed adjacency matrix to a new kind of Hermitian adjacency matrix called $\omega-$Hermitian adjacency matrix of mixed graphs $\mathbb{X}$ by replacing the complex number i with the sixth root of unity $\omega = e^{\frac{\pi}{3}i}$. The fact that these adjacency matrices (H_i and H_ω) are Hermitian, open a hot research topic nowadays, moreover the new kind of Hermitian adjacency matrix proposed by Mohar in [2] has been generalized with considering a unit complex number α instead of ω. In this paper we study the cospectrality of Hermitian adjacency matrix of a mixed graph.

2 Cospectrality of Hermitian Adjacency Matrices of Mixed Graphs that have Same Underlying Graph

In this section we investigate when two mixed graphs with same underlying graphs are co-spectral. To this end, fix a weakly connected mixed graph $\mathbb{X}$ and a unit complex number α. Let $u, v \in V(\mathbb{X})$

and let W be a walk in $\mathbb{X}$, say $W = (u = r_1, ..., r_k = v)$. Then, define a function f_W^α recursively as follows: assign a value $f_W^\alpha(j)$ to the j^{th} vertex (i.e. to r_j) along W by

$$f_W^\alpha(1) = 1,$$

$$f_W^\alpha(j+1) = \left\{ \begin{array}{ll} f_W^\alpha(j) & \text{if } r_j r_{j+1} \in E_0(\mathbb{X}) \\ \alpha f_W^\alpha(j) & \text{if } r_j r_{j+1} \in E_1(\mathbb{X}) \\ \bar{\alpha} f_W^\alpha(j) & \text{if } r_{j+1} r_j E_1(\mathbb{X}) \end{array} \right\}$$

for $j = 1, ..., k$. We shall write $f_W^\alpha(*)$ for the final value $f_W^\alpha(k)$.

Let $\mathbb{X}$ be a mixed graph, $u \in V(\mathbb{X})$, then for a vertex $v \in V(X)$ define the α-store of the vertex v with respect to the vertex u by:

$$\mathbb{T}_u^\alpha(v) = \{f_W^\alpha(v) : W \text{ is a walk from } u \text{ to } v\},$$

and the store size of the vertex v by $\mathfrak{t}_u^\alpha(v) = |\mathbb{T}_u^\alpha(v)|$.

Theorem 1. *Let $\mathbb{X}$ be a weakly connected mixed graph, $u \in V(\mathbb{X})$ and $H_\alpha = [h_{ij}]$ be its α-Hermitian adjacency matrix. Then the following statements are equivalent:*

1. *$\mathfrak{t}_u^\alpha(v) = 1$ for a vertex $v \in \mathbb{X}$*

2. *$\mathfrak{t}_u^\alpha(m) = 1$ for every vertex $m \in \mathbb{X}$.*

3. *For every cycle $C = (r_1, r_2, \ldots, r_{k-1}, r_1)$ in $\mathbb{X}$, the weight of C*

$$h_\alpha(C) = h_{r_1 r_2} h_{r_2 r_3} \ldots h_{r_{k-1} r_1}$$

   in H_α equals one.

Proof. (1) $\to$ (2) Suppose that $\mathfrak{t}_u^\alpha(v) = 1$ for a vertex v of $\mathbb{X}$ and let m be a vertex of $\mathbb{X}$ with $\mathfrak{t}_u^\alpha(m) > 1$. Suppose further W_{uv} is a walk from u to v. Then, since $\mathfrak{t}_u^\alpha(m) > 1$ there are two walks W_{um} and W'_{um} from u to m with $f_{W_{um}}^\alpha(m) \neq f_{W'_{um}}^\alpha(k)$. Obviously the walk starts from u to m through W_{um} then from m to u through the reverse walk of W'_{um} $(Rev(W'_{um}))$ then from u to v through W_{uv} is a walk from u to v. Now, since $f_{W_{um}}^\alpha(m) \neq f_{W'_{um}}^\alpha(k)$ we have $f_{W_{um} \cup Rev(W'_{um}) \cup W_{uv}}^\alpha(v)$ is the same value as $f_{W_{uv}}^\alpha(v)$ but with different initial value of $f_{W_{uv}}^\alpha(u)$, that is $f_{W_{uv}}^\alpha(u) \neq 1$. Which means $f_{W_{um} \cup Rev(W'_{um}) \cup W_{uv}}^\alpha(v) \neq f_{W_{uv}}^\alpha(v)$, Contradiction.

(2) $\to$ (3) Suppose that C is a cycle in $\mathbb{X}$. Let v be a vertex of the cycle C and W be the closed walk consists of the path from u to v, P_{uv}, then the cycle C then the reverse path from v to u $(Rev(P_{uv}))$. Using the definition of $f_W^\alpha(u)$ we have, Obviously

$$\begin{aligned} f_W^\alpha(u) &= h_\alpha(P_{uv}) h_\alpha(C) h_\alpha(Rev(P_{uv})) && (1) \\ &= h_\alpha(P_{uv}) h_\alpha(C) \overline{h_\alpha(P_{uv})} && (2) \\ &= h_\alpha(C). && (3) \end{aligned}$$

Finally, Using the definition of f_W^α and the assumption that $\mathfrak{t}_u^\alpha(v) = 1$ for all vertices of $\mathbb{X}$, we have $f_W^\alpha(u) = 1$.
(3) $\to$ (1) Obvious. $\square$

Definition 1. *A mixed graph (X) is called α-monostore mixed graph if for every vertex v, $\mathfrak{t}_u^\alpha(v) = 1$.*

Obviously, changing the initial vertex u of an α-monostore mixed graph $\mathbb{X}$ will not change the store size of the vertices of $\mathbb{X}$, but it may change the values in the store set of the vertices.

Theorem 2. *Let $\mathbb{X}$ be an α-monostore graph, $H_\alpha = [h_{st}]$ be its α-Hermitian adjacency matrix of $\mathbb{X}$ and for a vertex u in $\mathbb{X}$,*
mathbbXelta be the diagonal matrix $\Delta = diag\{t_v : t_v \in \mathbb{T}_u^\alpha(v)$ and $v \in V(\mathbb{X})\}$. Then, $\Delta H_\alpha \Delta^* = A(\Gamma(\mathbb{X}))$. Where $A(\Gamma(\mathbb{X}))$ is the traditional adjacency matrix of the graph $\Gamma(\mathbb{X})$.

Proof. Since the matrix Δ is a diagonal matrix it's enough to prove that $\Delta H_\alpha \Delta^*$ is $0,1$-matrix. To do this, suppose that rs is an arc in $\mathbb{X}$, then

$$[\Delta H_\alpha \Delta^*]_{rs} = t_r h_{rs} \overline{t_s}.$$

Since rs is an arc in $\mathbb{X}$ and $\mathbb{X}$ is α-monostore mixed graph, $t_s = h_{rs} t_r$. Therefore, $[\Delta H_\alpha \Delta^*]_{rs} = 1$. □

Definition 2. *If $\mathbb{X}$ is α-monostore graph, then a diagonal matrix Δ_α that satisfies $\Delta_\alpha H_\alpha \Delta_\alpha^* = A(\Gamma(\mathbb{X}))$ is called an orienting matrix of $\mathbb{X}$.*

Corollary 1. *If $\mathbb{X}$ is α-monostore mixed graph then $\mathbb{X}$ is cospectral with its underlying graph.*

Corollary 2. *If $\mathbb{X}$ is a tree mixed graph then the spectrum of $\mathbb{X}$ is completely determined by its underlying graph $\Gamma(\mathbb{X})$.*

Lemma 1. *let H_α be the α-Hermitian adjacency matrix of a mixed graph $\mathbb{X}$, $\Delta = diag(d_u \in \mathbb{C} : u \in V(\mathbb{X})$ and $|d_u| = 1)$ and $\mathfrak{K} = [k_{uv}] = \Delta H_\alpha \Delta^*$. Then, for every cycle C in $\mathbb{X}$ the weights of the cycle C in H_α and in $\mathfrak{K}$ are equal.*

Proof. Suppose that $C = u_1 u_2 \dots u_n u_1$ is a cycle in $\mathbb{X}$. Then, observing that $[\Delta H_\alpha \Delta^*]_{uv} = d_u h_{uv} \overline{d_v}$, we have

$$\begin{aligned} h_\alpha(C) &= h_\alpha(u_1u_2)h_\alpha(u_2u_3)\dots h_\alpha(u_nu_1) && (4)\\ &= d_{u_1} h_\alpha(u_1u_2)\overline{d_{u_2}} d_{u_2} h_\alpha(u_2u_3)\overline{d_{u_3}} \dots d_{u_n} h_\alpha(u_nu_1)\overline{d_{u_1}} && (5)\\ &= k(C). && (6) \end{aligned}$$

□

Example 1. *Let $\mathbb{X}$ be the mixed graph shown in Figure 1, α be the primitive third root of unity $e^{\frac{2\pi}{3}i}$ and H_α its α-Hermitian adjacency matrix. Obviously, the tree mixed subgraph T illustrated in the sold lines is α-monograph with the orienting matrix $\Delta_\alpha = diag(\mathfrak{t}_0^\alpha(v) : v \in V(\mathbb{X}))$. Then, the matrix $\Delta_\alpha H_\alpha \Delta_\alpha^*$ can be considered as the adjacecny matrix of the complex weighted graph $\mathbb{X}_\alpha$ (or the unit gain graph $\mathbb{X}_\alpha$, see[3]) shown in Figure 2 with weight 1 for each edge of the tree $\Gamma(T)$, furthermore according to Lemma 1, the weight of the edge 53 in $\mathbb{X}_\alpha$, say $k(53)$, equals to the weight of the fundamental cycle $C_{53} = 32453$ in H_α. And thus,*

$$k(53) = \overline{k(35)} = h(C_{35}) = \alpha^2.$$

Similar calculations can be done to get, $k(89) = \alpha^2$. Therefore, the matrix $\Delta_\alpha H_\alpha \Delta_\alpha^*$ is the adjacecny matrix of the complex weighted graph $\mathbb{X}_\alpha$.

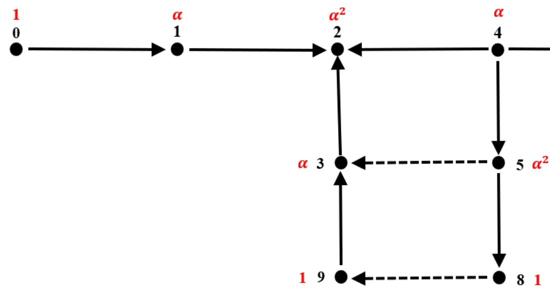

Figure 1: The mixed graph $\mathbb{X}$ with the values of $\mathfrak{t}_0^{\alpha}(v)$.

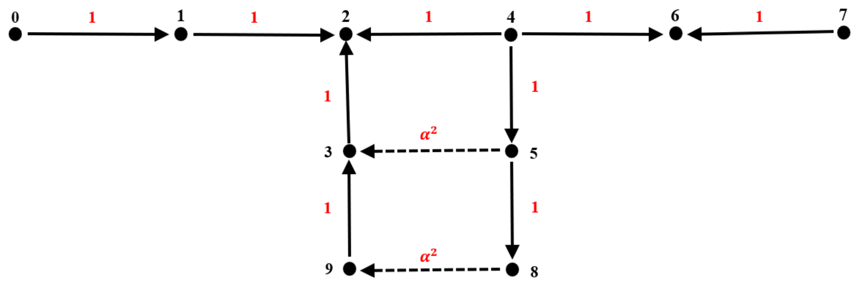

Figure 2: The complex weighted graph $\mathbb{X}_\alpha$ with the weight of the edges

Theorem 3. *Let $\mathbb{X}$ be a mixed graph, α and γ be two unit complex numbers. If for all cycles C in $\mathbb{X}$ one of $h_\alpha(C) = h_\gamma(C)$ or $h_\alpha(C) = \overline{h_\gamma(C)}$ holds then $H_\alpha(\mathbb{X})$ and $H_\gamma(\mathbb{X})$ are cospectral ($\mathbb{X}$ is $\alpha - \gamma$ cospectral).*

Proof. Suppose that T is spanning tree of $\mathbb{X}$, u is a vertex of $\mathbb{X}$ and, Δ_α and Δ_γ is the α-orienting and γ-orienting matrix of T respectively. Then,

$$\Delta_\alpha H_\alpha(T)\Delta_\alpha^* = \Delta_\gamma H_\gamma(T)\Delta_\gamma^* = A(\Gamma(T))$$

which means that the matrices $\Delta_\alpha H_\alpha(\mathbb{X})\Delta_\alpha^*$ ($\Delta_\gamma H_\gamma(\mathbb{X})\Delta_\gamma^*$) can be considered as a Hermitian adjacency matrices of the weight mixed graph $\mathbb{X}_\alpha$ (respectively $\mathbb{X}_\gamma$) with underlying graph $\Gamma(\mathbb{X})$, all edges of the tree T have weight 1. Furthermore, suppose that rs is an edge in $\mathbb{X}$ that is not in T, s belongs to the path from u to r in T and C_{rs} be the fundamental cycle of $\mathbb{X}$ corresponding to rs. Then, using Lemma 1 the weight of rs in $\mathbb{X}_\alpha$(respectively $\mathbb{X}_\gamma$) equals $h_\alpha(C_{rs})$(respectively $h_\gamma(C_{rs})$). Therefore, either $\Delta_\alpha H_\alpha(\mathbb{X})\Delta_\alpha^* = \Delta_\gamma H_\gamma(\mathbb{X})\Delta_\gamma^*$ or $\Delta_\alpha H_\alpha(\mathbb{X})\Delta_\alpha^* = \Delta_\gamma \overline{H}_\gamma(\mathbb{X})\Delta_\gamma^*$, which means $H_\alpha(\mathbb{X})$ and $H_\gamma(\mathbb{X})$ are cospectral. $\square$

References

[1] Guo K, Mohar B. Hermitian adjacency matrix of digraphs and mixed graphs. Journal of Graph Theory. 2017;85(1):217-48.

[2] Mohar B. A new kind of Hermitian matrices for digraphs. Linear Algebra and its Applications. 2019; (1): 343-352.

[3] Nathan Reff Spectral properties of complex unit gain graphs Linear Algebra and its Applications. 2012; (9): 3165-3176