Minimal obstructions for properties related to polarity on cograph superclasses

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Abstract

Let s and k be nonnegative integers. An (s,k)-polar graph is a graph G whose vertex set admits a partition (A,B) such that $G[A]$ and $\overline{G[B]}$ are complete multipartite graphs with at most s and k parts, respectively. If s or k is replaced by ∞ it means that the number of parts of $G[A]$ or $\overline{G[B]}$, respectively, is unbounded.

Complete lists of cograph minimal (s,k)-polar obstructions are known when $s = k = \infty$, $s = 1$ and $k = \infty$, $s = 1$ and k is any arbitrary fixed integer, or $s = k = 2$. In this work we generalize all these results giving complete lists of P_4-sparse and P_4-extendible minimal (s,k)-polar obstructions for the said cases.

Additionally, we show that any hereditary property of graphs has only finitely many minimal obstructions when restricted to either P_4-sparse or P_4-extendible graphs, generalizing analogous results for cographs and P_4-reducible graphs.

1 Introduction

All graphs in this work are simple and finite. For graphs G and H, we use $G + H$ and $G \oplus H$ to denote the disjoint union and the join of G and H, respectively. The disjoint union of n copies of a graph G is denoted by nG. We write $H \le G$ to denote that H is an induced subgraph of the graph G. We say that G is *H-free* if H is not an induced subgraph of G. For a family $\mathcal{H}$ of graphs, we say that G is *$\mathcal{H}$-free* if it is H-free for every $H \in \mathcal{H}$. A property of graphs is *hereditary* if it is closed under taking induced subgraphs. Given a hereditary property $\mathcal{P}$ of graphs, a *minimal $\mathcal{P}$-obstruction* is a graph that does not have the property $\mathcal{P}$ but such that every vertex-deleted subgraph does.

Given $s,k \in \mathbb{N}$, a partition (A,B) of V_G is said to be an *(s,k)-polar partition* if A induces a complete multipartite graph with at most s parts and B induces the complement of a multipartite graph with at most k parts. An *(s,k)-polar graph* is a graph whose vertex set admits an (s,k)-polar partition. Analogous definitions can be given when s or k is replaced by ∞. Usually, $(1,\infty)$- and (∞,∞)-polar graphs are simply called *monopolar graphs* and *polar graphs*, respectively. An (s,s)-polar graph will be called an *s-polar graph.* A polar partition (A,B) of V such that A is a clique is called a *unipolar partition*, and a graph whose vertex set admits a unipolar partition is naturally called a *unipolar graph.* It is known that for any pair of fixed nonnegative integers s and k, (s,k)-polar graphs have finitely many minimal obstructions [8], so they can be efficiently recognized. Unipolar graphs also have been shown to be efficiently recognizable [2]. In contrast, the problem of deciding whether a given graph is polar and the problem of deciding whether it is monopolar are known to be NP-complete problems [1, 7]. Such results motivated the study of properties related to polarity in graph classes like cographs (P_4-free graphs) [3, 4, 6, 10].

A graph is said to be a *P_4-sparse graph* if any set of five vertices induces at most one P_4. A *P_4-extendible graph* is a graph such that, for any vertex subset W inducing a P_4, there exists at most one vertex $v \notin W$ which belongs to a P_4 sharing vertices with W.

In this work we generalize several results related to cograph minimal (s,k)-polar obstructions on both, P_4-sparse and P_4-extendible graphs. Specifically, we give complete list of minimal $(s,1)$-polar obstructions, minimal 2-polar obstructions, minimal unipolar obstructions, minimal monopolar obstructions, and minimal polar obstructions on said cograph superclasses. In addition, we show that any hereditary property of graphs has only a finite number of minimal obstructions in the families of P_4-sparse and P_4-extendible graphs. For the rest of this text, $\mathcal{G}$ will be used to denote either the family of P_4-sparse graphs or the family of P_4-extendible graphs.

2 Minimal $(s,1)$-polar obstructions

In the following theorem, we completely characterize the minimal $(s,1)$-polar obstructions on the classes of P_4-sparse and P_4-extendible graphs. It is worth noticing that the ten graphs mentioned in items 1 and 2 are the only disconnected minimal $(s,1)$-polar obstructions on general graphs.

Theorem 1. *Let s be an integer, $s \geq 2$. If G is a graph in the class $\mathcal{G}$, then G is a minimal $(s,1)$-polar obstruction if and only if G satisfies exactly one of the following assertions:*

1. *G is isomorphic to one of the graphs depicted in Figure 1.*
2. *G is isomorphic to some of $2K_{s+1}, K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$.*
3. *The complement of G is disconnected with components $G_1, \dots, G_t$, where each G_i is a minimal $(1, s_i)$-polar obstruction whose complement is not a graph in Figure 1, and $s = t - 1 + \sum_{i=1}^{t} s_i$.*

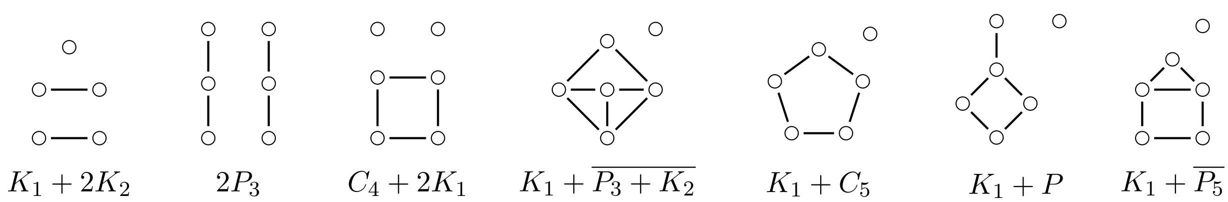


Figure 1: Some minimal $(\infty, 1)$-polar obstructions.

3 Minimal unipolar, monopolar and polar obstructions

The next two results provide complete lists of minimal unipolar and minimal monopolar obstructions on P_4-sparse and P_4-extendible graphs.

Theorem 2. *If G is a graph in the class $\mathcal{G}$, then G is a minimal unipolar obstruction if and only if $G \in \{2P_3, K_{2,3}, C_5\}$.*

Theorem 3. *If G is a graph in the class $\mathcal{G}$, then G is a minimal monopolar obstruction if and only if $\overline{G}$ is one of the graphs depicted in Figure 1.*

As we show in the next lemma, for general graphs, disconnected minimal polar obstructions are closely related to minimal monopolar obstructions.

Lemma 4. *If G is a graph, then G is a disconnected minimal polar obstruction if and only if $G \cong P_3 + H$ where H is a minimal monopolar obstruction which is not a minimal polar obstruction.*

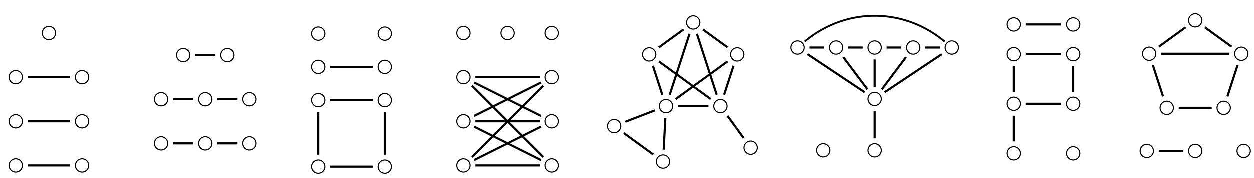
We conclude this section by showing that the only minimal polar obstructions on the families of P_4-sparse and P_4-extendible graphs are the graphs described in Lemma 4 and their complements. Observe that, since there is only a finite number of minimal unipolar, monopolar, and polar obstructions on such classes, the associated recognition problems are efficiently solvable on the classes of P_4-sparse and P_4-extendible graphs.

Theorem 5. *If G is a graph in the class $\mathcal{G}$, then G is a minimal polar obstruction if and only if either G or its complement is the join of $\overline{P_3}$ with one of the graphs depicted in Figure 1.*

4 Minimal 2-polar obstructions

A *partial complement* of a graph H is either the usual complement of H, or a graph H' obtained by choosing a partition (H_1, H_2) of the set of connected components of H, and taking the disjoint union of the complements of H_1 and H_2, that is, $H' = \overline{H_1} + \overline{H_2}$. Partial complements are the key for the following compact characterization of minimal 2-polar obstructions on P_4-sparse and P_4-extendible graphs.

Theorem 6. *Let $\mathcal{F}$ be the family of graphs depicted in Figure 2. If G is a graph in $\mathcal{G}$, then G is a minimal 2-polar obstruction if and only G can be obtained from some graph in $\mathcal{G} \cap \mathcal{F}$ by a finite sequence of partial complementations.*



F_1 F_6 F_{13} F_{21} F_{26} F_{27} F_{32} F_{37}

Figure 2: Some minimal 2-polar obstructions.

5 Hereditary properties

Peter Damaschke [5] proved that if a graph family $\mathcal{F}$ can be constructed under certain conditions, the poset obtained by considering $\mathcal{F}$ under the induced subgraph relation is a well-quasi-ordering, or equivalently, that for any hereditary property $\mathcal{P}$, there is only a finite number of minimal $\mathcal{P}$-obstructions in $\mathcal{F}$. Providing constructive characterizations for the classes of P_4-sparse and P_4-extendible graphs that satisfy the hypotheses of Damaschke's theorem, we conclude the following result generalizing those for cographs and P_4-reducible graphs originally obtained in [5].

Theorem 7. *For any hereditary property $\mathcal{P}$, there are finitely many minimal $\mathcal{P}$-obstructions in $\mathcal{G}$.*

6 Concluding remarks

We find interesting that any P_4-sparse minimal obstruction for unipolarity, monopolarity, polarity, $(s, 1)$-polarity and 2-polarity is a cograph. This leads us to ask the following question.

Problem 1. *For any positive integers s and k, is every P_4-sparse minimal (s, k)-polar obstruction a cograph?*

Moreover, it was shown in [9] that any P_4-sparse minimal obstruction for (k, ℓ)-coloring is a cograph too, so we have the next more general question.

Problem 2. *Which hereditary properties $\mathcal{P}$ satisfy that every P_4-sparse minimal $\mathcal{P}$-obstruction is a cograph?*

Additionally, Hannnebauer [9] proved that any P_4-sparse minimal (s, k)-polar obstruction has at most $(s + 1)(k + 1)$ vertices, so we pose the following question.

Problem 3. *Can we establish an $O(sk)$ upper bound for the order of the P_4-extendible minimal (s, k)-polar obstructions?*

As a future line of work, we propose to extend our results to other cograph superclasses having few induced P_4's, for instance, P_4-tidy graphs or extended P_4-laden graphs, any of which contains both P_4-sparse and P_4-extendible graphs. Another possible line of work is to characterize some other hereditary properties on cograph superclasses by their sets of minimal obstructions.

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