Turán numbers and anti-Ramsey numbers for short cycles in complete 3-partite graphs

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Abstract

We call a 4-cycle in K_{n_1,n_2,n_3} multipartite, denoted by C_4^{multi}, if it contains at least one vertex in each part of K_{n_1,n_2,n_3}. The Turán number $ex(K_{n_1,n_2,n_3}, C_4^{\text{multi}})$ (respectively, $ex(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\})$) is the maximum number of edges in a graph $G \subseteq K_{n_1,n_2,n_3}$ such that G contains no C_4^{multi} (respectively, G contains neither C_3 nor C_4^{multi}). We call an edge-colored C_4^{multi} rainbow if all four edges of it have different colors. The anti-Ramsey number $ar(K_{n_1,n_2,n_3}, C_4^{\text{multi}})$ is the maximum number of colors in an edge-colored K_{n_1,n_2,n_3} with no rainbow C_4^{multi}. In this paper, we determine that $ex(K_{n_1,n_2,n_3}, C_4^{\text{multi}}) = n_1n_2 + 2n_3$ and $ar(K_{n_1,n_2,n_3}, C_4^{\text{multi}}) = ex(K_{n_1,n_2,n_3}, \{C_3, C_4^{\text{multi}}\}) + 1 = n_1n_2 + n_3 + 1$, where $n_1 \geq n_2 \geq n_3 \geq 1$.

1 Introduction

After extended study of Ramsey problems - finding monochromatic subgraphs - anti-Ramsey questions were raised and discussed first by Erdős, Simonovits and Sós [7]. A subgraph of an edge-colored graph is *rainbow*, if all of its edges have different colors. For graphs G and H, the anti-Ramsey number $ar(G, H)$ is the maximum number of colors in an edge-colored G with no rainbow copy of H. Erdős, Simonovits and Sós [7] first studied the anti-Ramsey number in the case when the host graph G is a complete graph K_n and showed the close relationship between it and the Turán number. Since then, the anti-Ramsey numbers for various special graph classes - e.g., paths, cycles, cliques and matchings - in the complete graph have been determined, see e.g. [1, 5, 9, 12, 13, 15]. Later, many different graphs have been considered as the host graph G: complete bipartite or r-partite graphs, complete hypergraphs, complete split graphs and triangulations, etc. In this paper, we consider the anti-Ramsey number of C_4^{multi} in the complete 3-partite graphs.

We consider only nonempty simple graphs. Let G be such a graph, the vertex and edge set of G is denoted by $V(G)$ and $E(G)$, the number of vertices and edges in G by $\nu(G)$ and $e(G)$, respectively. We denote the neighborhood of v in G by $N_G(v)$, and the degree of a vertex v in G by $d_G(v)$, the size of $N_G(v)$. Let U_1, U_2 be vertex sets, denote by $e_G(U_1, U_2)$ the number of edges between U_1 and U_2 in G. We write $d(v)$ instead of $d_G(v)$, $N(v)$ instead of $N_G(v)$ and $e(U_1, U_2)$ instead of $e_G(U_1, U_2)$ if the underlying graph G is clear.

Given a graph family $\mathcal{F}$, we call a graph H an $\mathcal{F}$-free graph, if H contains no graph in $\mathcal{F}$ as a subgraph. The Turán number $ex(G, \mathcal{F})$ for a graph family $\mathcal{F}$ in G is the maximum number of edges in a graph $H \subseteq G$ which is $\mathcal{F}$-free. If $\mathcal{F} = \{F\}$, then we denote $ex(G, \mathcal{F})$ by $ex(G, F)$.

An old result of Bollobás, Erdős and Szemerédi [3] showed that $ex(K_{n_1,n_2,n_3}, C_3) = n_1n_2 + n_1n_3$ for $n_1 \geq n_2 \geq n_3 \geq 1$ (also see [4, 2, 6]). Lv, Lu and Fang [10, 11] constructed balanced 3-partite graphs which are C_4-free and $\{C_3, C_4\}$-free respectively and showed that $ex(K_{n,n,n}, C_4) = (\frac{3}{\sqrt{2}} + o(1))n^{3/2}$ and $ex(K_{n,n,n}, \{C_3, C_4\}) \geq (\sqrt{3} + o(1))n^{3/2}$.

For further discussion, we need the definitions of the multipartite subgraphs and a function $f(n_1, n_2, \dots, n_r)$.

Definition 1. *[8] Let $r \geq 3$ and G be an r-partite graph with vertex partition $V_1, V_2, \dots, V_r$, we call a subgraph H of G* multipartite*, if there are at least three distinct parts V_i, V_j, V_k such that $V(H) \cap V_i \neq \emptyset, V(H) \cap V_j \neq \emptyset$ and $V(H) \cap V_k \neq \emptyset$. In particular, we denote a multipartite H by H^{multi}.*

Fang, Győri, Li and Xiao [8] recently showed that if $G \subseteq K_{n_1,n_2,\dots,n_r}$ and $e(G) \geq f(n_1, n_2, \dots, n_r)+1$, then G contains a multipartite cycle. Furthermore, they proposed the following conjecture.

Conjecture 1. *[8] For $r \geq 3$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, if $G \subset K_{n_1,n_2,\dots,n_r}$ and $e(G) \geq f(n_1, n_2, \dots, n_r) + 1$, then G contains a multipartite cycle C^{multi} of length at most $\frac{3}{2}r$.*

In this paper, we study the Turán numbers of C_4^{multi} and $\{C_3, C_4^{\text{multi}}\}$ in the complete 3-partite graphs and obtain the following results.

Theorem 1. *For $n_1 \geq n_2 \geq n_3 \geq 1$, $ex(K_{n_1,n_2,n_3}, C_4^{multi}) = n_1n_2 + 2n_3$.*

Lower bound: Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 (respectively, V_2) and V_3 are joined by an n_3-matching. Clearly, G is C_4^{multi}-free and $e(G) = n_1n_2 + 2n_3$. Therefore, $ex(K_{n_1,n_2,n_3}, C_4^{multi}) \geq n_1n_2 + 2n_3$.

Theorem 2. *For $n_1 \geq n_2 \geq n_3 \geq 1$, $ex(K_{n_1,n_2,n_3}, \{C_3, C_4^{multi}\}) = n_1n_2 + n_3$.*

Lower bound: Let $G \subseteq K_{n_1,n_2,n_3}$ be a graph, such that V_1 and V_2 are completely joined, V_1 and V_3 are joined by an n_3-matching and there is no edge between V_2 and V_3. Clearly, G is $\{C_3, C_4^{multi}\}$-free and $e(G) = n_1n_2 + n_3$. Therefore, $ex(K_{n_1,n_2,n_3}, \{C_3, C_4^{multi}\}) \geq n_1n_2 + n_3$.

Notice that Theorem 2 confirms Conjecture 1 for the case when $r = 3$.

Theorem 3. *For $n_1 \geq n_2 \geq n_3 \geq 1$, $ar(K_{n_1,n_2,n_3}, C_4^{multi}) = n_1n_2 + n_3 + 1$.*

Lower bound: We color the edges of K_{n_1,n_2,n_3} as follows. First, color all edges between V_1 and V_2 rainbow. Second, for each vertex $v \in V_3$, color all the edges between v and V_1 with one new distinct color. Finally, we assign a new color to all edges between V_2 and V_3. In such way, we use exactly $n_1n_2 + n_3 + 1$ colors, and there is no rainbow C_4^{multi}.

The following lemma plays an important role in our proof.

Lemma 1. *Let G be a 3-partite graph with vertex partition X, Y and Z, such that for all $x \in X$, $N(x) \cap Y \neq \emptyset$ and $N(x) \cap Z \neq \emptyset$.*

(i) If G is C_4^{multi}-free, then $e(G) \leq |Y||Z| + 2|X|$;

(ii) If G is $\{C_3, C_4^{multi}\}$-free, then $e(G) \leq |Y||Z| + |X|$.

Proof. (i) Since G is C_4^{multi}-free, $G[N(x)]$ is $K_{1,2}$-free for each $x \in X$. Therefore,

$$e(G[N(x)]) = e\bigg(N(x) \cap Y, N(x) \cap Z\bigg) \leq \min\bigg\{|N(x) \cap Y|, |N(x) \cap Z|\bigg\}. \tag{1}$$

For $x \in X$, we let e_x be the number of missing edges of G between $N(x) \cap Y$ and $N(x) \cap Z$. By (1), we have

$$\begin{aligned} e_x &= |N(x) \cap Y| \cdot |N(x) \cap Z| - e\Big(N(x) \cap Y, N(x) \cap Z\Big) \\ &\geq |N(x) \cap Y| \cdot |N(x) \cap Z| - \min\Big\{|N(x) \cap Y|, |N(x) \cap Z|\Big\} \\ &\geq |N(x) \cap Y| + |N(x) \cap Z| - 2, \end{aligned} \tag{2}$$

where the last inequality holds since $|N(x) \cap Y| \geq 1$ and $|N(x) \cap Z| \geq 1$ for all $x \in X$.

By (2), we get

$$\sum_{x \in X} e_x \geq \sum_{x \in X} \Big(|N(x) \cap Y| + |N(x) \cap Z| - 2\Big) = e(X, Y) + e(X, Z) - 2|X|. \tag{3}$$

Notice that for any two distinct vertices $x_1, x_2 \in X$, they cannot have common neighboors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Thus each missing edge between Y and Z is calculated at most once in the sum $\sum_{x \in X} e_x$. Hence the number of missing edges between Y and Z is at least $\sum_{x \in X} e_x$. Then we have

$$e(Y, Z) \leq |Y||Z| - \sum_{x \in X} e_x \leq |Y||Z| - (e(X, Y) + e(X, Z) - 2|X|). \tag{4}$$

By (4), we get

$$e(G) = e(X, Y) + e(X, Z) + e(Y, Z) \leq |Y||Z| + 2|X|.$$

(ii) Since G is C_3-free, for each $x \in X$,

$$e\Big(N(x) \cap Y, N(x) \cap Z\Big) = 0. \tag{5}$$

Since for each $x \in X$, $|N(x) \cap Y| \geq 1$ and $|N(x) \cap Z| \geq 1$ hold, by (5), the number of missing edges between $N(x) \cap Y$ and $N(x) \cap Z$ is $|N(x) \cap Y| \cdot |N(x) \cap Z|$. Notice that for any two distinct vertices $x_1, x_2 \in X$, they cannot have common neighboors in both Y and Z at the same time, otherwise we find a copy of C_4^{multi} in G. Hence, the number of missing edges between Y and Z is at least $\sum_{x \in X} |N(x) \cap Y| \cdot |N(x) \cap Z|$. Thus,

$$\begin{aligned} e(Y, Z) &\leq |Y||Z| - \sum_{x \in X} |N(x) \cap Y| \cdot |N(x) \cap Z| \\ &\leq |Y||Z| - \sum_{x \in X} (|N(x) \cap Y| + |N(x) \cap Z| - 1) \\ &= |Y||Z| + |X| - e(X, Y) - e(X, Z), \end{aligned} \tag{6}$$

the second inequality holds since $|N(x) \cap Y| \geq 1$ and $|N(x) \cap Z| \geq 1$ for $x \in X$.

By (6), we have $e(G) = e(Y, Z) + e(X, Y) + e(X, Z) \leq |Y||Z| + |X|$. □

The proof of the general case when the conditions of this lemma do not hold is several pages, we cannot include it in this extended abstract.

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