On k-community structures in special graph classes

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Abstract

A k-community structure in a graph is a partition of its vertex set into k sets, called communities, such that every vertex of the graph has proportionally at least as many vertices in its community as in any other. In this paper, we introduce several properties that have to be satisfied by the vertices of a graph whenever it admits a 2-community structure. We show that a tree T admits a k-community structure that can be found in polynomial time if and only if T has a matching of size at least k, for $k \geq 2$. Furthermore, we define a subfamily of threshold graphs and show that it always admits a 2-community structure that can be found in linear time. We finally introduce an infinite family of graphs, which can have odd or even size, that does not admit any 2-community structure.

1 Introduction

Throughout the paper, we denote by $[\![t]\!]$, with $t \in \mathbb{Z}$, the set of all integers greater than or equal to 1 and at most t, that is the set $\{1, 2, \ldots, t\}$. Two vertices are *false twins* if they are adjacent and have the same neighbourhood. Two vertices are *true twins* if they are non-adjacent and have the same neighbourhood.A *k-community structure* of a graph $G = (V, E)$ is a k-partition $\Pi = \{C_1, \ldots, C_k\}$ of the vertex set V such that:

- $k \geq 2$,
- for all $i \in \{1, \ldots, k\}$, $|C_i| \geq 2$,
- for all $v \in C_i$, for all $C_j \in \Pi$, $i \neq j$, the following property holds:

$$\frac{|N_{C_i}(v)|}{|C_i| - 1} \geq \frac{|N_{C_j}(v)|}{|C_j|}. \tag{1}$$

Each set $C_i \in \Pi$, for $i \in [\![k]\!]$, is called a community. Formally, the k-community problem is defined as follows:

k-COMMUNITY
Input: A graph $G = (V, E)$.
Question: Does the graph G admit a k-community structure?

In [1], the authors showed that graphs of maximum degree 3, graphs of minimum degree at least $|V| - 3$ and trees always admit a 2-community structure and that it can be found in polynomial time. In [2], the authors introduced an infinite family of graphs that does not admit a 2-community structure. Notice, that any graph of this family has an even number of vertices. Furthermore, in [3] it was shown that the k-community problem is solvable in polynomial time in graphs of bounded clique-width. In this paper, we first introduce properties that have to be satisfied by the vertices of a graph G whenever it admits a 2-community structure. Then, we show that a tree T admits a k-community structure if and only if T has a matching of size at least k, for any $k \geq 2$, and it can be found in polynomial time. We also define a subfamily of threshold graphs and show that it always admits a 2-community structure that can be found in linear time. Finally, we introduce an infinite family of graphs that does not admit any 2-community structure. The graphs in this family can have an even or odd number of vertices.

2 Properties

Property 1. *Let u and v be false twins in some graph $G = (V, E)$, such that $N(v) \neq V \backslash \{v\}$. Then, in any 2-community structure of G, u and v must belong to the same community.*

Let $G = (V, E)$ be a graph for which there exists a 2-community structure, denoted by $\Pi = \{C_1, C_2\}$. Then, we have the following properties.

Property 2. *Let C'_1 (resp. C'_2) be the subset of vertices in G which have already been assigned to C_1 (resp. C_2). Furthermore, let U be the vertices which have not yet been assigned to any of these two sets, i.e. $U = V \setminus (C'_1 \cup C'_2)$. If assigning some $v \in U$ and all its neighbours in U to C_i as well as all non-neighbours of v in U to C_{3-i} for $i \in \{1, 2\}$, makes (1) fail, then we cannot find any 2-community structure $\Pi = \{C_1, C_2\}$ of G such that $C'_1 \subseteq C_1$, $C'_2 \subseteq C_2$.*

Property 3. *Let $v \in V$ and $i \in \{1, 2\}$. If $N_{C_i}(v) = \emptyset$, then $v \in C_{3-i}$.*

Property 4. *Let $v \in V$ and $i \in \{1, 2\}$. If $C_i \subseteq N[v]$ and $C_{3-i} \not\subseteq N[v]$, then $v \in C_i$.*

3 k-community in trees

Let $\Pi = \{C_1, \ldots, C_k\}$ be a k-partition of a graph G. Then *a size tuple of* Π is defined as a k-tuple $(s_1, \ldots, s_k)$, where $s_i = |C_i|$ and such that $s_i \leq s_{i+1}$ and $s_i \geq 2$ for all $i \in [\![k-1]\!]$. A (connected) k-partition Π of G is *uniform* if its size tuple is lexicographically largest

Lemma 1. *Let $T = (V, E)$ be a tree such that $|V| \geq 2k$ for some positive integer $k \geq 2$. Let Π be a connected uniform k-partition of T. Then Π forms a connected k-community structure.*

Proof. First, note that since $\Pi = \{C_1, \ldots, C_k\}$ is a connected uniform k-partition, we have that $|C_j| \leq |C_{j+1}|$, for all $j \in [\![k-1]\!]$. Assume that there exists a vertex $v \in C_i$, for some $i \in [\![k]\!]$, such that the (1) fails for v. Notice, that $v \in C_i$, for some $i \in [\![k]\!]$, has at most one neighbor in C_j, for any $i \neq j \in [\![k]\!]$. Hence, the following inequality holds for the vertex v and some C_j, for $i \neq j \in [\![k]\!]$: $|C_i| - 1 \geq d_{C_i}(v) \cdot |C_j| + 1$. Moreover, $|C_i \setminus \{v\}| \geq d_{C_i}(v) \cdot |C_j| + 1$, and

$$|V(T_{v'})| \geq \frac{|C_i \setminus \{v\}|}{d_{C_i}(v)} > |C_j| \,, \tag{2}$$

for some tree T_v in the forest induced by the vertices in $|C_i \setminus \{v\}|$. Hence, by the definition of Π we have that $j < i$. Assume without loss of generality, that $|C_j| < |C_{j+1}|$. Let $A = (C_i \setminus V(T_v)) \cup C_j$ and $B = V(T_v)$. We define a k-partition Π' of T as follows: $\Pi' = (\Pi \backslash \{C_i, C_j\}) \cup \{A, B\}$. Clearly, Π' is a connected k-partition of T, since both $T[A]$ and $T[B]$ induce connected subgraphs of T. Then, we know the following: (i) Since $V(T_v) \subset C_i$, we have that $|A| = |C_i| - |V(T_v)| + |C_j| > |C_j|$; (ii) We know that $|B| = |V(T_v)| > |C_j|$ from the eq. 2. Furthermore, we rename the elements of Π' in such a way that $|C'_\ell| \leq |C'_{\ell+1}|$ for all $\ell \in [k-1]$. By the arguments above let us set $P'_\ell = P_\ell$ for all $\ell \in [j-1]$. Since Π is a connected uniform k-partition of G, we have that $|C'_j| \leq |C_j|$. And since $C_i, C_j \notin \Pi'$, we have that either $C'_j = C_\ell$ for some $\ell \in \{j+1, \ldots, k\} \setminus \{i\}$ or $C'_j \in \{A, B\}$. Let us consider the first possibility. If $C'_j = C_\ell$ for some $\ell \in \{j+1, \ldots, k\} \setminus \{i\}$, then $|C'_j| > |C_j|$, a contradiction from the fact that Π is a connected uniform k-partition. We conclude that $C'_j \in \{A, B\}$. However, by the arguments above we know that $|A|, |B| > |C_j|$, and hence it implies that $|C'_j| > |C_j|$, a contradiction from the fact that the size tuple of Π' is lexicographically larger than the size tuple of Π and Π is a connected uniform k-partition of T. Thus, every $v \in V$ is satisfied with respect to Π, and hence Π is a connected k-community structure of T. □

Theorem 5. *Let $T = (V, E)$ be a tree and $k \geq 2$ be some positive integer. Then T admits a connected k-community structure that can be computed in time $\mathcal{O}(n^k)$ if and only if T has a matching of size at least k.*

We get the complexity of $\mathcal{O}(n^k)$ in the theorem above by scaning all possible sets of $k-1$ edges. For each of the sets we consider k partition obtained by deleting those edges. Further, we compute the size of each community and check whether vertices satisfy the inequality 1.

4 Threshold graphs

Let $G = (C \cup I, E)$ be a threshold graph not isomorphic to a star such that $|C \cup I| = n$, where $C = \{v_1, \ldots, v_c\}$ is a clique and $I = \{w_1, \ldots, w_s\}$ is an independent set. Let $c = (s-1)j + k$, where $k = d(w_1)$ and $j = d(w_{i+1}) - d(w_i)$, for $i \in \{1, \ldots, s-1\}$. And let $\Pi = \{C_1, C_2\}$ be a 2-partition of the vertex set V such that $|C_i| \geq 2$, for $i = 1, 2$, and $C_1 = \{v_1, w_1, \ldots, w_i\}$ for some $i \in \{1, \ldots, s\}$ and $C_2 = V \setminus C_1$.

Some observations:

- Vertex $v_1 \in V$ is adjacent to all vertices in $V \setminus \{v_1\}$. Hence, (1) always holds for v_1.
- Each vertex in $\{v_2, \ldots, v_{k+(i-1)j}\}$ is adjacent to some $w_\ell \in C_1$ for some $\ell \in \{1, \ldots, i\}$. Then every vertex in $\{v_2, \ldots, v_{k+(i-1)j}\}$ is adjacent to all $v \in C_2$ and hence (1) holds for all of them.
- Let $\{v_{k+(i-1)j+1}, \ldots, v_c\}$ be a subset of vertices that are not adjacent to any $w_\ell \in C_1$, for $\ell \in \{1, \ldots, i\}$. Then if (1) holds for the vertex v_c, then it holds for all vertices in $\{v_{k+(i-1)j+1}, \ldots, v_{c-1}\}$.
- If (1) holds for w_i then it holds for all w_ℓ where $\ell < i$. Moreover, if (1) holds for w_{i+1} then it holds for all w_m where $m > i$.

Theorem 6. *Let G be a threshold graph defined as above. Then G has a connected 2-community structure. Furthermore, it can be found in linear time.*

Proof. We show that if $i^2 - i \leq \frac{n-1-ki}{j}$ and $i^2 + i \geq \frac{n-1-k-ki}{j}$ holds then $\{C_1, C_2\}$ defined above is a connected 2-community structure. From the observations introduced above it follows that it suffices to show that the property holds for vertices v_c, w_i and w_{i+1}.

Assume that (1) does not hold for vertex v_c. Then the following inequality holds: $\frac{d(w_s)}{d(w_s)-2+s-i} < \frac{1}{i+1}$ and $(s-1)j+k < \frac{s-i-2}{i}$, a contradiction. Assume that (1) does not hold for vertex w_i. Then the following inequality holds: $\frac{1}{i} < \frac{k+(i-1)j-1}{d(w_s)+s-i-1}$ and $\frac{d(w_s)+s-1-ki}{j} = \frac{n-1-ki}{j} < i^2 - i$, a contradiction. And finally, assume that (1) does not hold for vertex w_{i+1}. Then the following inequality holds: $\frac{k+ij-1}{d(w_s)+s-i-2} < \frac{1}{i+1}$ and $\frac{d(w_s)+s-1-ki-k}{j} = \frac{n-1-ki-k}{j} > i^2 + i$, a contradiction. Hence we conclude that $\Pi = \{C_1, C_2\}$ is a connected 2-community structure. □

5 Graphs with no 2-community structure

Let us introduce an infinite family of graphs that does not admit any 2-community structure. Let $k, l \in \mathbb{N}^+$. The graph $G_{k,l} = (V, E)$ (see Figure 1) is defined as follows: (i) $V = \{u, v_0, \ldots, v_4\} \cup F \cup T$, where F is a set of k vertices $f_1, \ldots, f_k$ which are pairwise false twins and T is a set of ℓ vertices $t_1, \ldots, t_\ell$ that are pairwise true twins; (ii) u is adjacent to all vertices in $V \setminus \{v_0\}$; (iii) E contains in addition the edges $v_0v_1, v_1v_2, v_1v_3, v_2v_4$ as well as either the edge v_2v_3 or the edge v_3v_4; (iv) finally, all the vertices of F are adjacent to all the vertices of $T \cup \{u, v_1, v_3\}$.

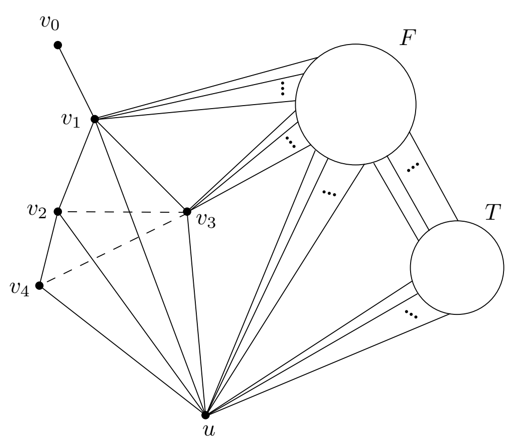

Figure 1: The graph $G_{k,l}$.

Based on this description, we may define the infinite family of graphs $\mathcal{G} = \{G_{k,\lceil \frac{k}{2} \rceil} : \ k \geq 3\}$. A tedious case analysis allows to show that graphs in this family admit no 2-community structure.

Theorem 7. *For $k \geq 3$, $G_{k,\lceil \frac{k}{2} \rceil} = (V, E)$ does not admit any 2-community structure.*

References

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