Vizing's conjecture holds

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Abstract

In 1964 Vizing proved that to properly color the edges of a graph G, one need at most $\Delta+1$ colors, where Δ is the maximum degree of G. In his paper, Vizing actually proves that one can transform any proper edge coloring into a $(\Delta+1)$-edge-coloring using only Kempe changes. Soon after his paper, he asked the following question: is an optimal edge-coloring always reachable from any proper edge-coloring using only Kempe changes? Bonamy & al. proved that the conjecture holds for triangle free graphs, following their work, we prove that it holds for all graphs.

1 Introduction

Vizing proved in 1964 [1] that to properly color the edges of a graph G, one needs at most $\Delta(G)+1$ colors where $\Delta(G)$ denotes the maximum degree of G.

Theorem 1 ([1]). *Any simple simple graph G satisfies $\chi'(G) \leq \Delta(G) + 1$.*

Here, $\chi'(G)$ denotes the chromatic index of G, *ie* the minimum integer k such that G admits a proper k-edge-coloring. The proof of Theorem 1 heavily relies on Kempe changes, a concept invented in 1879 by Kempe in his failed attempt to prove the four-color theorem. Given a graph G and a proper edge-coloring α of G, a *Kempe chain* is a maximal bichromatic component of G. Applying a *Kempe change* (or Kempe swap) on a Kempe chain consists in switching the colors of the edges in the Kempe chain. The result obtained is a new coloring α', called *equivalent* to α, which is also guaranteed to be proper (unless the Kempe chain is spanning all the edges of the two selected colors, the partition of the edges will be changed after the swap).

Note that a Kempe chain can consist in a single edge, and if this edge is the only edge colored c, then applying a Kempe swap on this edge may reduce the number of colors in the coloring as no more edges will be colored c after the swap. In his paper of 64, Vizing actually proves that, starting from any proper edge-coloring, one can find a sequence of Kempe changes to transform the original coloring into a $(\Delta + 1)$-edge-coloring.

Theorem 2 ([1]). *For any simple graph G, for any integer k, and any k-edge-coloring α of G, there exists a $(\Delta + 1)$-edge-coloring α' equivalent to α.*

In 1965, Vizing extends this result to multigraphs, and ask the following question: for any k-edge-coloring α, is there always an optimal coloring α' equivalent to α? Note that this question does not imply that all optimal edge-colorings are equivalent, *ie* we do not have the choice of the target optimal coloring. Asratian et al. answer by the affirmative to a slightly stronger question (we can choose the target optimal coloring) for the case graphs with maximum degree 4 [4], and Bonamy et al. proved that it is also true for triangle-free graphs [2].

Theorem 3 ([2]). *For every triangle-free graph G and for any integer $k > \chi'(G)$, any k-edge-coloring of G is equivalent to any $\chi'(G)$-edge-coloring of G.*

Our contribution is to prove that the conjecture holds in the general case.

Theorem 4. *For every graph G and for any integer $k > \chi'(G)$, any k-edge-coloring of G is equivalent to any $\chi'(G)$-edge-coloring of G.*

2 Result

2.1 General setting

The general approach toward Theorem 4 follows that of [2] which itself follows that of [4] and [5]. Mohar proved that any k-edge-coloring of a graph G (with $k > \chi'(G) + 1$) is equivalent to any $(\chi'(G) + 1)$-edge-coloring of G, so it suffices to prove that any $(\chi'(G) + 1)$-edge-coloring of G is equivalent to any optimal edge-coloring of G. The general setting of the proof is as follows: we first prove that we can reduce the problem to the case of regular graphs. Indeed, any graph G is the induced subgraph of a $\chi'(G)$-regular graph G' such that:

- Any $(\chi' + 1)$-edge-coloring of G can be extended to G', and
- if two $(\chi' + 1)$-edge-colorings of G' are equivalent, then their restriction to G are also equivalent.

This allows us to only consider the following case: let G be a $\Delta(G)$-regular graph, α a $\chi'(G)$-edge coloring of G and β a $(\chi'(G) + 1)$-coloring of G. Our goal is to show that β is equivalent to α (*ie* we must find a sequence of Kempe changes to transform β into α). Note that in any $(\chi'(G)+1)$-edge-coloring, each vertex v is incident to all but one color, we call this color the *missing* color at v.

The general approach is an induction on the chromatic index. Let c be a color used in α, the edges of G colored c in α induce a perfect matching M in G, we prove that there exists a coloring β' equivalent to β such that for any edge e, $\beta'(e) = c \Leftrightarrow \alpha(e) = c$. We then consider the graph $G' = G - M$, the graph G' is $(\Delta(G) - 1)$-regular, and we have $\chi'(G') = \chi'(G) - 1$, thus we can apply the induction. To find such a coloring β', we first need some terminology to quantify how far we are from this coloring. Given any edge-coloring γ we call an edge e:

- *good* if $e \in M$ and $\gamma(e) = c$,
- *bad* if $e \notin M$ and $\gamma(e) = c$, and
- *ugly* if $e \in M$ and $\gamma(e) \neq c$

We then consider an edge-coloring γ equivalent to β which minimizes the number of ugly edges among the edge-colorings equivalent to β that minimizes the number of bad edges; we call *minimal* such a coloring. Our goal is to find a coloring equivalent to γ with fewer bad edges, or with the same number of bad edges, and fewer ugly edges. To do so, we use a technical tool invented by Vizing himself in his proof of 64 that allows us to apply Kempe changes in a very controlled way.

2.2 Vizing's fans

To define *Vizing's fans*, we first need to define an auxiliary graph. Given a $(\chi'(G)+1)$-edge-coloring γ, and a vertex v of G we define the directed graph D_v with vertex set $\{vw|w \in N(v)\}$, and where there is an arc between vw_1, and vw_2 if the missing color at w_1 is $\gamma(vw_2)$. The fan $X_v(w)$ around a vertex v starting at the edge vw is the maximal component of D_v reachable from vw. In the digraph D_v, each vertex has outdegree at most 1, so we only have 3 cases:

- $X_v(w)$ is a directed path

- $X_v(w)$ is a directed cycle
- $X_v(w)$ is a directed comet, *ie* a path with an additional arc between the sink of the path, and another vertex of the path which is neither the sink nor the source of the path.

If a fan $X_v(w_1) = (vw_1, \cdots, vw_k)$ is a cycle, and there exists an edge-coloring γ' equivalent to γ such that for any i:

- w_i is missing the color $\gamma(vw_i)$ in γ', and
- the edge vw_i is colored $\gamma(vw_{i-1})$ in γ'

then $X_v(vw_1)$ is called *invertible*, and γ' is called the invert of $X_v(w_1)$. Observation 2.1, and Lemma 2.4 of [2] already handle the case of paths and comets, and Lemma 3.2 of the same paper shows that we only need to prove that every cycle is invertible.

2.3 Our contribution

Our contribution is indeed to prove that any cycle is invertible.

Theorem 5. *Given any $(\chi' + 1)$-coloring of G, any fan X which is a cycle is invertible*

The general outline of the proof is an induction on the size of the cycles. A cycle of size 2 is trivially invertible as it consists in a 2-edge Kempe chain. Let $X = (vw_1, \cdots, vw_k)$ be a minimal non-ivertible fan around a vertex v in a coloring γ of G. We prove that G consists in an even clique, and that the coloring γ is such that each vertex of G misses a different color. This leads to a contradiction: in any $(\chi'(G) + 1)$-coloring of an even clique, for any color c, there is always an even number of vertices missing the color c.

To prove Theorem 5 we precisely study the structure of the fans around v and the neighbors of v. We prove that all these fans are always cycles, and that they pairwise always share at least one vertex of G. Theorem 5 is a direct consequence of the three following Lemmas.

Lemma 1. *Let w_j and $w_{j'}$ be two neighbors of v in X, and e be the edge incident with w_j colored with the color missing at $w_{j'}$, then the fan around w_1 starting at the edge e is a cycle containing w_2.*

Lemma 2. *Let w be a neighbor of v, then the fan around v starting at vw is a cycle.*

Lemma 3. *Let w be a neighbor of v, and X' a fan around w which is a cycle, for any color c different from the missing color at v, the fan around v starting at the edge colored c is a cycle Y, and the fan around w starting at the edge colored c is a cycle containing the vertex missing the color c in the fan Y.*

The proofs of theses lemmas heavily relies on two simple yet useful observations:

- In an edge-coloring, a Kempe chain is either a path or an even cycle.
- Two Kempe changes respectiveley involving the colors $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are commutative if $|\{i_1, j_1, i_2, j_2\}| = 4$.

3 Concluding remarks

This result answer by the affirmative to Vizing's question, however we mentioned in the introduction that Vizing generalized his theorem on simple graphs to multigraphs in 1965.

Theorem 6. *For any multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Where $\mu(G)$ denotes here the maximum multiplicity of the multiedges of G. Generalizing the main result to multigraphs would thus be an interesting question. Furthermore, Cranston et al. recently proved the Kempe equivalence of the Δ-list-vertex-coloring of Δ-regular graphs [3], another interesting question would hence be: can the Vizing's conjecture be generalized to list-edge-coloring.

References

[1] V. Vizing. On an estimate of the chromatic class of a p-graph. Discret Analiz, 1964.

[2] M. Bonamy, O. Defrain, T. Klimošová, A. Lagoutte, and J. Narboni. On Vizing's edge colouring question. arXiv preprint arXiv:2107.07900

[3] D. W. Cranston, R. Mahmoud. Kempe Equivalent List Colorings. arXiv preprint arXiv:2112.07439

[4] A. Asratian, C. Casselgren. Solution of Vizing's Problem on Interchanges for the case of Graphs with Maximum Degree 4 and Related Results. Journal of Graph Theory, Wiley Online Library, 2016

[5] B. Mohar. Kempe equivalence of colorings. Graph Theory in Paris, Springer, 2006