Infinite families of non-superperfect graphs based on even antiholes

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Abstract

Given a graph G with non-negative integral node weights $\mathbf{w}$, an interval coloring is an assignment of w_v consecutive colors to the nodes v of G such that adjacent nodes receive different colors. A graph G is superperfect if the maximum weight of a clique of G equals the minimum number of colors needed in an interval coloring of G for all non-negative integral node weights $\mathbf{w}$. A characterization of superperfect graphs by minimal non-superperfect graphs is not yet known. Hoffman showed in 1974 that every comparability graph is superperfect, hence all non-superperfect graphs are in particular non-comparability. Gallai provided 1967 a complete list of minimal non-comparability graphs, and results of Golumbic 1980 and Andreae 1985 show which minimal non-comparability graphs are also minimal non-superperfect and which are superperfect. According to Golumbic, even antiholes are minimal non-comparability superperfect graphs. In this work, we provide infinite families of non-superperfect graphs having even antiholes as only minimal non-comparability subgraphs.

1 Introduction

Let $G = (V, E)$ be a graph with weight function $\mathbf{w} : V \to \mathbb{N}_0$. A *weighted coloring* of a weighted graph $(G, \mathbf{w})$ is an assignment of w_v colors to the nodes v of G such that adjacent nodes receive different colors. The *weighted chromatic number* of $(G, \mathbf{w})$ is defined as the smallest number of colors needed for a weighted coloring, and is denoted by $\chi(G, \mathbf{w})$.

A generalization of graph coloring in weighted graphs $(G, \mathbf{w})$ was introduced in [7] as *interval coloring* where the task is to assign intervals of width w_v to the nodes v of G such that adjacent nodes receive disjoint intervals, see e.g. [4, 6, 8] for applications. In other words, an interval coloring of $(G, \mathbf{w})$ is a weighted coloring where all w_v colors assigned to each node v are consecutive. The *interval chromatic number* $\chi_I(G, \mathbf{w})$ is the minimum spectrum width such that $(G, \mathbf{w})$ has a proper interval coloring. The *weighted clique number* $\omega(G, \mathbf{w})$ is the maximum weight of a clique in $(G, \mathbf{w})$. Clearly, the weighted clique number $\omega(G, \mathbf{w})$ is a lower bound on the weighted chromatic number $\chi(G, \mathbf{w})$ (as all nodes v in a clique have to receive w_v different colors), and $\chi(G, \mathbf{w})$ is a lower bound on the interval chromatic number $\chi_I(G, \mathbf{w})$ (as all w_v colors assigned to each node v have to be consecutive). Thus, we obtain for any weighted graph $(G, \mathbf{w})$ that

$$\omega(G, \mathbf{w}) \leq \chi(G, \mathbf{w}) \leq \chi_I(G, \mathbf{w})$$

holds. Berge [2] called a graph G *perfect* if $\omega(G, \mathbf{w}) = \chi(G, \mathbf{w})$ for all 0/1-valued weights $\mathbf{w}$, and Lovász [9] showed that a graph G is perfect if and only if $\omega(G, \mathbf{w}) = \chi(G, \mathbf{w})$ holds for all weights $\mathbf{w} \in \mathbb{N}_0$. (Note that including zero-weights is equivalent to requiring the property for all induced subgraphs). Berge observed that all chordless odd cycles C_{2k+1} with $k \geq 2$, called *odd holes*, and their complements, the *odd antiholes* $\overline{C}_{2k+1}$, are imperfect. Chudnovsky et al. [3] verified a famous conjecture of Berge that a graph G is perfect if and only if G has no odd hole or odd antihole as induced subgraph [2].

Graphs where weighted clique number and interval chromatic number coincide for all possible non-negative integral weights are called *superperfect*, see e.g. [6]. This shows in particular that every superperfect graph is perfect.

A graph $G = (V, E)$ is *comparability* if and only if there exists a partial order $\mathcal{O}$ on $V \times V$ such that $uv \in E$ if and only if u and v are comparable w.r.t. $\mathcal{O}$. Hoffman [7] proved that every comparability graph is superperfect, and Gallai [5] provided a complete list of minimal non-comparability graphs, that are

- odd holes C_{2k+1} for $k \geq 2$ and antiholes $\overline{C}_n$ for $n \geq 6$,
- the graphs J_k and J'_k for $k \geq 2$ and the graphs J''_k for $k \geq 3$ (see Fig. 1),
- the complements of D_k for $k \geq 2$ and of E_k, F_k for $k \geq 1$ (see Fig. 2),
- the complements of $A_1, \ldots, A_{10}$ (see Fig. 3).

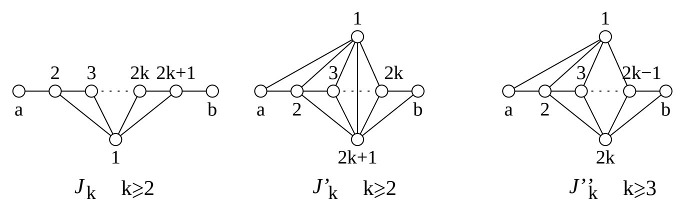

Figure 1: Minimal non-comparability graphs: J_k, J'_k for $k \geq 2$ and J''_k for $k \geq 3$.

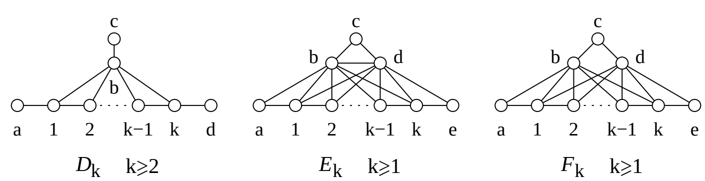

Figure 2: Minimal non-comparability graphs: the complements of D_k, E_k, F_k.

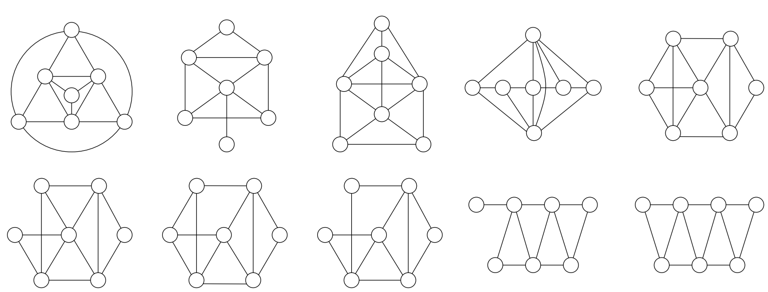

Figure 3: Minimal non-comparability graphs: $\overline{A}_1, \ldots, \overline{A}_{10}$ (from top left to bottom right).

The above results imply that every non-superperfect graph is non-comparability, which raises the question which minimal non-comparability graphs are also minimal non-superperfect. Clearly, odd holes and odd antiholes are minimal non-superperfect (as they are minimal non-perfect). It has been shown by Golumbic [6] that $\overline{A}_1$, $\overline{D}_2$, $\overline{E}_1$, $\overline{E}_2$ and J_2 are non-superperfect, but that even antiholes $\overline{C}_{2k}$ for all $k \geq 3$ are superperfect. Furthermore, Andreae showed in [1], that the graphs

J_2', J_k'' for $k \geq 3$ and the complements of $A_3, \ldots, A_{10}$ are superperfect, but that the graphs J_k for $k \geq 2$ and J_k' for $k \geq 3$ as well as $\overline{D}_k$ for $k \geq 2$ and $\overline{E}_k$, $\overline{F}_k$ for $k \geq 1$ are non-superperfect.

In contrary to comparability and perfect graphs, there is no characterization of minimal non-superperfect graphs known yet. It is, therefore, of interest to find new examples of such graphs. As noted in [8], all such graphs must not contain any known minimal non-superperfect graph, but must contain a minimal non-comparability superperfect graph as proper induced subgraph, thus, one of the following graphs: even antiholes $\overline{C}_{2k}$ for $k \geq 3$, J_2', the graphs J_k'' for $k \geq 3$, or $\overline{A}_3, \ldots, \overline{A}_{10}$. Based on this observation, some new minimal non-superperfect graphs have been already presented in [8]. That are graphs containing $\overline{A}_6$, $\overline{A}_7$, and $\overline{A}_{10}$ and a further node v, see Fig. 4 for the graphs and node weights $\mathbf{w}$ causing a gap between weighted clique and interval chromatic number.

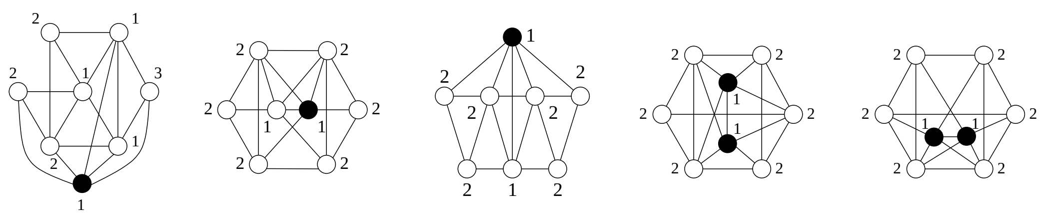

Figure 4: Minimal non-superperfect graphs containing $\overline{A}_6$, $\overline{A}_7$, $\overline{A}_{10}$ (left), and $\overline{C}_6$ (right) where white nodes induce the minimal non-comparability subgraphs.

Each further minimal non-superperfect graph is of interest, but we are particularly interested in *infinite families* of such graphs. Note that they have to be based on one of the infinite families of minimal non-comparability superperfect graphs: even antiholes $\overline{C}_{2k}$ for $k \geq 3$ and J_k'' for $k \geq 3$. We are going to present infinite families of new non-superperfect graphs based on even antiholes.

2 Non-superperfect graphs based on even antiholes

Let $\overline{C}_{2k}$ be an even antihole for some $k \geq 3$ and let $\overline{C}_{2k,j}$ be the graph obtained from $\overline{C}_{2k}$ by adding two adjacent nodes x and y, where x is adjacent to all nodes of $\overline{C}_{2k}$ but 1 and 2, and y is adjacent to all nodes of $\overline{C}_{2k}$ but $2+j$ and $3+j$, for all even j with $0 \leq j < k$. Note that $\overline{F}_1$ can be considered to be $\overline{C}_{4,0}$; the graphs $\overline{C}_{6,0}$ and $\overline{C}_{6,2}$ are the two rightmost graphs shown in Fig. 4.

Theorem 1. *$\overline{C}_{2k,j}$ is minimal non-superperfect for all $k \geq 3$ and all even j with $0 \leq j < k$.*

Proof. (sketch) Indeed, $\overline{C}_{2k,j}$ is non-superperfect due to $\omega(\overline{C}_{2k,j}, \mathbf{w}) < \chi_I(\overline{C}_{2k,j}, \mathbf{w})$ with $w_x = w_y = 1$ and $w_i = 2$ for all nodes i in $\overline{C}_{2k}$. Minimality follows since $\overline{C}_{2k,j}$ has the even antihole $\overline{C}_{2k}$ as only minimal non-comparability subgraph, and we can show that $\overline{C}_{2k,j} - \{x\}$ and $\overline{C}_{2k,j} - \{y\}$ are superperfect. $\square$

Moreover, we observe that $\overline{C}_{6,0}$ and $\overline{A}_7 + v$ from Fig. 4 differ in exactly one edge. In fact, removing edge 25 from $\overline{C}_{6,0}$ yields a graph isomorphic to $\overline{A}_7 + v$. Next we examine the removal of which edges from $\overline{C}_{2k,j}$ results in further infinite families of non-superperfect graphs.

Theorem 2. *The graphs $\overline{C}_{2k,j} - (2, 2k-1)$ are non-superperfect for all $k \geq 3$ and all even j with $0 \leq j < k$ and contain $\overline{A}_7$ (for $k = 3$) and $\overline{C}_{2(k-1)}$ (for $k \geq 4$) as only minimal non-comparability subgraph.*

Proof. (sketch) Indeed, $\overline{C}_{2k,j} - (2, 2k-1)$ is non-superperfect due to $\omega(\overline{C}_{2k,j} - (2, 2k-1), \mathbf{w}) < \chi_I(\overline{C}_{2k,j} - (2, 2k-1), \mathbf{w})$ with $w_x = w_y = 1$ and $w_i = 2$ for all other nodes i. Using Gallai's characterization of minimal non-comparability graphs, we can verify that $\overline{C}_{6,0} - 25$ and $\overline{C}_{6,2} - 25$ have $\overline{A}_7$ and that $\overline{C}_{2k,j} - (2, 2k-1)$ has the even antihole $\overline{C}_{2(k-1)}$ induced by nodes $2, 3, \ldots, 2k-1$ as only minimal non-comparability subgraph. □

This implies that $\overline{C}_{6,0} - 25$ and $\overline{C}_{6,2} - 25$ are minimal non-superperfect. We further conjecture:

Conjecture 1. $\overline{C}_{2k,j} - (2, 2k-1)$ *is minimal non-superperfect for* $k \geq 4$ *and even* j *with* $0 \leq j < k$.

Moreover, we further observe that $\overline{C}_{6,2} - 25$ is isomorphic to $\overline{C}_{6,2} - 14$. Hence, also $\overline{C}_{6,2} - 14$ is minimal non-superperfect and the smallest graph in another family of new non-superperfect graphs:

Theorem 3. *The graphs* $\overline{C}_{2k,j} - (1, 4)$ *are non-superperfect for all* $k \geq 3$ *and all even* j *with* $2 \leq j < k$ *and contain* $\overline{A}_7$ *(for* $k = 3$*) and* $\overline{C}_{2(k-1)}$ *(for* $k \geq 4$*) as only minimal non-comparability subgraph.*

Proof. (sketch) Indeed, $\overline{C}_{2k,j} - (1, 4)$ is non-superperfect due to $\omega(\overline{C}_{2k,j} - (1, 4), \mathbf{w}) < \chi_I(\overline{C}_{2k,j} - (1, 4), \mathbf{w})$ with $w_x = w_y = 1$ and $w_i = 2$ for all other nodes i. Using Gallai's characterization of minimal non-comparability graphs, we can verify that $\overline{C}_{6,2} - 14$ has $\overline{A}_7$ and $\overline{C}_{2k,j} - (1, 4)$ the even antihole $\overline{C}_{2(k-1)}$ induced by nodes $4, 5, \ldots, 2k, 1$ as only minimal non-comparability subgraph. □

This implies that $\overline{C}_{6,2} - 14$ is minimal non-superperfect and we further conjecture:

Conjecture 2. $\overline{C}_{2k,j} - (1, 4)$ *is minimal non-superperfect for all* $k \geq 4$ *and even* j *with* $2 \leq j < k$.

3 Concluding remarks

We presented in this paper three infinite families of non-superperfect graphs based on even antiholes. Our future research includes to prove the above conjectures about the minimality of the latter two families and to identify further minimal non-superperfect graphs containing only superperfect minimal non-comparability subgraphs. The final goal shall be to characterize superperfect graphs by giving a complete list of minimal non-superperfect graphs.

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