Exactly Hittable Interval Graphs

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Abstract

Given a set system (also well-known as a hypergraph) $H = \{\mathcal{U}, \mathcal{X}\}$, where $\mathcal{U}$ is a set of elements and $\mathcal{X}$ is a set of subsets of $\mathcal{U}$, an exact hitting set S is a subset of $\mathcal{U}$ such that each subset in $\mathcal{X}$ contains exactly one element in S. We refer to a set system as *exactly hittable* if it has an exact hitting set. In this paper, we study interval graphs which are the intersection graphs of set systems that are exactly hittable. We refer to these interval graphs as *exactly hittable interval graphs* (EHIG). We present a forbidden structure characterization for EHIG and also show that the class of proper interval graphs is a strict subclass of EHIG. Finally, we give an algorithm that runs in polynomial time to recognize graphs belonging to the class of EHIG.

1 Introduction

An interval is a set of consecutive elements belonging to the set $\{1, \ldots, n\}$. In [1], it is shown that a conflict free coloring of a set of intervals is exactly a partition into sets of intervals such that each set has an exact hitting set. This motivates the question of characterizing those sets of intervals which have an exact hitting set. A natural characterization is obtained by writing the hitting set linear program with one constraint per interval. This system is totally unimodular and thus defines an integer polytope [2]. Thus the intervals have an exact hitting set if and only if the polytope defined by the exact hitting set linear program is non-empty. Thus it is possible to find if the interval hypergraph is exactly hittable in polynomial time [2]. In this work we consider a related graph theoretic version of this question- which interval graphs are the intersection graphs of a set of intervals that have an exact hitting set? We refer to this class as the class of *Exactly Hittable Interval Graphs* (EHIG). We believe that the two characterization questions are different. The reason is that a given set of intervals defines a unique interval graph, but an interval graph can have many interval representations. We present a characterization of the class of interval graphs which can be represented as the intersection graph of a set of intervals which has an exact hitting set. One natural observation is that proper interval graphs are a strict subclass of EHIG. An example is that the graph $K_{1,3}$ is an EHIG, but it is not a proper interval graph. Further, a constraint propagation algorithm naturally computes an exact hitting set for a proper interval representation of an interval graph.

1.1 Our results

We introduce the class EHIG, which is the set of interval graphs that have an exactly hittable representation. We present a forbidden structure characterization for EHIG.

Definition 1. *For each $k \geq 1$, $\mathcal{F}_k$ denotes the set of connected interval graphs whose vertex set can be partitioned into an induced path P consisting of k vertices and the open neighbourhood of P (consisting of only those vertices which are not in P), which is an independent set of size $k+3$. Further, $\mathcal{F}$ is defined to be $\bigcup\limits_{k\geq 1} \mathcal{F}_k$.*

We show that every graph in $\mathcal{F}$ is a forbidden structure for EHIG. See Figure 1 for examples of forbidden structures.

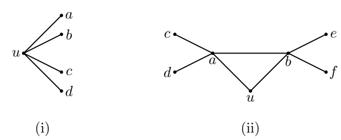

Figure 1: Two simple instances of forbidden induced subgraphs

Theorem 1. *An interval graph $G = (V, E)$ is exactly hittable if and only if it does not contain any graph from the set $\mathcal{F}$ as an induced subgraph.*

We consider a *canonical interval representation* for a given interval graph. This representation is crucial in proving Theorem 1. Given an interval graph G, a canonical interval representation H_G is an **interval hypergraph** denoted by $H_G = ([n], \mathcal{I})$, where $[n] = \{1, \ldots, n\}$ and $\mathcal{I} \subseteq \{\{i, i+1, \ldots, j\} \mid i \leq j, i, j \in [n]\}$, and all intervals have distinct left endpoints and distinct right endpoints. Further, for each $v \in G$, $I_v \in \mathcal{I}$ denotes the corresponding interval. H_G is obtained by *stretching* intervals obtained from the well known linear ordering of maximal cliques associated with an interval graph [3, 4]. We use H_G to show that if the interval graph G does not have any of the forbidden structures in $\mathcal{F}$, then H_G has an exact hitting set. We show that the class EHIG is positioned between the class of proper interval graphs and the class of interval graphs in the graph classes containment hierarchy.

Theorem 2. *Proper interval graphs $\subset$ EHIG $\subset$ Interval Graphs.*

This result suggests a generalization of exactly hittable interval graphs, and thus opens up the possibility of an interesting classification of interval graphs. For $k \geq 1$, an interval graph is called a k-hittable interval graph, if it is the intersection graph of an interval hypergraph which has a k-membership hitting set. The k-membership hitting set, which is a natural generalization of the exact hitting set, is a hitting set that hits each interval at most k times. This is the dual of the minimum membership set-cover problem studied by [5]. In this paper we present a characterization for $k = 1$, however, we do not know of a characterization of k-hittable interval graphs for $k \geq 2$. Further, we do not know of polynomial time algorithms to recognize if a given interval graph is a k-hittable interval graph.

2 Characterizing Exactly Hittable Interval Graphs

Theorem 1 is proved using the following lemmas.

Lemma 1. *Let G be an interval graph. Let $\mathtt{F} \in \mathcal{F}$. If G contains $\mathtt{F}$ as an induced subgraph, then G is not an Exactly Hittable Interval Graph.*

We prove this lemma as a simple application of the pigeonhole principle.

Lemma 2. *If an interval graph G does not contain a graph in $\mathcal{F}$ as an induced subgraph, then G is an Exactly Hittable Interval Graph.*

Let $\mathcal{O} = \{Q_1, Q_2 \ldots Q_t\}$ be a linear ordering of maximal cliques in G, which each interval graph has. As mentioned in the introduction, we can assume that H_G is obtained from $\mathcal{O}$. To prove Lemma 2 we use a minimum clique cover of the closed neighbourhood of a vertex v which satisfies the property in Observation 3. We denote this minimum clique cover by $C(N[v])$.

Observation 3. *For a vertex $v \in V$, let $\{Q_i \ldots Q_j\}$, $i, j \in [1, t], i \leq j$ be the set of maximal cliques containing v. Then there is a minimum clique cover of $N[v]$ which contains Q_j.*

Let $|C(N[v])|$ denote the number of cliques in $C(N[v])$. We denote the minimum clique cover of vertices in the maximal cliques Q_i to Q_j (in the ordering $\mathcal{O}$, $i < j$) by $C(Q_i, \ldots, Q_j)$. By an abuse of notation, we use $C(N[v_i, v_{i+1}, \ldots, v_j])$, $i < j$ to denote a clique cover of the graph induced on the vertex set $N[v_i] \cup N[v_{i+1}] \cup \ldots \cup N[v_j]$ which is implicitly constructed in Algorithm 1. Note that since the intervals in H_G associated with the vertices all have distinct left end points, it is well-defined to consider $v_i, \ldots, v_j$, $i < j$ in increasing order of the left-end ponts. Let $\mid C(N[v_i, v_{i+1}, \ldots, v_j]) \mid$ to denote the number of cliques in $C(N[v_i, v_{i+1}, \ldots, v_j])$. Note that this may not be the minimum clique cover. Our proof is based on the structural properties of a path P in G, the construction of which is described in Algorithm 1.

Algorithm 1: Construction of path P.

$i = 1$
$v_1 \leftarrow$ `Interval in` Q_1 `with largest right endpoint`
$P \leftarrow v_1$
$Q_r^1 =$ `last maximal clique containing` v_1
$C(N[v_1]) =$ `Minimum clique cover of` $N[v_1]$
$Q_{r'}^1 =$ `Maximal clique previous to` Q_r^1 `in` $C(N[v_1])$
while $Q_r^i \neq Q_t$ **do**
  $i = i + 1$
  $v_i =$ `vertex associated with that interval` $I \in Q_r^{i-1} \setminus Q_{r'}^{i-1}$ `which has largest right endpoint`
  $P \leftarrow P \cup v_i$
  $Q_r^i =$ `last maximal clique containing` v_i
  Let Q_{r+1}^{i-1} denote the clique just to the right of Q_r^{i-1} in $\mathcal{O}$. Note that $r+1$ is not the increment of an index, but points to the clique just to the right of Q_r^{i-1}
  $C(N[v_1, \ldots, v_i]) = C(N[v_1, \ldots, v_{i-1}]) \cup C(Q_{r+1}^{i-1}, \ldots, Q_r^i)$
  $Q_{r'}^i =$ `Maximal clique previous to` Q_r^i `in` $C(N[v_1, \ldots, v_i])$
end while
$\mathcal{K} = C(N[v_1, \ldots, v_i])$
return P

Let $\{v_1, v_2, \ldots, v_p\}$ be the set of vertices in path P constructed by Algorithm 1, with respect to the linear ordering $\mathcal{O}$. The maximal cliques in $\mathcal{K}$ are denoted by $\{K_1, K_2, \ldots, K_{\alpha'}\}$. We next state some important properties of path P.

- P is an induced path, and $N[P] = V(G)$.
- For $1 \leq i \leq p$, $\mid C(N[v_i]) \mid \leq 3$.
- Since G is a connected interval graph, for each vertex $v \in P \setminus v_p$, $\mid C(N[v]) \mid \geq 2$.

- In path P, there is at most one vertex which has minimum clique cover of size 3. If for any vertex v_i, $1 \leq i \leq p$, $\mid C(N[v_i]) \mid= 3$, then $|C(N[v_j])| \leq 2$, $\forall j \neq i$.
- α' is either $p+1$ or $p+2$.

We now use the properties of the P listed above to construct a clique cover of G with the property that the cliques are all vertex disjoint. The clique cover is denoted by $\mathcal{B} = \{B_1, B_2 \ldots B_{\alpha'}\}$. We outline the steps in the procedure below.

Let $v_l, 1 \leq l \leq p$ be the vertex in P such that $|C(N[v_l])| = 3$. If no such vertex exists in P, then let $l = p+1$, and let K_{p+2} be the empty set. Further, K_{p+3} is the empty set. We know that there is at most one such vertex, and the construction below will also take care of the case when for all $v_i, 1 \leq i \leq p$, $|C(N[v_i])| = 2$.

1. For $1 \leq i \leq l-1$, $B_i = K_i \setminus K_{i+1}$.
2. For $i = l \leq p$, we define B_l, B_{l+1}, B_{l+2}.
    - $B_l = K_l \setminus K_{l+1}$.
    - $B_{l+1} = K_{l+1} \setminus (K_l \cup K_{l+2})$
    - $B_{l+2} = K_{l+2} \setminus B_{l+1}$.
3. For $i \geq l+1$, $B_{i+2} = K_{i+2} \setminus K_{i+3}$.

Since G does not have the forbidden structure, it follows that $\mathcal{B}$ is a clique cover, and by construction, it is a parition of the vertex set. Further, the number of cliques in $\mathcal{B}$ is α'.

To complete the proof of Theorem 1, we use the crucial property of the canonical representation H_G that no two intervals in H_G have the same left end point or the same right end point. Using this propetry, we show that for each B_i there is a point p_i in H_G such that the intervals in H_G which contain p_i are exactly those which correspond to the vertices in B_i. These points $p_i, 1 \leq i \leq \alpha'$ form an exact hitting set of H_G. This completes the characterization of EHIG. Further, given G, H_G can be constructed in polynomial time, and thus it is possible in polynomial time to check using the result of [2] whether G is an EHIG.

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