Maker-Maker Domination Game

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Abstract

We introduce the Maker-Maker Domination Game, a perfect information game played on an undirected graph G. Two players color alternately an uncolored vertex of the graph. The first player that colors a dominating set wins, or the game is a draw if none of them manage to complete one. This game takes place in the positional game theory and is the natural Maker-Maker variant of the Maker-Breaker Domination Game, introduced by Duchene et al. [1]. As any Maker-Maker positional game, the second player cannot win. Therefore the best outcome that the second player can hope is a draw. We prove that deciding whether the first player has a winning strategy or not on a graph G is a PSPACE-complete problem even if G is a split or bipartite graph. We study some local structures that can give information on the graph, and we prove that deciding the outcome of the game be can be done in polynomial time if G is a cycle or a tree.

***Keywords* -** Positional game, Maker-Maker game, Domination game, PSPACE-complete.

1 Introduction

Positional games are games where two players, usually named Alice and Bob, play on a hypergraph $G = (V, F)$. The set of hyperedges F of G are the winning sets. The two most studied variants are Maker-Breaker, in which Alice tries to get all the vertices of any hyperedge $X \in F$ while Bob tries to prevent her from achieving this; and Maker-Maker, in which both Alice and Bob try to claim a winning set by taking all the elements in it. The Maker-Maker version of a game is more natural, as it represents more games that can be met in real life, as the famous Tic Tac Toe, but the Maker-Breaker is more studied. In the Maker-Breaker version, Alice only tries to take an hyperedge and Bob prevents her to do so, while in Maker-Maker they both have in the same time to take an hyperedge and to disable their opponent to take one. Therefore, Maker-Breaker games are easier to study as both players have to focus on a single task. This phenomenon explains why Maker-Breaker games behave well with some graph operations. Namely, the outcome of a disjoint union of two graphs can be computed from the outcome of the different graphs in the union or we know that adding a winning set is always good for Maker. This is not the case with the Maker-Maker version, which has the extra-set paradox: if an instance is winnable for the first player, adding a winning set can change the outcome of the game. However, despite the differences between these two classes of games, some results in Maker-Breaker can be used in Maker-Maker. Namely, if the outcome of a game in Maker-Breaker is that Breaker wins, then necessarily, this game ends by a draw in the Maker-Maker variant. Note that the contrary is false: for example, Tic-Tac-Toe is a draw in Maker-Maker but a first player win in Maker-Breaker.

The Maker-Breaker Domination Game, introduced by Duchêne *et al.* [1], takes place in the Maker-Breaker theory. Alice and Bob alternately color the vertices of a graph and Alice starts. Alice colors her vertices in red and Bob in blue. Alice wins if she colors red a dominating set of G. Otherwise, Bob wins. Duchêne *et al.* have proven some results on this version of the game, for instance, computing the outcome of the game is PSPACE-complete on bipartite or split graphs, but it can be computed in polynomial time in trees, cographs, and cycles. It seems natural to wonder which of theses results can still be applied in the Maker-Maker version of the game.

We prove here that, given a graph G, determining the winner of the Maker-Maker domination game on G is still PSPACE-complete, even if G is supposed bipartite or split. Furthermore, the outcome of the game can still be determined in polynomial time on cycles and trees. Contrary to the Maker-Breaker version, the complexity for cographs is still open in Maker-Maker. Moreover, we studied some substructures of graphs that provide some information on any graph in which they appear. Note that even if in both conventions, the outcome can be computed in polynomial time for trees, the characterisation of the trees in which Alice wins is much more simpler in Maker-Breaker than in Maker-Maker.

The two possible outcomes in our game are either that the first player has a winning strategy, or that for any strategy of the first player, the second player can ensure a draw (or to win first). We say that $G \in FP$ in the first case and that $G \in D$ in the second one.

2 Complexity

We prove that the Maker-Maker Domination Game is PSPACE-complete even restricted to split graphs or bipartite graphs. Recall first that a graph $G = (V, E)$ is bipartite if its vertices V can be partitioned into two subsets V_1 and V_2 such that $E \subset V_1 \times V_2$; and it is split if its vertices can be partitioned into two subsets V_1 and V_2 such that V_1 is a clique and V_2 is a stable set.

Theorem 1. *Given a graph G, the decision problem : "Does Alice has a winning strategy ?" is* PSPACE*-complete, even if G is supposed bipartite or split.*

Sketch of the proof. The proof is straightforward, as the Maker-Breaker version has been proven PSPACE-complete in [1]. Starting from an instance of the Maker-Breaker version, we add an isolated vertex that Alice is forced to play first. After this move, Bob cannot dominate the graph any longer. Therefore the game is Draw if and only if it is a breaker win in the Maker-Breaker version. $\triangleleft$

3 Substructures

We will present in this section structures on which some moves are known to be optimal or which enable us to know the outcome on any graph containing it. First, note that if G has a universal vertex, Alice can play it first and win in a single move, thus $G \in FP$. A natural extension of this statement shows that if a graph G has two universal vertices, it is neutral for the union. I.e, for any graph H, we have $H \in FP$ if and only if $H \cup G \in FP$.

A *leaf* of a graph G is a degree-1 vertex of G. A *cherry* of a graph G is a vertex v which is connected two at least two leaves.

Theorem 2. *Let G be a graph, If G has at least two isolated vertices, two cherries or one cherry and one isolated vertex, then $G \in D$.*

If G has only one isolated vertex or one cherry (and not both at the same time), then any winning strategy for Alice (if it exists), should start by playing it.

Sketch of the proof. If Alice has a winning strategy that starts by coloring one of the leaves attached to v. The same strategy starting by coloring v is also a winning strategy as the neighborhood of the leaves is strictly included in the neighborhood of v.

By playing first on an isolated vertex or a cherry, Bob can trivially isolate a vertex that Alice will never be able to dominate. So Alice has to color first an isolated vertex or a cherry. $\triangleleft$

Until now, the results that we gave are symmetrical as the optimal moves shown are the same for Alice and for Bob. As Alice tries to win while Bob can only hope for a draw, some results are not symmetrical as the following one. We first introduce a *tail* of G as a vertex t such that one of the neighbour of t is a leaf.

Theorem 3. *Let G be a graph after some optimal moves of the players. Suppose it is Bob's turn and suppose G has a tail t attached to a leaf l such that neither t or l are colored. There exists an optimal strategy for Bob that starts by coloring t.*

Sketch of the proof. First note that if $G \in FP$, whatever Bob does, he will lose. Therefore, any strategy for Bob is optimal.

If $G \in D$, Alice and Bob will have to play until all the vertices are colored. Therefore t and l will be played. As the neighborhood of l is included in the one of t it is always better to play t first. If it is Bob's turn, playing t forces the next move of Alice on l. $\triangleleft$

This result is quite strong as it can be applied in any graph. It enables us to only study strategies after these moves have been made. Especially in the case of trees, which are studied in next section, this result directly implies that some moves are known to be optimal for Bob: if Alice has not played a neighbor of a leaf, it is optimal for Bob to play it.

4 Polynomial time solvable games: cycles and trees

Even if the game is difficult complexity-wise to handle in the general case, in simple class of graphs, it is possible to find the outcome in polynomial time. We first present it in paths and cycles.

4.1 Path and cycles

Let $n \geq 1$ be an integer. We denote by P_n (C_n resp.) the path (cycle resp.) of n vertices.

Theorem 4. *Let $n \geq 1$ be an integer. We have $P_n \in FP$.*

Sketch of the proof. Denote by $x_1, \dots, x_n$ the vertices of P_n such that the edges are exactly $(x_1, x_2), \dots, (x_{n-1}, x_n)$. Alice starts by coloring x_2. Then, while Bob colors a vertex x_k with $k \neq 1$, Alice colors x_{k+1} if possible, otherwise she colors x_{k-1} if possible. When Bob colors x_1, she colors a vertex x_k she does not dominate with k as small as possible. By construction its left neighbor x_{k-1} is not dominated by Bob. She considers now the graph as the path starting at x_{k-1} and applies the same strategy by induction. $\triangleleft$

Theorem 5. *Let $n \geq 1$ be an integer. We have $C_n \in FP$ if and only if $n \leq 9$ or $n \not\equiv 1 \mod 3$.*

Note that this result differs with the Maker-Breaker version where Maker as a first player is always winning.

Sketch of the proof. The cycles with less then nine vertices can be handled by testing exhaustively all the strategies. If $n \geq 10$, is such that $n \not\equiv 1 \mod 3$, Alice can play in such a way that the cycle is divided into paths with no path having two blue extremities and at least one path of length $\not\equiv 0 \mod 3$ with two red extremities. Keeping this invariant after each move, she manages to dominate the graph first. If $n \equiv 1 \mod 3$, we prove that Bob can ensure a draw by induction. If $n = 10$ or $n = 13$, we test that Bob can ensure a draw. If $n \geq 16$, Bob can, by playing a vertex at distance five from the first vertex played by Alice, force moves for Alice that will make the cycle equivalent to one with six vertices less. ◁

4.2 Trees

Trees in the Maker-Breaker convention can be solved as follows (see [1]): Remove recursively all the *pendant* P_2 (i.e., a leaf u and its neighbour v with v of degree 2). Then Maker wins first if only if it is remaining at the end a star or a single vertex. The situation is much more complicated in the Maker-Maker convention, but we are still able to solve the game in polynomial time.

Theorem 6. *The outcome of the Maker-Maker Domination Game in trees can be computed in polynomial time.*

Sketch of the proof. Consider a tree T. If T has at least two cherries, the outcome is D. If it has exactly one, note it c, the generalization of the result in the Maker-Breaker version on trees shows that, $T \in FP$ if and only if it contains a perfect matching, omitting if required some vertices that will be dominated by c.

In T has no cherries, we first prove that, except if T has diameter smaller or equal to 8, Alice's first move has to be on a tail of T, otherwise Bob has a winning strategy obtained by forcing all the moves of Alice by always threatening to isolate a vertex. Trees with small diameters are some particular cases but they can be handled by hand.

If T has diameter larger than 8, only few configurations are in FP. Most of them are paths on which some vertices are added. The main idea of Alice's strategy on theses configurations is to consider the algorithm played on a path and to apply it by considering the tree as a union (not disjoint) of paths. ◁

References

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