Circular orderings of graphs

Santiago Guzmán-Pro – Facultad de Ciencias, UNAM, México
Pavol Hell – School of Computing Science, Simon Frasers University, Canada
César Hernández-Cruz – Facultad de Ciencias, UNAM, México

Abstract

Each hereditary property can be characterized by its set of minimal obstructions; these sets are often unknown, or known but infinite. By allowing extra structure it is sometimes possible to describe such properties by a finite set of forbidden objects. This has been studied most intensely when the extra structure is a linear ordering of the vertex set. For instance, it is known that a graph G is k-colourable if and only if $V(G)$ admits a linear ordering $\leq$ with no vertices $v_1 \leq \cdots \leq v_{k+1}$ such that $v_iv_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k\}$. In this paper, we study such characterizations when the extra structure is a circular ordering of the vertex set. We show that the classes that can be described by finitely many forbidden circularly ordered graphs include forests, circular-arc graphs, and graphs with circular chromatic number less than k. In fact, every description by finitely many forbidden circularly ordered graphs can be translated to a description by finitely many forbidden linearly ordered graphs. Nevertheless, our observations underscore the fact that in many cases the circular ordering descriptions are simpler and more natural.

1 Introduction

A natural way to characterize or define a hereditary property is by exhibiting its set of minimal obstructions. For instance, bipartite graphs are characterized as those graphs with no induced odd cycles, while the class of evenhole-free graphs is defined as the class of graphs that contain no even cycle as an induced subgraph. Unfortunately exhibiting the set of minimal obstructions might be a highly complex task; as of today, the set of minimal obstructions to the class of k-colourable graphs is unknown for every positive integer k greater than 2.

In 1990, Damaschke [1] proposed to study characterizations of hereditary properties $\mathcal{P}$ by exhibiting a finite set of linearly ordered graphs F such that $\mathcal{P}$ is the class of graphs that admit an F-free linear ordering. He observed that, chordal graphs, bipartite graphs, and interval graphs are characterized by a forbidden set of linearly ordered graphs on three vertices; also in [1] he asked if the class of circular-arc graphs can be described by finitely many forbidden linearly ordered graphs. We will see that we can reinterpret a (known) characterization of circular-arc graphs in our context to obtain a positive answer to Damaschke's question.

Around 2014, Hell, Mohar, and Rafiey [4] showed that for every set F of linearly ordered graphs on three vertices, the class of graphs that admit an F-free linear ordering can be recognized in polynomial time. Recently, Habib and Feuilloley published a thorough survey [2] on the subject, where they characterized all hereditary properties defined by forbidden linear ordered graphs on three vertices. Moreover, they showed that all but two of these classes can be recognized in linear time. In their work, Habib and Feuilloley stated that an obvious next step is to study graph properties described by forbidden linear orderings on more vertices. All of our results can be translated to this context.

A classical result is the following characterization of k-colourable graphs in terms of certain forbidden linear orderings.

Proposition 1. *[2, 4] Let k be a positive integer. A graph G is k-colourable if and only if there is a linear ordering $\leq$ of $V(G)$ such that there are no $k+1$ vertices $v_1 \leq \cdots \leq v_{k+1}$ such that $v_i v_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k\}$.*

In this work, we start the study of circularly ordered graphs, attempting to obtain a development parallel to the one described in the above paragraphs for linearly ordered graphs.

2 Circularly ordered graphs

A *circularly ordered graph* (G, C) is a graph G together with a circular ordering C of its vertices. In this case we say that (G, C) is a circular ordering of the graph G. A natural way to represent a circularly ordered graph is by depicting its set of vertices on circumference S where C is recovered by traversing S in a clockwise motion. Notice that each graph on two or three vertices defines a unique circularly ordered graph; in Figure 1 we depict some circularly ordered graphs on four vertices.

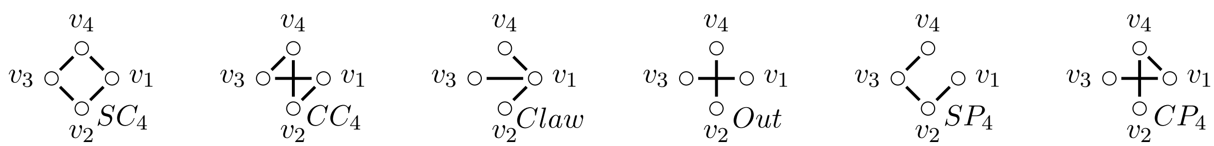

Figure 1: Some circularly ordered graphs on 4 vertices. In all cases, the circular ordering is $v_1 \leq v_2 \leq v_3 \leq v_4 \leq v_1$.

Induced circular ordered subgraphs of (G, C) are induced subgraphs of G whose set of vertices respects the circular ordering C. For a set F of circularly ordered subgraphs, we say that a circularly ordered graph (G, C) is *F-free* if (G, C) does not contain any induced circularly ordered subgraphs in F. A graph G *admits an $\mathcal{F}$-free circular ordering* if there exists an $\mathcal{F}$-free circularly ordered graph (G, C).

A *pattern* consists of a set V together with a set of edges E and a set of non-edges NE with the restriction that $NE \cap E = \varnothing$. A pattern (V, E, NE) *represents* all graphs $(V(G), E(G))$ such that $V(G) = V$ and $E \subseteq E(G)$ but $E(G) \cap NE = \varnothing$. So a circularly ordered pattern (G, C) consists of a pattern G together with a circular ordering of its vertices, and it represents all circularly ordered graphs obtained by a graph represented by G and ordering its vertices by C. We denote by $\langle (G, C) \rangle$ the set of circularly ordered graphs represented by (G, C).

We begin by noticing a relation between circularly ordered graphs and linearly ordered graphs. Consider a linear ordering $\leq$ of a set X, and depict the elements of X as points in the interior of a closed interval I according to $\leq$. The *circular closure* of $\leq$ is the circular ordering $c(\leq)$ recovered by identifying the endpoints of I and traversing the resulting circumference in a clockwise motion. Conversely, for every circular ordering C there is a linear ordering $\leq$ such that $C = c(\leq)$. So we can always describe a circular ordering of X as the circular closure of a linear ordering on X. Let c be the function that maps a linearly ordered graph $(G, \leq)$ to the circularly ordered graph $(G, c(\leq))$, i.e., $c(G, \leq) = (G, c(\leq))$. It is convenient to define the "linearizing operator" L for a set of circularly ordered graphs F as $L(F) = c^{-1}[F]$. For instance, in Figure 2 we depict a circularly ordered pattern CA and a pair of linearly ordered patterns that represent $L(\langle CA \rangle)$.

Observation 1. *Let F be a set of circularly ordered graphs and let $\mathcal{P}$ be the class of graphs that admit an F-free circular ordering. Then, $\mathcal{P}$ is the class of graphs that admit a $L(F)$-free linear ordering.*

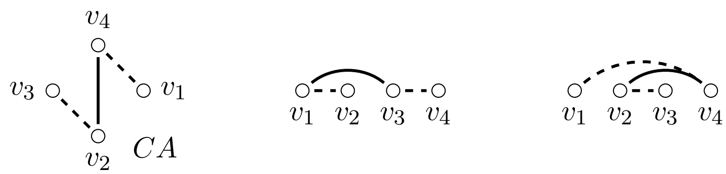

Figure 2: The circularly ordered pattern CA and a pair of linearly ordered patterns that represent $L(\langle CA \rangle)$.

We can naturally reinterpret a result due to Tucker [5] and obtain a characterization of circular-arc graphs by forbidden circular orderings. Outerplanar graphs also have a natural characterization in this context. We also show that other graph families described by forbidding "small" circularly ordered graphs include linear forests, caterpillar forests, and forests. We summarize these results in Table 2. We denote by OUT the set of spanning circularly ordered supergraphs of Out.

Forbidden circularly ordered graphs	Graph family
$SC_3, SP_4, CP_4, Claw$	Linear Forests.
SC_3, SP_4, CP_4	Caterpillar Forests.
$SC_3, SC_4, CC_4, Out, CP_4, SC_5, SP_5$	Forests.
OUT	Outerplanar graphs.
$\langle CA \rangle$	Circular-arc graphs.

Table 1: Some graph families described by forbidden circularly ordered graphs on at most 5 vertices.

By Observation 1 and Table 2, we recover an observation mentioned in [2] that states that there is a finite set of linearly ordered patterns that characterizes outerplanar graphs. With the same arguments we conclude that the patterns that represent $L(CA)$ (Figure 2) characterize circular ordered graphs by finitely many forbidden linearly ordered graphs. This remark positively answers a question posed by Damaschke [1]: is there a finite set of linearly ordered graphs that describes the class of circular-arc graphs?

The descriptions by forbidden circular arrangements of outerplanar graphs and circular-arc graphs proposed here are simpler (and more intuitive) than their descriptions by forbidden linearly ordered graphs. On the contrary, describing forests, linear forests, and caterpillar forest by linearly ordered graphs yields simpler expressions (and proofs) than describing these classes by forbidden circularly ordered graphs. Nonetheless, these results show that circularly ordered graphs can describe several natural graph classes. Moreover, these observations raise the following question.

Question 1. *Is there a hereditary property described by finitely many forbidden linearly ordered graphs that does not admit a characterization by finitely many forbidden circularly ordered graphs?*

We believe that the classes of k-colourable graphs are possible candidates to answer the previous question in the negative. More generally, we are interested in the following problem.

Problem 2. *Find a (relatively well-known) hereditary property that cannot be described by a finite set of forbidden circularly ordered graphs.*

Question 3. *For which positive integer k the class of k-colourable graphs can be described by finitely many forbidden circularly ordered graphs? In particular, is there a finite set of circularly ordered graphs that describes the class of bipartite graphs?*

Regarding larger circularly ordered graphs, we propose a characterization that relates the circular chromatic number of graphs and certain forbidden linear ordering of graphs. We interpret this characterization as an analogous version of Proposition 1 for circular orderings.

Lemma 1. *Let k be a positive integer. A graph G has circular chromatic number strictly less than k if and only if there is a circular ordering $c(\leq)$ of $V(G)$ such that there are no vertices $v_1 \leq \cdots \leq v_{k+1} \leq v_1$ such that $v_i v_{i+1} \in E(G)$ for every $i \in \{1, \ldots, k\}$.*

Let SP_{k+1} denote the path $v_1 \cdots v_{k+1}$ together with the circular ordering $v_1 \leq v_2 \leq \cdots \leq v_{k+1} \leq v_1$, and let SC_k denote the cycle $v_1 \ldots v_k$ together with the circular ordering $v_1 \leq v_2 \leq \cdots \leq v_k \leq v_1$. Finally, let $\mathcal{H}_{k+1}$ denote the set of spanning circularly ordered supergraphs of SP_{k+1} and SC_k and let $\chi_c(G)$ the circular chromatic number of a graph G. In terms of forbidden circularly ordered graphs the previous lemma reads as follows.

Theorem 4. *Let k be a positive integer and G a graph. Then, $\chi_c(G) < k$ if and only if G admits an $\mathcal{H}_k$-free circular ordering.*

Given a set F of circularly ordered graphs, the F-free circular ordering problem consists of determining if an input graph G admits an F-free circular ordering. As a consequence of Theorem 4 and a result of Hatami and Tuserkani [3] we conclude the following statement.

Corollary 1. *For every positive integer k, $k \geq 5$, there is a set F of circularly ordered graphs on k vertices such that the F-free circular ordering problem is NP-complete.*

Trivially, for every set of circularly ordered graphs on at most 3 vertices, the F-free circular ordering problem is polynomial time solvable. It is only natural to ask the following questions, where the second one is a particular case of the first one due to Theorem 4.

Question 5. *Is there a set F of circularly ordered graphs on at most 4 vertices such that the F-free circular ordering problem is NP-complete?*

Question 6. *Given a graph G, is the problem of determining if $\chi_c(G) < 3$ an NP-complete problem?*

References

[1] P. Damaschke. Forbidden ordered subgraphs. Topics in Combinatorics and Graph Theory (1990) 210–229.

[2] L. Feuilloley, and M. Habib. Graph classes and forbidden patterns on three vertices. SIAM Journal of Discrete Mathematics 35(1) (2020) 55–90.

[3] H. Hatami and R. Tuserkani. On the complexity of the circular chromatic number. Journal of Graph Theory 47(3) (2004) 226–230.

[4] P. Hell, B. Mohar, A. Rafiey. Ordering without forbidden patterns. Algorithms - ESA 2014, 8737 (2014), 554–565.

[5] A. Tucker. Characterizing circular-arc graphs. Bull. AMS 76 (1970) 1257–1260.