Partitioning into degenerate graphs in linear time

Alexandre Talon — Univ. Grenoble Alpes, CNRS, Grenoble INP, G-SCOP, 38000 Grenoble, France.
Timothée Corsini — Univ. Bordeaux, LaBRI, CNRS, Bordeaux INP, Talence, France.
Quentin Deschamps — Univ. Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France.
Carl Feghali — Univ. Lyon, EnsL, UCBL, CNRS, LIP, F-69342, Lyon Cedex 07, France.
Daniel Gonçalves — LIRMM, Univ Montpellier, CNRS, Montpellier, France.
Hélène Langlois — CERMICS, École des Ponts ParisTech, 77455 Marne-la-Vallée, France.

Abstract

Let G be a connected graph with maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$. Generalizing Brooks' Theorem, Bollobás and Manvel proved that if $s_1, \ldots, s_t$ are non-negative integers such that $s_1 + \cdots + s_t \geq \Delta - t$, then G admits a vertex partition into parts $A_1, \ldots, A_t$ such that, for $1 \leq i \leq t$, $G[A_i]$ is s_i-degenerate. Here we show that such a partition can be performed in linear time. This generalizes previous results that treated subcases of a conjecture of Abu-Khzam, Feghali and Heggernes [2], which our result addresses in full.

1 Introduction

Brooks' Theorem is a fundamental theorem in graph coloring that draws a connection between the chromatic number and the maximum degree of a graph.

Theorem 1 (Brooks' Theorem [6]). *Every connected graph with maximum degree $\Delta \geq 3$ that is distinct from $K_{\Delta+1}$ is Δ-colorable.*

A graph G is d-degenerate if every subgraph of G contains a vertex of degree at most d. Bollobás and Manvel [3] obtained the following generalization.

Theorem 2 (Bollobás and Manvel [3]). *Let G be a non-complete connected graph with maximum degree $\Delta \geq 3$. Let $s \geq 2$ and $p_1, \ldots, p_s \geq 0$ be integers such that $\sum_{i=1}^{s} p_i \geq \Delta - s$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_s$ such that, for each $i \in [s]$, $G[V_i]$ is (i) p_i-degenerate and (ii) has maximum degree at most $p_i + 1$.*

As usual, the notation $[n]$ stands for $\{1, \ldots, n\}$. Brooks' Theorem follows from Theorem 2 by noting that a d-degenerate graph is $(d+1)$-colorable. We should also mention that similar generalizations and variants of Brooks' Theorem exist: see for example [1] for a directed version or [9] for a distributed version.

From an algorithmic perspective, a very short proof of Brooks' Theorem due to Lovász [7] produces the coloring in linear time. The original proof of Theorem 2, as the alternative proof provided by Matamala [8], are not algorithmic, and do not seem to lead to a polynomial algorithm. An alternative proof of Theorem 2 is provided in [5], which is algorithmic with polynomial complexity: the runtime appears to be cubic in the number of vertices. This raises the question of whether one can possibly improve its time complexity to linear. In view of this, several groups improved the complexity of such a partition algorithm focusing on property (i) only. Bonamy et al. [4] showed that the complexity in the special case $s = 2$ with $p_1 = 0$ and $p_2 = \Delta - 2$ can be improved to quadratic for $\Delta \geq 4$ and to linear for $\Delta = 3$. Similarly, Abu-Khzam, Feghali and Heggernes [2] showed that in the special case $p_i \leq 1$ for all $i \in [s]$, it can be improved to linear.

The object of this paper is to obtain a common generalization of these results in linear time, via a much shorter argument.

Theorem 3. *Let G be a non-complete connected graph with maximum degree $\Delta \geq 3$. Let $s \geq 2$ and $p_1, \ldots, p_s \geq 0$ be integers such that $\sum_{i=1}^{s} p_i \geq \Delta - s$. There exists a linear-time algorithm partitionning $V(G)$ into sets $V_1, \ldots, V_s$ such that, for each $i \in [s]$, $G[V_i]$ is p_i-degenerate.*

Theorem 3 settles a conjecture of Abu-Khzam, Feghali and Heggernes [2] and, in the special case $s = 2$, a problem of Bonamy et al. [4].

2 Sketch of proof of Theorem 3

To prove Theorem 3, one should focus on the $s = 2$ case, which we present here. It corresponds to the following theorem (with notations slightly simpler than Theorem 2).

Theorem 4. *Let G be a connected graph with n vertices and maximum degree $\Delta \geq 3$ that is distinct from $K_{\Delta+1}$. For each pair d_A, d_B such that $d_A + d_B = \Delta - 2$, there is an $O(n)$-time algorithm partitioning $V(G)$ into sets A and B such that $G[A]$ is d_A-degenerate and $G[B]$ is d_B-degenerate.*

Given a graph G and a vertex ordering $v_1, v_2, \ldots, v_n$ of G let us denote $N^<(v_i)$ the neighbors of v_i with lower index, that is $N^<(v_i) = N(v_i) \cap \{v_j \mid j < i\}$. The following folklore observation shows us how such sets can help us construct a certificate that a graph is d-degenerate.

Observation 5. *If a graph G admits a vertex ordering $v_1, v_2, \ldots, v_n$ such that $|N^<(v_i)| \leq d$ for every vertex v_i, then G is d-degenerate.*

If G is not Δ-regular, let v be a vertex of degree at most $\Delta - 1$, and let T be a spanning tree of G rooted at v. Let $v_1, v_2, \ldots, v_n = v$ be a vertex ordering obtained by peeling off the leaves of T iteratively. The main property of this ordering is the following:

Fact 6. *Every vertex v_i has at most $\Delta - 1$ neighbors in $N^<(v_i)$.*

Indeed, this is clear for $v_n = v$. This is also clear for every vertex $v_i \neq v_n$, as its parent neighbor in T does not belong to $N^<(v_i)$. Note that such an ordering can be obtained in linear time as finding v and constructing T are clearly feasible in linear time and as the ordering considered can be a simple post-order traversal of T. Now, given such an ordering we partition $V(G)$ into A and B, using Algorithm 1, that is clearly linear.

In view of Observation 5, it remains to show that the ordering $v_1, v_2, \ldots, v_n$ ensures us that $G[A]$ is d_A-degenerate and $G[B]$ is d_B-degenerate. Phrased differently, we must show that for every vertex $v_i \in A$, we have $|A \cap N^<(v_i)| \leq d_A$ and, for every vertex $v_i \in B$, we have $|B \cap N^<(v_i)| \leq d_B$. The former is clearly implied by Algorithm 1. For the latter, it follows from

$$|B \cap N^<(v_i)| = |N^<(v_i)| - |N^<(v_i) \cap A| \leq (\Delta - 1) - (d_A + 1) \leq d_B.$$

Algorithm 1

```
A ← ∅
B ← ∅
for v_i from v_1 to v_n do
    if |N^<(v_i) ∩ A| ≤ d_A then
        A ← A ∪ v_i
    else
        B ← B ∪ v_i
    end if
end for
return (A, B)
```

To complete the proof of Theorem 4, it remains to deal with the case where G is Δ-regular. For this case, we have to introduce a particular graph that we denote K_*^-. This graph is obtained from $K_{\Delta+1}$ by subdividing exactly one edge (see Figure 1). Note that this graph has a degree two vertex denoted v and that all the $\Delta + 1$ remaining vertices have degree Δ.

In the case where G is not 2-connected and where one of its end-blocks is a K_*^- with v as its cut-vertex one should proceed as follows. We can first partition $G \setminus V'$, where $V' \cup \{v\}$ is the vertex set of K_*^-, viewed as a non-Δ-regular graph (thanks to Algorithm 1), and then we can easily extend the partition to the whole graph G (by appropriately partitioning K_*^-).

A key ingredient for the rest of the proof is the following lemma.

Lemma 1. *If there exists a vertex z in G and a set $X \subset N(z)$ such that*

a) $|X| = d_B + 1$,

b) $G[X]$ *is not a complete graph, and*

c) $G \setminus X$ *is connected,*

then there is a linear-time algorithm partitionning G into sets A, B such that $G[A]$ is d_A-degenerate and $G[B]$ is d_B-degenerate.

This lemma follows from a variant of Algorithm 1, where one starts by setting $B \leftarrow X$.

In the remaining case, if G is 2-connected, or if none of its end-blocks is a K_*^-, it is always possible to compute a pair (z, X) fulfilling the conditions of Lemma 1.

Finally, to prove Theorem 3, one just has to note that it follows from Theorem 4 (by bi-partitioning G into a p_1- and a $(\Delta - p_1 - 2)$-degenerate graph) and from the following lemma (by setting $d_A = p_i$ and $d_B = (\sum_{i<j\leq s} p_j) + (s - 1 - i)$, for i taking values from 2 to $s - 1$).

Lemma 7. *Let G be a d-degenerate graph, given with a vertex ordering $v_1, \ldots, v_n$, such that $|N^<(v_i)| \leq d$ for every v_i. For any pair d_A, d_B such that $d = d_A + d_B + 1$, Algorithm 1 partitions $V(G)$ into sets A and B such that $G[A]$ and $G[B]$ are d_A- and d_B-degenerate, respectively.*

The proof of this lemma also relies on Algorithm 1.

Acknowledgements. Supported by Agence Nationale de la Recherche (France) under research grant ANR DIGRAPHS ANR-19-CE48-0013-01, and ANR GATO ANR-16-CE40-0009.

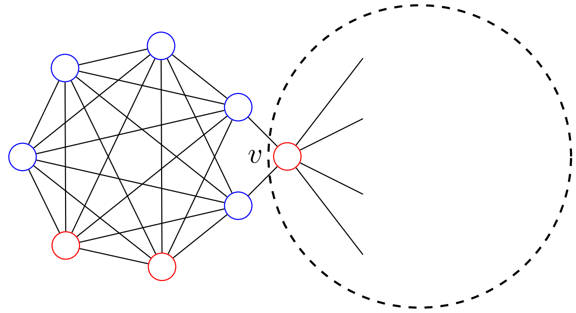

Figure 1: The graph K_*^- for $\Delta = 6$, and some vertex partitionings for $d_A = 1$ and $d_B = 3$ where the vertices in $A \ni v$ are represented in red and vertices in B in blue.

References

[1] Aboulker, Pierre and Aubian, Guillaume, Four proofs of the directed Brooks' Theorem, arXiv preprint arXiv:2109.01600, 2021.

[2] Abu-Khzam, Faisal N and Feghali, Carl and Heggernes, Pinar, Partitioning a graph into degenerate subgraphs, European Journal of Combinatorics, 83, 103015, 2020.

[3] Bollobás, Béla and Manvel, Bennet, Optimal vertex partitions, Bulletin of the London Mathematical Society, 11, 2, 113–116, 1979.

[4] Bonamy, Marthe and Dabrowski, Konrad K and Feghali, Carl and Johnson, Matthew and Paulusma, Daniël, Recognizing graphs close to bipartite graphs with an application to colouring reconfiguration, Journal of Graph Theory, 98, 1, 81–109, 2021.

[5] Borodin, Oleg V and Kostochka, Alexandr V and Toft, Bjarne, Variable degeneracy: extensions of Brooks' and Gallai's theorems, Discrete Mathematics, 214, 1-3, 101–112, 2000.

[6] Brooks, Rowland Leonard, On colouring the nodes of a network, Mathematical Proceedings of the Cambridge Philosophical Society, 37, 2, 194–197, 1941.

[7] Lovász, László, Three short proofs in graph theory, Journal of Combinatorial Theory, Series B, 19, 3, 269–271, 1975.

[8] Matamala, Martín Vertex partitions and maximum degenerate subgraphs, Journal of Graph Theory, 55, 3, 227–232, 2007.

[9] Panconesi, Alessandro and Srinivasan, Aravind, Improved distributed algorithms for coloring and network decomposition problems, Proceedings of the twenty-fourth annual ACM symposium on Theory of computing, 581–592, 1992.

[10] Schweser, Thomas and Stiebitz, Michael, Vertex partition of hypergraphs and maximum degenerate subhypergraphs, arXiv preprint arXiv:1807.02308, 2018.

[11] Tarjan, Robert E and Vishkin, Uzi, An efficient parallel biconnectivity algorithm, SIAM Journal on Computing, 14, 4, 862–874, 1985.