Generalized outerplanar Turán number of short paths

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Abstract

Let H be a graph. The generalized outerplanar Turán number of H, denoted by $f_{\mathcal{OP}}(n, H)$, is the maximum number of copies of H in an n-vertex outerplanar graph. Let P_k denotes a path on k vertices. In this paper we give an exact value of $f_{\mathcal{OP}}(n, P_4)$ and a best asymptotic value of $f_{\mathcal{OP}}(n, P_5)$. Moreover, we characterize all outerplanar graphs containing $f_{\mathcal{OP}}(n, P_4)$ copies of P_4.

1 Introduction

In 1941, Turán [13] proved a classical result in the field of extremal graph theory. He determined exactly the maximum number of edges an n-vertex K_r-free graph may contains. After his result, for a graph H, the maximum number of edges in an n-vertex H-free graph, denoted by $\mathrm{ex}(n, H)$, is named as *Turán number of H*. A major breakthrough in the study of the Turán number of graphs came in 1966, with the proof of the famous theorem by Erdős, Stone and Simonovits [6, 7]. They determined an asymptotic value of the Turán number of any non-bipartite graph H. In particular, they proved $\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2)$, where $\chi(H)$ is the chromatic number of H.

Since then, researchers have been interested working on Turán number of class of bipartite (degenerate) graphs and extremal graph problems with some more generality. Determining the maximum number of copies of H in an n-vertex F-free graph, denoted by $\mathrm{ex}(n, H, F)$, is among such problems. Since we count the number of copies of a given graph which is not necessarily an edge, such an extremal graph problem is commonly named as *generalized Turán problem.* The results on $\mathrm{ex}(n, K_r, K_t)$ by Zykov [14] (and independently by Erdős [5]), $\mathrm{ex}(n, C_5, C_3)$ by Győri [10] and $\mathrm{ex}(n, C_3, C_5)$ by Bollobás and Győri [9] were sporadic initial contributions.

Recently, some researchers were interested in extremal graph problems in some particular family of graphs, for instance, the family of planar graphs.

The study of generalized extremal problems in the family of planar graphs was initiated by Hakimi and Schmeichel [11] in 1979. Define the *generalized planar Turán number* of a graph H, denoted by $f_{\mathcal{P}}(n, H)$, as the maximum number of copies of H in an n-vertex planar graph. Hakimi and Schmeichel [11] determined the exact value of $f_{\mathcal{P}}(n, C_3)$ and $f_{\mathcal{P}}(n, C_4)$. Recently, this topic is active and many exact and best asymptotic values were obtained for different planar graphs. We refer [1, 4, 8] for more results.

Recently, Matolcsi and Nagy in [12] initiated the study of the generalized planar Turán number version in the family of outerplanar graphs.

Definition 1. *The generalized outerplanar Turán number of a graph H, denoted by $f_{\mathcal{OP}}(n, H)$, is the maximum number of copies of H in an n-vertex outerplanar graph. i.e.,*

$$f_{\mathcal{OP}}(n, H) = \max\{\mathcal{N}(H, G) : G \text{ is } n\text{-vertex outerplanar graph}\},$$

where $\mathcal{N}(H, G)$ is the number of copies of H as a subgraph in G.

Matolcsi and Nagy in [12] determined sharp and asymptotically sharp bounds of $f_{\mathcal{OP}}(n, H)$ for certain families of graphs H and described extremal graphs attaining the values. In particular, they determined the exponential growth of the generalized outerplanar Turán number of P_k, a k-vertex path, as a function of k. They also determined the exact value of $f_{\mathcal{OP}}(n, P_3)$ and characterized the extremal graphs.

Theorem 1. *(Matolcsi and Nagy [12])*

$$h(k)\binom{n}{2} < f_{\mathcal{OP}}(n, P_k) \leq 4^k\binom{n}{2}, \text{ where } \lim_{k\longrightarrow\infty} \sqrt[k]{h(k)} = 4.$$

Theorem 2. *(Matolcsi and Nagy [12]) Suppose that $n \geq 3$. Then*

$$f_{\mathcal{OP}}(n, P_3) = \frac{n^2 + 3n - 12}{2},$$

and the unique extremal outerplanar graph containing $f_{\mathcal{OP}}(n, P_3)$ P_3's is $K_1 + P_{n-1}$.

In this paper, we determine the exact value of $f_{\mathcal{OP}}(n, P_4)$ and an asymptotic value $f_{\mathcal{OP}}(n, P_5)$. Moreover, we characterize all outerplanar graphs containing $f_{\mathcal{OP}}(n, P_4)$ copies of P_4 as a subgraph. The following two theorems summarize our results. The extremal constructions attaining the $f_{\mathcal{OP}}(n, P_4)$ is given in Figure 1. The n-vertex outerplanar graph construction in Figure 2 verifies that the value of $f_{\mathcal{OP}}(n, P_5)$ given in Theorem 4 is sharp.

Theorem 3. *For $n \geq 8$,*

$$f_{\mathcal{OP}}(n, P_4) = 2n^2 - 7n + 2.$$

Moreover, for $n \geq 9$, the only n-vertex outerplanar graph containing $f_{\mathcal{OP}}(n, P_4)$ number of P_4's is $K_1 + P_{n-1}$.

Theorem 4.

$$f_{\mathcal{OP}}(n, P_5) = \frac{17}{4}n^2 + \Theta(n).$$

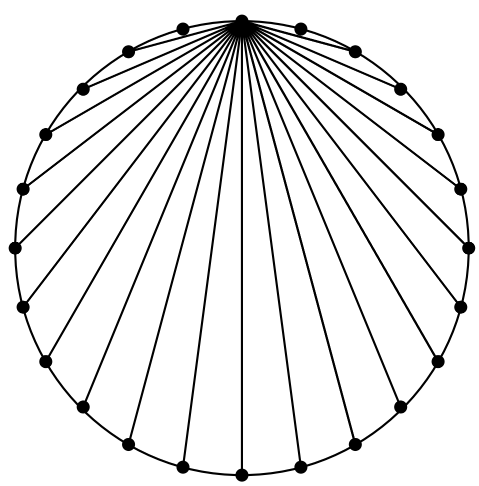

Figure 1: An n-vertex maximal outerplanar graph $G_n = K_1 + P_{n-1}$.

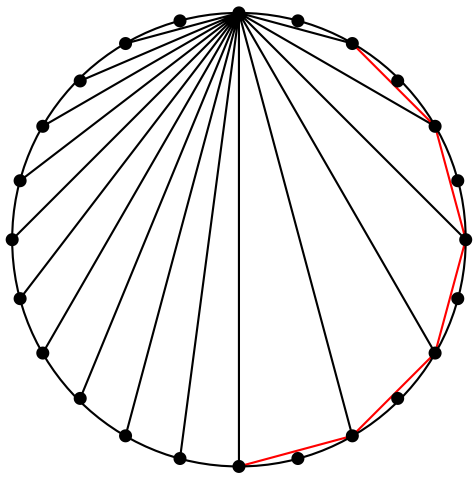

Figure 2: An n-vertex maximal outerplanar graph G_n.

2 Concluding remarks and conjecture

Considering the complexity of the proof we have for a best asymptotic value of $f_{\mathcal{OP}}(n, P_5)$, it might be not easy to determine a best asymptotic value of the generalized outerplanar Turán number of short paths. We pose the following conjecture related to the generalized outerplanar Turán number of the P_6.

Conjecture 1. $f_{\mathcal{OP}}(n, P_6) = 11n^2 + \Theta(n)$.

The following construction of an n-vertex maximal outerplanar graph G_n verifies the lower bound is attainable. G_n contains $\frac{n}{2}$ degree-2 vertices and all the remaining $\frac{n}{2} - 1$ vertices are adjacent to a vertex, say v, in G_n (see Figure 3). It can be checked that $\mathcal{N}(P_6, G_n) = 11n^2 + \Omega(n)$.

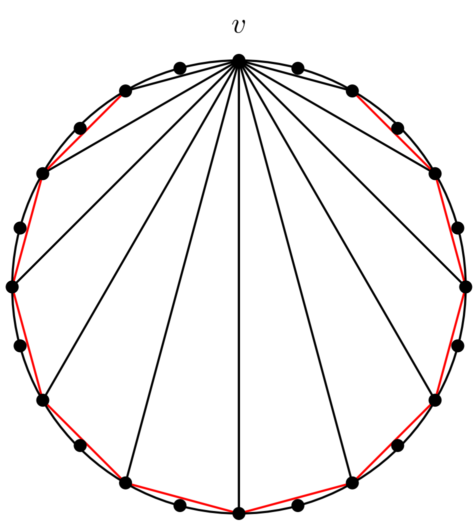

Figure 3: A maximal outerplanar graph G_n containing roughly $11n^2$ P_6's.

Acknowledgments

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