On the Inverse of $\alpha-$Hermitian Adjacency Matrix of Mixed Graphs

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Abstract

A graph X with both edges and arcs is called a mixed graph. This can be considered as a hybrid of an oriented graph and an unoriented graph. Let α be the primitive n^{th} root of unity $e^{\frac{2\pi}{n}i}$. Then, for a mixed graph X, the $\alpha-$Hermitian adjacency matrix of X is defined to be the matrix $H_\alpha(X) = [h_{rs}]$ where $h_{rs} = \alpha$ if rs is an arc in D, $h_{rs} = \overline{\alpha}$ if sr is an arc in D, $h_{rs} = 1$ if sr is an unoriented edge in D and $h_{rs} = 0$ otherwise. Accordingly, in this paper we study the inverse of $\alpha-$hermitian adjacency matrix of a mixed graph.

1 Introduction

A digraph D that can be obtained from a graph after orienting some or all of its edges is called mixed graph. The unoriented edges of D are called digons. The graph obtained from a mixed graph D after unorienting all of its edges, denoted by $\Gamma(D)$, is called the underlying graph of the mixed graph D. A perfect matching of a mixed graph is just a perfect matching of its underlying graph. To be formal, a set M of edges and arcs is called perfect matching if all elements of M are non-incident and cover the vertices of D. If X is a subgraph of a mixed graph D then by $D\backslash X$ we mean induced mixed graph over $V(D)\backslash V(X)$.
Let D be a mixed graph and α be a complex number with $|\alpha| = 1$, then the α-hermitian adjacency matrix of D is the matrix $H_\alpha(D) = [h_{uv}]$ where its rows and columns correspond to the vertices of D and,

$$h_{uv} = \begin{cases} 1 & \text{if } uv \text{ is digon in } D, \\ \alpha & \text{if } uv \text{ is an arc in } D, \\ \overline{\alpha} & \text{if } vu \text{ is an arc in } D, \\ 0 & \text{otherwise.} \end{cases}$$

Godsil in [1] studied the inverse of adjacency matrix of bipartite graphs. In fact this problem is related to a chemistry problem. The most interesting theorem in this paper was: the inverse of the adjacency matrix of a tree T is similar to the adjacency matrix of another graph G, the similarity matrix is ± 1 diagonal matrix and the graph G should contain a copy of T. More research papers appeared after [1] that continued on Godsil's work see [2], [3] and [4] for example. Yet, all research appears in this direction were based on the adjacency matrix of unoriented graphs. One of the reasons for that is digraphs with nonsingular traditional adjacency matrix are infrequent. Motivated by this and to overcome the challenge of such infrequency, in this paper we study the inverse of $\alpha-$hermitian adjacency matrix of a mixed graphs D. In fact, we give a formula of entries of the inverse matrix in terms of mixed subgraphs of D. In order to do that we need the following definitions and theorems:

Definition 1. *[5] Let D be a mixed graph and $H_\alpha = [h_{uv}]$ its α-hermitian adjacency matrix.*

- *A mixed subgraph X of D is called elementary mixed subgraph of D if for every component X' of X, $\Gamma(X')$ is either the complete graph K_2 or a cycle C_k (for some $k \geq 3$).*

- *For an elementary mixed subgraph X of D. The rank of X is defined as $r(X) = n - c$, where $n = |V(X)|$ and c is the number of its components. The co-rank of X is defined as $s(X) = m - r(X)$, where m is the number of digons and arcs in X.*

- *The value $h_\alpha(W)$ of a mixed walk W with vertices $v_1, v_2, \ldots, v_k$ is defined as*

$$h_\alpha(W) = (h_{v_1v_2}h_{v_2v_3}h_{v_3v_4}\ldots h_{v_{k-1}v_k}) \in \{\alpha^r\}_{r\in\mathbb{Z}}$$

Recall that a permutation η of a set of n elements V, is just a bijective function from V to itself. The set of all permutations of V form a group under the functions composition. Let η be a permutation of a set of n elements V, then $sgn(\eta)$ is defined to be $(-1)^k$, where k is the number of transpositions when η is decomposed as a product of transpositions. The following theorem is a well known result in linear algebra.

Theorem 1. *If $A = [a_{ij}]$ is an $n \times n$ matrix then*

$$det(A) = \sum_{\eta\in S_n} sgn(\eta)a_{1,\eta(1)}a_{2,\eta(2)}a_{3,\eta(3)}\ldots a_{n,\eta(n)}$$

2 Inverse of the α-Hermitian adjacency matrix of a Mixed Graph

In this section we give a general description of the inverse of $\alpha-$Hermitian adjacency matrix of mixed graphs. The following theorem can be found in [5].

Theorem 2. *(Determinant expansion for H_α) [5] Let D be a mixed graph and H_α its α-hermitian adjacency matrix, then*

$$det(H_\alpha) = \sum_{D'}(-1)^{r(D')}2^{s(D')}Re(\prod_C h_\alpha(\vec{C}))$$

where the sum ranges over all spanning elementary mixed subgraphs D' of D, the product ranges over all mixed cycles C in D', and $\vec{C}$ is any mixed closed walk traversing C.

In the following theorem we give a general description of the non-diagonal entries of the inverse of α-hermitian adjacency matrix of mixed graphs in terms of elementary mixed subgraphs.

Theorem 3. *Let D be a mixed graph, H_α be its α-hermitian adjacency matrix and for $i \neq j$, $\Im_{ij} = \{P : P$ is a mixed path from the vertex i to the vertex $j\}$. If $\det(H_\alpha) \neq 0$, then*

$$[H_\alpha^{-1}]_{ij} = \frac{1}{\det(H_\alpha)}\sum_{P_{i\to j}}\left[(-1)^{|E(P_{i\to j})|}\ h_\alpha(P_{i\to j})\left(\sum_{D'}(-1)^{r(D')}2^{s(D')}Re\left(\prod_C h_\alpha(\vec{C})\right)\right)\right]$$

where the first sum is taken over all paths $P_{i\to j} \in \Im_{ij}$ and the second sum ranges over all spanning elementary mixed subgraphs D' of $D\backslash P$, the product is being taken over all mixed cycles C in D' and $\vec{C}$ is any mixed closed walk traversing C.

Proof. Suppose that $i \neq j$, then

$$[H_\alpha^{-1}]_{ij} = \frac{m_{ji}}{\det(H_\alpha)},$$

where

$$m_{ji} = (-1)^{i+j}\det((H_\alpha)_{(j,i)}),$$

and $(H_\alpha)_{(j,i)}$ is the matrix obtained from $H_\alpha(D)$ after removing the j^{th} row and i^{th} column.

Now let M_{ji} be the matrix obtained from H_α by replacing the (ji)-entry with 1 and all other entries of j^{th} row and i^{th} column by 0, then

$$m_{ji} = \overline{\det(M_{ji})} \tag{1}$$

On the other hand, using Theorem 1 we have,

$$\det(M_{ji}) = \sum_{\eta \in S_n} sgn(\eta) h_{1\eta(1)} h_{2\eta(2)} \dots h_{n\eta(n)}$$

Now for any $\eta \in S_n$, since (j,k)-entries of M_{ji} are zeros and the (j,i)-entry is one, if η does not take j to i then η contributes zero in the expansion of $\det(M_{ji})$. Let

$$\psi_{j\to i} = \{\phi \in S_n : \phi \text{ is a permutation that takes } j \text{ to } i\}.$$

For each $\phi \in \psi_{j\to i}$ let δ_ϕ be the cycle in ϕ that permutes j to i and δ_ϕ^c be all other cycles in ϕ, then

$$\begin{aligned}
\det(M_{ji}) &= \sum_{\phi \in \psi_{j\to i}} sgn(\phi) \prod_{k \in V(G)\backslash\{j\}} h_{k\phi(k)} \\
&= \sum_{\phi \in \psi_{j\to i}} sgn(\delta_\phi^c) sgn(\delta_\phi) \prod_{k \in \delta_\phi} h_{k\delta_\phi(k)} \prod_{k \in \delta_\phi^c} h_{k\delta_\phi^c(k)} \\
&= \sum (-1)^{|E(P_{j\to i})|} h_\alpha(P_{j\to i}) \det(H_\alpha(X)) \\
&= \sum (-1)^{|E(P_{j\to i})|} \overline{h_\alpha(P_{i\to j})} \det(H_\alpha(X))
\end{aligned}$$

where X is the induced mixed graph over $V(D)\backslash V(P_{i\to j})$ and $P_{i\to j} \in \Im_{ij}$. Therefore using Equation 1 together with Theorem 2 we have,

$$\det(M_{ij}) = \sum_{P_{i\to j} \in \Im_{ij}} \left[(-1)^{|E(P_{i\to j})|} h_\alpha(P_{i\to j}) \sum_{D'} (-1)^{r(D')} 2^{S(D')} Re \left(\prod_C h_\alpha(\vec{C}) \right) \right]$$

where the second sum is taken over all spanning elementary mixed subgraphs of $D\backslash P_{i\to j}$, the product ranges over all mixed cycles C in D', and $\vec{C}$ is any mixed closed walk traversing C $\square$

Example 1. *Consider the mixed graph D shown in Figure 1 and let H_i be its i- Hermitian adjacency matrix. Observing that D has unique perfect matching and using Theorem 2 we get $det(H_i) = (-1)^{8-4} = 1$.*
One can easily check that for any two vertices i and j in D there is at most one path with the property $D\backslash P_{i\to j}$ has spanning elementary mixed subgraphs. Furthermore, the D unique cycle cannot be part of any spanning elementary mixed subgraph of D. Since for any path P in D has i-weight 0, $\pm i$ or ± 1 and using Theorem 3, we have H_i^{-1} is $\{0, \pm 1, \pm i\}$-matrix. In fact a simple calculation can be done to show that H_i^{-1} is a Hermitian adjacency matrix of another mixed graph.

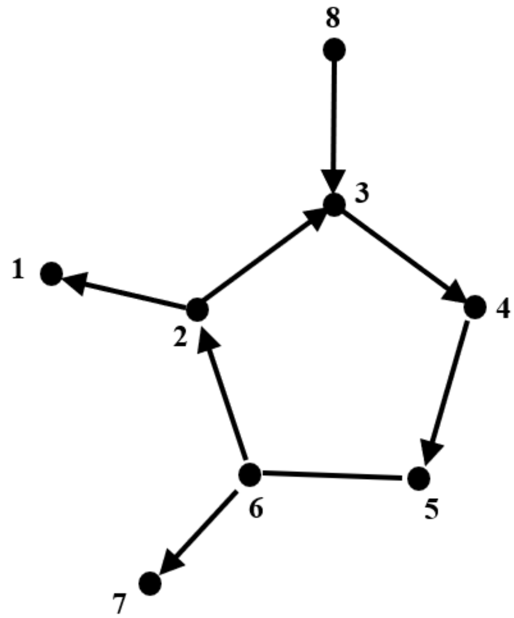

Figure 1: The mixed graph D

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