# The bondage number of chordal graphs

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**Abstract**

A set $S \subseteq V(G)$ of a graph $G$ is a dominating set if each vertex in $V(G) \setminus S$ has a neighbor in $S$. Let $\gamma(G)$ be the cardinality of a minimum dominating set in $G$. The bondage number $b(G)$ of a graph $G$ is the smallest cardinality of a set edges $A \subseteq E(G)$ such that $\gamma(G - A) = \gamma(G) + 1$. A chordal graph has no induced cycle of length four or more. We show that the bondage number of a chordal graph $G$ is at most the order of its maximum clique, that is, $b(G) \leq \omega(G)$. We show that this bound is best possible.

## 1 Preliminaries

The graphs considered are finite and simple, that is, without directed edges or loops or parallel edges. The reader is referred to [1] for definitions and notations in graph theory, and to the survey of Xu [6] for an overview on the bondage number and its related properties.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $v \in V(G)$ and $xy \in E(G)$. We say that $x$ and $y$ are the *endpoints* of the edge. Let $\delta(G)$ and $\Delta(G)$ denote its *minimum degree* and its *maximum degree*, respectively. The *degree* of $v$ in $G$ is $d_G(v)$ or simply $d(v)$ when the referred graph is obvious. If $d(v) = 0$, we say that $v$ is *isolated* in $G$. We denote by $d(u, v)$ the *distance* between two vertices, that is, the length of a shortest path between $u$ and $v$. Note that when $uv \in E$, $d(u, v) = 1$. We denote by $N_G(v)$ the *open neighborhood* of a vertex $v$ in $G$, and $N_G[v] = N_G(v) \cup \{v\}$ its *closed neighborhood* in $G$. When it is clear from context, we write $N(v)$ and $N[v]$. The *open neighborhood* of a set $U \subseteq V$ is $N(U) = \{N(u) \setminus U \mid u \in U\}$. For a subset $U \subseteq V$, let $G[U]$ denote the *subgraph* of $G$ induced by $U$ which has vertex set $U$ and edge set $\{uv \in E \mid u, v \in U\}$. We may refer to $U$ as an *induced subgraph* of $G$ when it is clear from the context. If a graph $G$ has no induced subgraph isomorphic to a fixed graph $H$, we say that $G$ is $H$-free. For $n \geq 1$, the graph $P_n = u_1 - u_2 - \cdots - u_n$ denotes the *cordless path* or *induced path* on $n$ vertices, that is, $V(P_n) = \{u_1, \ldots, u_n\}$ and $E(P_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\}$. For $n \geq 3$, the graph $C_n$ denotes the *cordless cycle* or *induced cycle* on $n$ vertices, that is, $V(C_n) = \{u_1, \ldots, u_n\}$ and $E(C_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{u_n u_1\}$. For $n \geq 4$, $C_n$ is called a *hole*. A set $U \subseteq V$ is called a *clique* if any pairwise distinct vertices $u, v \in U$ are adjacent. We denote by $\omega(G)$ the size of a maximum clique in $G$. The graph $K_n$ is the clique with $n$ vertices. A set $U \subseteq V$ is called a *stable set* or an *independent set* if any pairwise distinct vertices $u, v \in U$ are non adjacent.

We recall the two following results on the upper bound of the bondage number. They will be of use to prove Theorem 3 in the next section.

**Theorem 1** (Fink et al. [3])**.** *Let $G = (V, E)$ be a graph, and $u, v \in V$ such that $d(u, v) \leq 2$. Then $b(G) \leq d(u) + d(v) - 1$.*

**Theorem 2** (Hartnell and Rall [5])**.** *Let $G = (V, E)$ be a graph, and $uv \in E$. Then $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$.*

## 2 Our result

**Theorem 3.** *Let $G$ be a chordal graph. If $G$ is a clique, then $b(G) = \lceil \omega(G)/2 \rceil$. Else $b(G) \leq \omega(G) \leq \Delta(G)$.*

*Proof.* We can assume that $G$ is connected with at least two vertices. Note that $\Delta(G) \geq \omega(G) - 1$ and $\Delta(G) = \omega(G) - 1$ if and only if $G$ is a clique. When $G$ is an even clique, one can see that $b(G) = \omega(G)/2$ by removing a perfect matching of $G$. When $G$ is an odd clique, then one can see that $b(G) = (\omega(G) - 1)/2 + 1$ by removing a perfect matching of $G$ and any edge incident to the remaining universal vertex. So when $G$ is a clique, then $b(G) = \lceil \omega(G)/2 \rceil$. Therefore we can assume that $G$ is not a clique and so $\omega(G) \leq \Delta(G)$.

For the sake of contradiction, we suppose that $b(G) > \omega(G)$. Let $K$ be a clique of $G$. The *partition distance* in $G$ with respect to $K$ is the partition $(A_0, \ldots, A_k)$ of $V$ such that $A_0 = V(K)$ and $A_i = \{v \in V \mid v \in N(u), u \in A_{i-1}\}$, for $i = 1, \ldots, k$. Note that $A_i$ is the set of vertices at distance $i$ from $K$.

**Claim 1.** *Let $C \subseteq A_i$ where $i \neq 0$, be such that $G[C]$ is a connected component of $G[A_i]$, and let $Q = N(C) \cap A_{i-1}$. Then $G[Q]$ is a clique.*

For contradiction, suppose that $G[Q]$ is not a clique. Since $A_0$ is a clique, we can consider that $i \geq 2$. Let $u, u' \in Q$ such that $uu' \notin E$. There is a path from $u$ to $K$ and from $u'$ to $K$ in $G[A_0 \cup \ldots \cup A_{i-2} \cup \{u, u'\})]$. Therefore there is an induced path $P = u - \cdots - u'$ from $u$ to $u'$ in $G[A_0 \cup \ldots \cup A_{i-2} \cup \{u, u'\})]$. Let $P' = u - \cdots - u'$ be an induced path from $u$ to $u'$ in $G[C \cup \{u, u'\}]$. Then $G[V(P) \cup V(P')]$ is an induced cycle of length at least four, a contradiction. So $G[Q]$ is a clique. This proves Claim 1.

Let $W \subseteq A_i$, where $i = 0, \ldots, k$, such that $G[W]$ is a connected component of $G[A_i]$ with at least two vertices. We restrict $W$ such that $F = N(W) \cap A_{i+1}$ is either empty or an independent set of $G$, and such that $N(F) \cap A_{i+2} = \emptyset$. We choose $W$ so that $\psi(K) = |F \cup W|$ is minimum. When $W \neq V(K)$, we denote $Q = N(W) \cap A^{i-1}(K)$. Note that when $W = V(K)$, then $Q = \emptyset$.

We show that $W$ exists such as described above. Since $G$ is not a clique, it follows that $A_{k-1}, A_k \neq \emptyset$. If $A_k$ is not an independent set of $G$, then there is a connected component $C$ of $G[A_k]$ with at least two vertices. Since $|C| \geq 2$ and $N(C) \cap A_{k+1} = \emptyset$, it follows that $W$ exists. Now we can assume that $A_k$ is an independent set of $G$. Let $C$ be a connected component of $G[A_{k-1}]$ such that $N(C) \cap A_k \neq \emptyset$. If $|C| \geq 2$, then $W$ exists since $N(C) \cap A_k$ is an independent set of $G$ and $A_{k+1} = \emptyset$. Hence it remains the case where $|C| = 1$. Let $C = \{u\}$ and $v \in N(u) \cap A_k$. From Claim 1 $G[N(v) \cap A_{k-1}]$ is a clique. Thus $N(v) = \{u\}$ and $d(v) = 1$. From Claim 1 $N(u) \cap A_{k-2}$ is a clique. Therefore $d(u) \leq \omega(G)$. Then from Theorem 1 $b(G) \leq d(u) + d(v) - 1 \leq \omega(G)$, a contradiction. Hence $|C| \geq 2$ and so $W$ exists.

Let $K$ be a clique of $G$ such that $\psi(K) = min(\{\psi(K') \mid K' \textit{ is a clique of } G\})$. We consider the sets $A_0, \ldots, A_k, F, Q, W$ as described above in the partition distance with respect to $K$.

**Claim 2.** *For every $u \in W$ such that $Q = N(u) \cap A_{i-1}$, the sets $W \setminus \{u\}$ and $N(u) \cap (F \cup W)$ are independent in $G$, and $W = N[u] \cap W$.*

For contradiction, suppose that $W \setminus \{u\}$ or $N(u) \cap (F \cup W)$ is not an independent set of $G$. Let $K' = G[Q \cup \{u\}]$. Note that $Q$ is empty when $W = A_0$. From Claim 1 $Q$ is a clique and it follows that $K'$ is also a clique. Let $A'_0, A'_1, \ldots, A'_{k'}$ be the partition distance with respect to $K'$. Hence $A'_0 = K'$. Since $W \setminus \{u\}$ or $N(u) \cap (F \cup W)$ are not an independent set, there is $W' \subseteq A'_1 \cap (F \cup W)$ such that $W'$ is a connected component of $G[A'_1]$ with at least two vertices. Let $F' = N(W') \cap A'_2$. Note that $F' \subseteq F$. Therefore either $F' = \emptyset$ or $F'$ is an independent set of $G$, and $N(F') \cap A'_3 = \emptyset$. Then $|F' \cup W'| \leq |F \cup W| - 1$ and thus $\psi(K)$ is not minimum, a contradiction. Hence $W \setminus \{u\}$ and

$N(u) \cap (F \cup W)$ are two independent sets of $G$. Since $G[W]$ is connected, it follows that $W \subseteq N[u]$. This proves Claim 2.

**Claim 3.** *There exists $u \in W$ such that $Q = N(u) \cap Q$.*

For contradiction, suppose that for every vertex $u \in W$, we have $Q \neq N(u) \cap Q$ i.e. $Q \nsubseteq N(u)$. Let $u \in W$ such that $|N(u) \cap Q|$ is maximal. Since every vertex of $Q$ has a neighbor in $W$, there is $u' \in W$ such that $q'u' \in E$ and $q'u \notin E$, where $q' \in Q$. We choose $u'$ so that $d(u, u')$ is minimal. From the maximality of $|N(u) \cap Q|$, there is $q \in Q$ such that $qu \in E$ and $qu' \notin E$. Since $G[W]$ is connected, there is a shortest path $P = u - \cdots - u'$ between $u$ and $u'$ in $G[W]$. If $P = u - u'$, then $C_4 = q - q' - u' - u - q$ is an induced cycle of length four, a contradiction. Let $v \in V(P) \setminus \{u, u'\}$. Suppose that $q'v \in E$. From the minimality of $d(u, u')$, it follows that $N(u) \cap Q \subseteq N(v) \cap Q$. Then $|N(v) \cap Q| > |N(u) \cap Q|$ is a contradiction of the maximality of $|N(u) \cap Q|$. Hence for every $v \in V(P) \setminus \{u, u'\}$, we have $vq' \notin E$. Therefore if no vertex of $V(P) \setminus \{u, u'\}$ is a neighbor of q, it follows that $G[V(P) \cup \{q, q'\}]$ is an induced cycle of length at least five, a contradiction. So there is $v \in V(P) \setminus \{u, u'\}$ such that $qv \in E$. We choose $v$ such that $d(u', v)$ is minimum. Let $P' = v - \cdots - u'$ be a shortest path between $u'$ and $v$. Then $G[V(P') \cup \{q, q'\}]$ is an induced cycle of length at least four, a contradiction. This proves Claim 3.

**Claim 4.** *For every $u \in W$, $|N(u) \cap F| \leq 1$, and for every $v \in F$, $d(v) = 1$.*

For contradiction, suppose there exists $u \in W$ such that $v, v' \in N(u) \cap F$. From Claim 3 there is $w \in W$ such that $Q = N(w) \cap Q$. From Claim 2 $W = N[w] \cap W$, and $W \setminus \{w\}$, $(F \cup W) \cap N(w)$ are two independent sets of $G$. From Claim 1 $N(v) \cap A_i$, $N(v') \cap A_i$ are two cliques and therefore $N(v) \subseteq W$ and $N(v') \subseteq W$. If $d(v) \geq 2$ or $d(v') \geq 2$, then $(F \cup W) \cap N(w)$ is not an independent set. Hence $d(v), d(v') \leq 1$. Yet from Theorem 1 it follows that $b(G) \leq d(v) + d(v') - 1 \leq 1$, a contradiction. This proves Claim 4.

**Claim 5.** $|Q| \leq \omega(G) - 1$

From Claim 1 $Q$ is a clique and from Claim 3 there is $u \in W$ such that $Q = N(u) \cap Q$. Hence $Q \cup \{u\}$ is a clique and therefore $|Q| \leq \omega(G) - 1$. This proves Claim 5.

From Claim 3 there is $u \in W$ such that $Q = N(u) \cap Q$. Recall that $|W| \geq 2$ and that $G[W]$ is a connected. Suppose that there is $v \in W$, $u \neq v$, such that $Q = N(v) \cap Q$. From Claim 2 $W \setminus \{u\}$ and $W \setminus \{v\}$ are two independent sets of $G$. Thus $W = \{u, v\}$. From Claim 1 $Q$ is a clique, and therefore $|Q| \leq \omega(G) - 2$. From Claim 4 $|N(u) \cap F|, |N(v) \cap F| \leq 1$. Hence $d(u) \leq |Q \cup W \setminus \{u\}| + 1 \leq \omega(G)$ and $d(v) \leq |Q \cup W \setminus \{v\}| + 1 \leq \omega(G)$. Suppose that $u$ has a neighbor $x \in F$. It follows from Claim 4 that $d(x) = 1$. Thus from Theorem 1 $b(G) \leq d(u) + d(x) - 1 \leq \omega(G)$, a contradiction. Hence $N(u) \cap F, N(v) \cap F = \emptyset$. Therefore $d(u) = d(v) = \omega(G) - 1$. From Theorem 2 it follows that $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)| \leq \omega(G)$, a contradiction.

So we can assume that $u$ is the only vertex in $W$ such that $Q = N(u) \cap Q$. We show that $F$ is empty. Recall that from Claim 1 $G[Q]$ is a clique, from Claim 5 $|Q| \leq \omega(G) - 1$, and from Claim 4 every vertex of $W$ has at most one neighbor in $F$. Moreover from Claim 2 $W = N[u]$ and $(F \cup W) \setminus \{u\}$ is an independent set of $G$. Hence for every $v \in W \setminus \{u\}$, we have $d(v) \leq |Q| + 1 \leq \omega(G)$. Let $x \in F$. From Claim 4 $d(x) = 1$. If there is $v \in W \setminus \{u\}$ a neighbor of $x$, then from Theorem 1 it follows that $b(G) \leq d(v) + d(x) - 1 \leq \omega(G)$, a contradiction. Hence $x$ is a neighbor of $u$. Yet for every $v \in W \setminus \{u\}$, we have $d(v, x) \leq 2$. Therefore from Theorem 1 it follows that $b(G) \leq d(v) + d(x) - 1 \leq \omega(G)$, a contradiction. Hence $F = \emptyset$. It follows that for every $v \in W \setminus \{u\}$, we have $d(v) \leq |Q| \leq \omega(G) - 1$.

Let $S$ be a minimum dominating set of $G$. Suppose that $|S \cap W| \geq 2$. Then $(S \setminus W) \cup \{u\}$ is a dominating set, a contradiction. Hence for every minimum dominating set of $G$, we have $|S \cap W| \leq 1$. Let $v \in W \setminus \{u\}$ and $E_v = \{vv' \in E \mid v' \in N(v)\}$. Recall that $d(v) \leq \omega(G) - 1$, and therefore $|E_v| \leq \omega(G) - 1$. Let $w \in W \setminus \{v\}$ ($u = w$ is possible). Let $E_w = \{qw \in E \mid q \in (N(w) \cap Q) \setminus N(v)\}$, that is, the edges incident to $w$ with an extremity in $Q$ that is not a neighbor of $v$. Note that $|E_w| \leq |Q \setminus N(v)|$, and therefore $|E_v \cup E_w| \leq |Q| + 1 \leq \omega(G)$. We remove the edges $E_v \cup E_w$ from $G$ to construct $G' = (V, E - (E_v \cup E_w))$. Since $b(G) > \omega(G)$, it follows that $\gamma(G') = \gamma(G)$. Let $S'$ be a minimum dominating set of $G'$. Since $G'$ is the graph $G$ minus some edges, any dominating set of $G'$ is a dominating set of $G$. Hence $S'$ is a minimum dominating set of $G$. Therefore from previous arguments, we have $|S' \cap W| \leq 1$. Note that $v$ is isolated in $G'$, and thus $v \in S'$. If $S' \cap N_G(v) \neq \emptyset$, then $S' \setminus \{v\}$ is a dominating set of $G$, a contradiction. Hence $S' \cap N_G(v) = \emptyset$. Recall that $N_{G'}(w) \cap Q \subseteq N_G(v) \cap Q$. Hence $N_{G'}(w) \cap S' \cap W \neq \emptyset$. Yet it follows that $|S' \cap W| \geq 2$, a contradiction.

Hence $\gamma(G') > \gamma(G)$. Since we removed at most $\omega(G)$ edges from $G$ to construct $G'$, it follows that $b(G) \leq \omega(G)$. This completes the proof. □

We show that the bound of Theorem 3 is sharp. The corona $G_1 \circ G_2$ (introduced by Frucht and Harary in [4]) is the graph formed from $|V(G_1)|$ copies of $G_2$ by joining the ith vertex of $G_1$ to the ith copy of $G_2$. Let $G = K_n \circ K_1$. Note that $\omega(G) = \Delta(G) = n$. Carlson and Develin in [2] have shown that $\gamma(G) = \omega(G)$ and that $b(G) = \omega(G)$.

For non-chordal graphs, we show that there is an infinite family of graphs $\mathcal{C}$, where for every $G \in \mathcal{C}$, we have $b(G) > \omega(G)$, and its longest induced cycle has length four. The *cartesian product* $G \,\square\, H$ of two graphs $G$ and $H$ is the graph whose vertex set is $V(G) \times V(H)$. Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent in $G \,\square\, H$ if either $g_1 = g_2$ and $h_1 h_2$ is an edge in $H$ or $h_1 = h_2$ and $g_1 g_2$ is an edge in $G$. Consider $G = (P_2 \,\square\, P_k) \circ K_1$, where $k \geq 2$. The longest cycle of $G$ is four and $\omega(G) = 2$. Then one can easily check that $\gamma(G) = 2k$ and that $b(G) = 3 = \omega(G) + 1$. We remark that it would be of interest to know if there exists a graph $G$ for which the longest cycle is $C_4$, and such that $b(G) > \omega(G) + 1$. Graphs for which the longest cycle is $C_4$ may be known as the class of *quadrangulated* graphs (an extension of chordal graphs, that is, chordal graphs where $C_4$ are allowed).

# References


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