Matchings in Frameworks

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Abstract

We say that a pair $(G, \mathcal{M})$, where G is a graphs and $\mathcal{M} = (V(G), \mathcal{I})$ is a matroid on the vertex set of G, is a framework. We consider the Max Rank Matching problem that, given a framework $(G, \mathcal{M})$, asks for a matching M of maximum rank with respect to $\mathcal{M}$, that is, we maximize the rank of the set of vertices saturated by M. Our main combinatorial result is an analog of the classical Berge's lemma for frameworks. Using this result, we show that Max Rank Matching can be solved in $\mathcal{O}(n^4)$ time on frameworks $(G, \mathcal{M})$, where G is an n-vertex graph and $\mathcal{M}$ is given by the independence oracle.

1 Introduction

Berge's lemma [1] asserting that a matching in a graph is maximum if and only if the graph has no augmenting path plays crucial role in the algorithmic study of the Maximum Matching problem (see [10] for the introduction). In particular, now classical, Edmond's blossom algorithm [4] is essentially an efficient procedure finding an augmenting path. Maximum Matching has various generalization and one of the most important ones is the Matroid Parity problem introduced by Lawler [8] whose task is, given a matroid $\mathcal{M}$ and a family $\mathcal{P}$ of disjoint pairs of the elements of the ground set, find a largest independent set of paired elements. We introduce another matroid generalization Maximum Matching. Following Lovász [9], we say that a pair $(G, \mathcal{M})$, where G is a graphs and $\mathcal{M} = (V(G), \mathcal{I})$ is a matroid on the vertex set of G, is a *framework*. For a set of edges S of a graph G, we denote by $V_G(S)$ the set of end-vertices of the edges of S. We denote by $r\colon V(G) \to \mathbb{Z}_{\geq 0}$ the rank function of $\mathcal{M}$ and say that a *rank of a matching* M of G is $r(M) = r(V_G(M))$.

Max Rank Matching

Input:	A framework $(G, \mathcal{M})$.
Task:	Find a maximum rank matching M in G.

Note that Matroid Parity can be stated as a problem on frameworks: given a framework $(G, \mathcal{M})$, find a matching M in G of maximum size such that the end-vertices of the vertices of M compose an independent set of $\mathcal{M}$. While both Matroid Parity and Max Rank Matching are generalization of Maximum Matching, they are incomparable. Consider, for example, $(G, \mathcal{M})$, where G is a matching with m edges and $\mathcal{M}$ is a linear matroid of rank m with $2m$ elements such that for each edge e of G, one of its end-vertices is the zero vector, and the other end-points of the edges form an independent set of M. Then the unique solution of Matroid Parity is the empty set and the unique solution of Max Rank Matching is the set of edges of G.

Max Rank Matching can be reduced to the well-known Matroid Intersection problem[1]. Given a framework $(G, \mathcal{M})$, consider the matroid $\widehat{\mathcal{M}}$ with the ground set $V(G)$ whose bases are

[1] We are grateful to an anonymous reviewer who pointed to us this fact.

maximal vertex sets saturated by matchings. Then the maximum rank of a matching in G is the maximum size of a common independent set of $\mathcal{M}$ and $\widehat{\mathcal{M}}$. This implies that MAX RANK MATCHING can be solved in polynomial time [7]. However, the graph structure allows to construct a more efficient direct algorithm.

Our main combinatorial result, given in Lemma 2, is an analog of Berge's lemma [1] about matchings of maximum rank in a framework. Given a framework $(G, \mathcal{M})$ and a matching M in G, we construct an auxiliary graph G_M together with a matching M^* such that M is not a matching of maximum rank in G if and only if G_M has an M^*-alternating path P with prescribed end-vertices such that $\widehat{M} = (M \triangle E(P)) \cap E(G)$ is a matching whose rank is bigger that the rank of M. Using Lemma 2, we obtain our main algorithmic result in Theorem 1 that MAX RANK MATCHING can be solved in $\mathcal{O}(n^4)$ time on frameworks $(G, \mathcal{M})$, where G is an n-vertex graph and $\mathcal{M}$ is given by the independence oracle. We would like to point that Theorem 1 gives an interesting contrast of MAX RANK MATCHING with MATROID PARITY. It is well-known that MATROID PARITY can be solved in polynomial time for linear matroids (see [10, 2, 5]). However, if the input matroid is given by its independence oracle, there is no algorithm for MATROID PARITY whose number of oracle calls is bounded by a polynomial on the size of the matroid ground set [6].

In this abstract, we use standard graph-theoretic terminology and refer to the textbook of Diestel [3] for missing notions. For the introduction to Matroid Theory, we refer to the textbook of Oxley [11].

2 Rank-augmenting paths

Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid with the ground set V and the family of independent sets $\mathcal{I}$. Let also $X \subseteq V$. We say that x is *important* (with respect to X) if $r(X \setminus \{x\}) < r(X)$. We denote by $\mathsf{Imp}(X) \subseteq X$ the set of all important vertices with respect to X. Let x be an important vertex with respect to X and let $y \in V \setminus X$. We say that y is a *mate* of X with respect to X if $r((X \setminus \{x\}) \cup \{y\}) = r(X)$. We write $y \sim_X x$ to denote that y is a mate of x. Respectively, $x \not\sim_X y$ denotes that $y \in V \setminus X$ is not a mate of x with respect to X. We may omit "with respect to" if the set is clear from context. Using basic matroid properties, we show the following technical lemma.

Lemma 1. *Let $\mathcal{M} = (V, \mathcal{I})$ and let $X \subseteq V$ be such that $r(X) = r(\mathcal{M})$. Suppose that $x_1, \ldots, x_k \in \mathsf{Imp}(X)$ are distinct important elements, $y_1, \ldots, y_k$ are distinct elements of $V \setminus X$ such that (a) $y_i \sim_X x_i$ for all $i \in \{1, \ldots, k\}$ and (b) $y_j \not\sim_X x_i$ for every $i, j \in \{1, \ldots, k\}$ such that $i < j$. Then the following holds:*

(i) $r(X) = r((X \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\})$, *and*

(ii) $\mathsf{Imp}((X \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}) = (\mathsf{Imp}(X) \setminus \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}$.

Further, we use Lemma 1, to show our main combinatorial result. Given a framework $(G, \mathcal{M})$ and an inclusion maximal matching M, we construct a special auxiliary graph G_M and a matching M^*. Also we define some special subsets of vertices of G_M.

Definition 1 (Construction of G_M and M^*)**.** *Let $(G, \mathcal{M})$ be a framework and let M be an inclusion maximal nonempty matching in G. We define:*

- $X = V_G(M)$,
- $U = \{v \in V(G) \setminus X : r(X \cup \{v\}) > r(X)\}$,

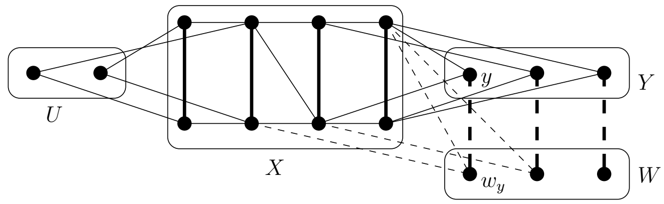

Figure 1: Construction of G_M. The edges of M are shown by thick lines, the edges of $M' = M^* \setminus M$ are shown by dashed thick lines, and the edges between W and X are shown by dashed thin lines.

- $Y = V(G) \setminus (U \cup X)$.

We construct G_M from G as follows:

- *For every $y \in Y$, construct a vertex w_y and make it adjacent to y. Set $W = \{w_y : y \in Y\}$.*
- *For every $w_y \in W$, make w_y adjacent to each $x \in X$ such that $y \sim_X x$.*

Finally, we define $M' = \{yw_y : y \in Y\}$ and set $M^* = M \cup M'$.

The construction is shown in Figure 1. We use the introduced notation in the following crucial lemma, which is our analog of Berge's lemma. We remind that given a matching M in a graph G, a path $v_0v_1 \cdots v_\ell$ is said to be *M-alternating* (or simply *alternating* if M is clear from context) if $\ell \geq 1$ and either $v_{2i}v_{2i+1} \in M$ for all $i \in \{0, \ldots, \lceil \ell/2 \rceil - 1\}$ or $v_{2i-1}v_{2i} \in M$ for all $i \in \{1, \ldots, \lfloor \ell/2 \rfloor\}$.

Lemma 2. *Let $(G, \mathcal{M})$ be a framework and let M is an inclusion maximal nonempty matching in G. Then M is not a matching of maximum rank in the framework if and only if G_M has an M^*-alternating path $P = v_0 \ldots v_\ell$ such that $v_0 \in U$ and one of the following is fulfilled:*

(i) $v_{\ell-1}v_\ell \in M$ *and* $v_\ell \notin \mathsf{Imp}(X)$,

(ii) $v_{\ell-1}v_\ell \notin M^*$ *and* $v_\ell \in U \cup Y$.

Furthermore, given P, an inclusion maximal matching $\widehat{M}$ in G with $r(\widehat{M}) > r(M)$ can be constructed in $\mathcal{O}(n^2)$ time.

To give some intuition behind the proof, suppose that P is a shortest M^*-alternating path with its end-vertex in U satisfying either (i) or (ii). Then Lemma 1 implies that $\widehat{M} = (M \triangle E(P)) \cap E(G)$ is a matching whose rank is bigger that the rank of M. For the opposite direction, assume that $\widehat{M}$ is a matching in G with $r(\widehat{M}) > r(M)$. Consider the family of nontrivial paths $\mathcal{P}$ that are the connected components of $(V(G), M \triangle \widehat{M})$. It can be seen that $\mathcal{P}$ contains paths with at least one end-vertex in U. We select these paths and argue that if they do satisfy neither (i) nor (ii), then we can append at least one new path from $\mathcal{P}$ to the end-vertex of one of the selected paths in G_M in such a way that we obtain an M^*-alternating path. By iteratively applying these arguments we construct a rooted forest of M^*-alternating paths with their roots in U. Since $\mathcal{P}$ has bounded size, at some moment we would be unable to iterate further and this means that the forest contains a root-leaf path P satisfying either (i) or (ii).

3 Algorithm for Max Rank Matching

By the classical result of Edmonds [4], a matching of maximum size can be found in $\mathcal{O}(n^2m)$ time. The crucial step of the algorithm is the subroutine that finds an augmenting path for a matching M (an M-alternating path as *augmenting* if its end-vertices are not saturated in M) if it exists. We use this subroutine of Edmond's algorithm as a blackbox and Lemma 2 to show our algorithmic result.

Theorem 1. Max Rank Matching *can be solved in* $\mathcal{O}(n^4)$ *time on frameworks* $(G, \mathcal{M})$*, where* G *is an* n*-vertex graph and* $\mathcal{M}$ *is given by the independence oracle.*

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