Almost balanced perfect matchings in balanced edge-colored complete graphs

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Abstract

We call a graph with a k-edge coloring c *balanced*, if every color appears equally often. We contribute to a question posed by Kittipassorn and Sinsap [arxiv:2011.00862v1], who asked whether every balanced complete graph K_{2kn} with a k-edge coloring contains a balanced perfect matching. While this question has previously been answered affirmatively for $k = 2$, we show that it is not true in general. However, we believe that there are always *almost* balanced perfect matchings. To measure the deviation of a matching from being balanced, we introduce the function $f(M) = f_{k,n,c}(M) = \sum_{i=1}^{k} \left| \left|c^{-1}(i) \cap M\right| - n\right|$, measuring the sum of deviations for each color class from being balanced.
We show that for any balanced complete graph, we can find a perfect matching M with $f(M) \leq 3k\sqrt{kn\ln(2k)}$ and for $k = 3$ a perfect matching M with $f(M) \leq 2$, which is tight in some cases. We conjecture that in fact there is always a perfect matching with $f(M) \leq \mathcal{O}(k^2)$.

1 Introduction

As a special question from the area of zero-sum Ramsey theory [3, 8], Caro, Hansberg, Lauri, and Zarb [4] asked whether every 2-edge-colored complete graph of order $4n$ with equally many edges of both colors always has a perfect matching that also contains equally many edges of both colors. This question was answered independently and affirmatively by Ehard, Mohr, and Rautenbach [5] and by Kittipassorn and Sinsap [11]. The motivation for the present work is an interesting problem formulated by Kittipassorn and Sinsap concerning possible generalizations of this result to settings with more than two colors. In particular, they [11] asked whether, for every k-edge-coloring of K_{2kn} with equally many edges of each color, there is a perfect matching M that also contains equally many edges of each color; in other words, whether there is a *color-balanced* perfect matching for every *color-balanced* edge-coloring of K_{2kn}. Note that the considered edge-colorings are not required to be proper, and that an obvious necessary condition for the existence of M is that the order of the complete graph is a multiple of $2k$. While the example in Figure 1 shows that there is not always such a matching, we believe that there are always perfect matchings that are very close to being color-balanced.

We propose the following conjecture.

Conjecture 1. *If n and k are positive integers, and $c : E(K_{2kn}) \to [k]$ is such that, for every i in $[k]$, there are equally many edges e with $c(e) = i$, then there is a perfect matching M of K_{2kn} with*

$$f(M) \leq \mathcal{O}(k^2),$$

where $f(M) = f_{k,n,c}(M) = \sum_{i=1}^{k} \left| \left|c^{-1}(i) \cap M\right| - n\right|$.

In order to upper-bound the value of $f_{k,n,c}(M)$ for some optimal matching M, we introduce the function

$$g(k, n) = \max_{c\in\mathcal{C}_{k,n}} \min_{M\in\mathcal{M}_{k,n}} f_{k,n,c}(M),$$

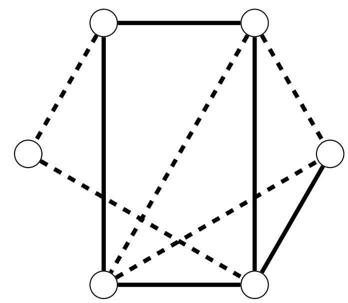

Figure 1: For the edge-coloring of the 15 edges of K_6 with three colors such that the 5 edges indicated by bold lines form one color class, the 5 edges indicated by dashed lines form a second color class, and the missing 5 edges form the third color class, there is no perfect matching containing one edge from each color class.

where $\mathcal{M}_{k,n}$ is the set of perfect matchings of K_{2kn} and $\mathcal{C}_{k,n}$ is the set of balanced k-edge-colorings of $E(K_{2kn})$. This allows us to rephrase Conjecture 1 as

$$g(k,n) \leq \mathcal{O}(k^2).$$

Pardey and Rautenbach [18] showed that $g(k,n) \leq 3k\sqrt{kn\ln(2k)}$, which still includes a dependency on n. For smaller values of k we know the following: Trivially, $g(1,n) = 0$ for all $n \in \mathbb{N}$. In [11] and [5], it was shown that $g(2,n) = 0$ for all $n \in \mathbb{N}$. In [18], Pardey and Rautenbach showed that $g(3,n) \leq 2$ for all $n \in \mathbb{N}$ and $g(3,1) = 2$, the latter resulting from the counterexample given in Figure 1.

2 Results

We will not show the results $g(2,n) = 0$ and $g(3,n) \leq 2$ from [18], but rather provide proofs for weaker results, bounding $g(2,n)$ and $g(3,n)$ by a constant, that is, a bound independent of n. This omits the detailed analysis of each case but rather shows the general strategies used to obtain some results for the conjecture $g(k,n) \leq \mathcal{O}(k^2)$.

Lemma 2. *$g(2,n) \leq 2$.*

For a perfect matching M and matching edges $uv, xy \in M$, we can *swap*, to obtain another perfect matching $M' = M - \{uv, xy\} + \{ux, vy\}$.

Proof of Lemma 2. For a K_{4n} with a balanced red-blue edge coloring, let M be a matching that minimizes $f(M) = ||r-n| + |b-n||$, where r and b denote the number of red and blue edges in M respectively. W.l.o.g. $r \geq b$. Suppose $r \geq n+2$. If there is a blue edge ux between 2 red matching edges $uv, xy \in M$, then swapping the matching edges uv and xy for ux and vy would yield a matching M' where $f(M') < f(M)$, a contradiction. Similarly, if for a red matching edge uv and a blue matching edge xy the edges ux and vy are both blue, swapping would again yield a matching M' where $f(M') < f(M)$. By symmetry, we obtain that there are no blue edges between red matching edges and at least half the edges between red and blue matching edges are red. Counting the total number of red edges in the graph we find at least

$$\binom{2r}{2} + \frac{1}{2}2r(4n-2r) = 2r^2 - r + 4nr - 2r^2 = r(4n-1) \geq (n+2)(4n-1)$$

red edges, which is a contradiction to the edge coloring being balanced, that is, there being exactly $\frac{1}{2}\binom{4n}{2} = n(4n-1)$ red edges in the graph. $\square$

Lemma 3. $g(3,n) \leq 8$.

Proof. Similarly as in Lemma 2, for some balanced red-green-blue edge coloring of K_{6n}, let M be a perfect matching that minimizes $f(M) = ||r-n| + |g-n| + |b-n||$, where again r, g, and b denote the number of red, green, and blue edges in M respectively. W.l.o.g. let $r \geq g \geq b$.
We first observe, that $|g-n| \leq 1$. Otherwise, if $g \geq n+2$, we can argue similarly as in the proof of Lemma 2, by observing that there are no blue edges between matching edges that are either red or green and at most half the edges between blue and red or blue and green matching edges are blue. Counting now the total number of blue edges in the graph, we find at most

$$\binom{2b}{2} + \frac{1}{2}2b(6n-2b) = 2b^2 - b + 6nb - 2b^2 = b(6n-1) \leq (n-4)(6n-1)$$

blue edges, which is strictly less than the actual total number of blue edges, $n(6n-1)$.
If $g \leq n-2$, a similar argument, where we count the total number of red edges, again yields a contradiction,implying that $g \in [n-1, n+1]$
Among all matchings M that minimize $f(M)$ pick one that minimizes $|r-g|$. Suppose that $r \geq n+3$. Then, $b \leq n-2$. If there is a blue or green matching edge between 2 red matching edges, we could swap to obtain a matching that either decreases $f(M)$ or decreases $|r-g|$, while maintaining the value of $f(M)$, both being contradictions. If there were 2 non-incident non-red edges between a red matching edge and a non-red matching edge, then we could again decrease either $f(M)$ or $|r-g|$ by swapping accordingly.
Hence, we can count the total number of red edges in the graph, obtaining at least

$$\binom{2r}{2} + \frac{1}{2}2r(6n-2r) = 2r^2 - r + 6nr - 2r^2 = r(6n-1) \geq (n+3)(6n-1)$$

red edges, which is strictly more than the actual total number of red edges, $n(6n-1)$. This contradiction completes our proof.

$\square$

If we analyse $g(k,n)$ for general k, let M be a matching that minimizes $f(M)$ and let m_i be the number of matching edges of color i for $i \in [k]$. A recurring observation from the previous proofs was that two consecutive color classes never were farther apart than 2, that is, if $m_1 \leq \cdots \leq m_i \leq \cdots \leq m_k$, then $m_{i+1} - m_i \leq 2$ for all $i \in [k-1]$. This motivates the exponent 2 in the conjecture $g(k,n) \leq \mathcal{O}(k^2)$.

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