Neighbour sum-distinguishing edge colorings with local constraints[1]

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Abstract

An edge coloring ω is neighbour sum-distinguishing if the vertex coloring defined by $\sigma_\omega(u) = \sum_{v\in N(u)} \omega(uv)$ is proper. The 1-2-3 Conjecture states that if ω is unrestricted, then ω having 3 colors is enough for any graph. Several variants have been defined, such as ω being proper.

We define a general framework for the problem and its variants, by having local constraints on the coloring. A neighbour sum-distinguishing edge coloring is d-relaxed if every vertex is incident with at most d edges of the same color. We study this variant on trees, complete and subcubic graphs for several values of d. Those results lead us to conjecture that $\lceil \frac{\Delta(G)}{d} \rceil + 2$ colors are enough for any graph. We also study a slightly different framework: each vertex of large enough degree has to be incident with a non-monochromatic set of edges. We prove results for graphs of different maximum degrees and for bipartite graphs.

1 Introduction

A *k-edge coloring* of a graph $G(V,E)$ is a function $\omega : E \to \{1,\dots,k\}$. A natural vertex coloring induced by ω is σ_ω, defined as $\sigma_\omega(u) = \sum_{v\in N(u)}(uv)$ for every vertex $u \in V$: each vertex is colored by the sum of the colors of its incident edges. An edge coloring (vertex coloring) is *proper* if adjacent edges (vertices) receive different colors. We say that the edge coloring ω *distinguishes* vertices $u, v \in V(G)$ if its induced vertex coloring σ_ω is proper; such an edge coloring is called a *neighbour sum-distinguishing edge coloring*.

Observe that every graph with no connected component isomorphic to K_2 admits a neighbour sum-distinguishing edge coloring. Graphs with no such connected component are called *nice graphs*. Furthermore, since the distinguishing is local, studying a non-connected graph amounts to independently study its connected components.

Neighbour sum-distinguishing edge colorings have garnered a lot of attention, particularly since Karoński, Łuczak and Thomason proposed the 1-2-3 Conjecture in 2004 [7]. By denoting by $\chi^e_{\sum}(G)$ the smallest k such that there is a neighbour sum-distinguishing k-edge coloring (not necessarily proper) of G, they stated the following:

Conjecture 1 ([7])**.** *For every nice graph G, $\chi^e_{\sum}(G) \le 3$.*

While the 1-2-3 Conjecture remains open as of today, many results have been found. Notably, the best current bound for $\chi^e_{\sum}$ is 5 [6], and Przybyło [9] recently proved that $\chi^e_{\sum}(G) \le 4$ for every d-regular graph with $d \ge 2$ and that the 1-2-3 Conjecture holds for d-regular graphs with $d \ge 10^8$. The conjecture also holds for graphs large and dense enough [8, 13]. More information on this topic can be found in the survey made by Seamone [10].

Several variants of this problem have also emerged, the most notable one being the variant introduced by Flandrin *et al.* [3] in which the underlying edge coloring must be proper. By denoting

[1] Part of the results presented here have been published in [2].

by $\chi'_{\sum}(G)$ the smallest k such that there is a proper neighbour sum-distinguishing k-edge coloring of G, they proposed the following conjecture:

Conjecture 2 ([3])**.** *For every nice graph G with $G \neq C_5$, $\chi'_{\sum}(G) \leq \Delta(G) + 2$.*

Flandrin *et al.* [3] proved the validity of this conjecture for trees, complete graphs and complete bipartite graphs. The best known bound for $\chi'_{\sum}$ is $\lceil \frac{10\Delta(G)+2}{3} \rceil$ [11], and a bound of 6 has been proved for subcubic graphs [4, 12].

We investigate neighbour sum-distinguishing edge colorings with local constraints, with two main directions. First, we present a general framework that includes both the 1-2-3 Conjecture and its proper variant. Next, we study neighbour sum-distinguishing edge colorings in which vertices of large enough degree are incident with a non-monochromatic set of edges. Several of our proofs use Alon's Combinatorial Nullstellensatz [1].

2 The d-relaxed neighbour sum-distinguishing edge coloring

A neighbour sum-distinguishing k-edge coloring is *d-relaxed* if every vertex is incident with at most d edges of the same color. We denote by $\chi'^{d}_{\sum}(G)$ the smallest k such that G admits a neighbour sum-distinguishing d-relaxed k-edge coloring. Note that, when $d = \Delta(G)$, we get the setting of the 1-2-3 Conjecture, and when $d = 1$, we get the proper variant. Our goal is to give a general conjecture that encapsulates Conjectures 1 and 2, and our study leads us to propose the following:

Conjecture 3. *For every nice graph G with $G \neq C_5$, $\chi'^{d}_{\sum}(G) \leq \left\lceil \frac{\Delta(G)}{d} \right\rceil + 2$.*

We prove that this conjecture holds for several graph classes and values of d.

Theorem 4. *Let T be a nice tree with maximum degree Δ and $1 \leq d \leq \Delta$. We have:*
$$\chi'^{d}_{\sum}(T) = \begin{cases} \frac{\Delta}{d} + 1, & \text{if } \Delta \equiv 0 \pmod{d} \text{ and there are two adjacent vertices of degree } \Delta, \\ \lceil \frac{\Delta}{d} \rceil, & \text{otherwise.} \end{cases}$$

Theorem 5. *Let $n \geq 4$ and $d \in \{\lceil \frac{n-1}{2} \rceil, \dots, n-2\}$ be two integers. We have*
$$\chi'^{d}_{\sum}(K_n) \leq 4.$$

Proof idea. This proof is by induction on n. Starting from K_4, we add a new vertex to K_n the following way: we order the vertices by their weight, and link the new vertex to the half having the smallest (resp. largest) weights with an edge colored either 1 or 3 (resp. either 2 or 4), depending on the parity of n. □

Theorem 6. *Let $n \geq 4$. We have $\chi'^{2}_{\sum}(K_n) = \lceil \frac{n-1}{2} \rceil + 1$ if $n \not\equiv 3 \pmod{4}$ and $\chi'^{2}_{\sum}(K_n) = \lceil \frac{n-1}{2} \rceil + 2$ otherwise.*

Proof idea. This proof has two parts. The first is showing that such a coloring exists, and the second is proving that this number of colors is necessary.

For the first part, the idea is to consider the vertices of K_n as the vertices of an n-gon, and to label them $u_1, \dots, u_{\lfloor \frac{n}{2} \rfloor}$ and $u_{-1}, \dots, u_{-\lceil \frac{n}{2} \rceil}$. The edges are then colored color by color, in order to obtain a 2-relaxed coloring that distinguishes all vertices except the pairs (u_i, u_{-i}). A recoloring step is then required to make the coloring neighbour-sum distinguishing. At each step of this construction, there are small differences depending on the value of n mod 4.

The second part is proved by contradiction: a d-relaxed edge coloring using fewer colors cannot be neighbour sum-distinguishing since it has a limited set of possible weights for the vertices. □

3 Stronger local constraints for neighbour sum-distinguishing edge colorings

We now prove results on neighbour sum distinguishing edge colorings with stronger local constraints for graphs of a given maximal degree. Namely, we want to ensure that vertices of large enough degree are not incident with a monochromatic set of edges. For subcubic graphs, we prove the following result result, which implies that if G is a subcubic graph, then $\chi'^{2}_{\sum}(G) \leq 4$:

Theorem 7. *Let G be a nice subcubic graph such that $G \neq C_5$. There is a neighbour sum distinguishing 2-relaxed 4-edge coloring of G such that all the vertices of degree 2 have their two incident edges of different colors.*

Proof idea. The proof is by induction on the number of edges of G. First, we identify an "interesting" vertex u, use the induction hypothesis to get such a coloring of $G - u$, and then we extend the coloring to G. This extension is made under different constraints, which will restrict the colors that can be used on the edges incident with u, and which induce a polynomial. Verifying those constraints becomes equivalent to finding values for the colors of the edges incident with u that allow the polynomial to be non-zero. We then use the Combinatorial Nullstellensatz to prove that the constraints can be verified, and thus that we can extend the coloring.

The choice of the vertex u depends on several factors, such as the girth or the minimum degree of G. Hence, there are several cases, each leading to several subcases, although most of them work on the same idea. Some subcases require a recoloring before the extension of the coloring to G. □

We use similar proof methods to extend this result to graphs of maximum degree 4 and 5:

Theorem 8. *Every nice graph G with $\Delta(G) \leq 4$ (resp. $\Delta(G) \leq 5$) admits a neighbour sum distinguishing 6-edge coloring (resp. 7-edge coloring) such that all the vertices of degree at least 2 are incident with at least two edges of different colors.*

Note that this implies that if G verifies $\Delta(G) \leq 4$ (resp. $\Delta(G) \leq 5$), then $\chi'^{\Delta(G)-1}_{\sum}(G) \leq 6$ (resp. $\chi'^{\Delta(G)-1}_{\sum}(G) \leq 7$). We then prove the following result on general graphs, which, together with Theorems 7 and 8, implies that every nice graph G verifies $\chi'^{\Delta(G)-1}_{\sum}(G) \leq 7$:

Theorem 9. *Every nice graph G admits a neighbour sum distinguishing 7-edge coloring such that all the vertices of degree at least 6 are incident with at least two edges of different colors.*

Proof idea. The proof is based on an algorithm designed by Kalkowski [5], which has been reused several times in order to obtain several results on the 1-2-3 Conjecture and some of its variants. First, we define a vertex ordering with specific properties, and then we construct the coloring. At first, all the edges receive the color 4, and then, each vertex is processed one after another. At each step, the only edges that can be modified are those between the considered vertex and some of its neighbours (namely, its predescessors and first sucessor in the vertex ordering). The weights are modified under strict constraints that guarantee both that the resulting coloring is neighbour sum distinguishing and that every vertex of degree at least 6 is incident with a non-monochromatic set of edges. Several cases and subcases are considered, depending on whether the considered vertex has a successor or not, and other conditions. □

Finally, we similarly show that every nice bipartite graph G verifies $\chi'^{\Delta(G)-1}_{\sum}(G) \leq 6$:

Theorem 10. *Every nice bipartite graph admits a neighbour sum distinguishing* 6*-edge coloring such that every vertex of degree at least two is incident with at least two edges of different colors.*

Proof idea. First, recall that the 1-2-3 Conjecture holds for nice bipartite graphs, even if we consider the weights of the vertices modulo 3 [7]. We then prove that every bipartite graphs admits a 2-edge coloring such that every vertex of degree at least 2 is incident with two edges of different colors. We can then cross such a 2-coloring with a neighbour sum distinguishing 3-edge coloring by relabeling edges colored 1 (resp. 2 and 3) with either 1 and 4 (resp. either 2 and 5 and either 3 or 6), keeping the vertices distinguished modulo 3 and making sure that no vertex of degree at least two is incident with a monochromatic set of edges. $\square$

Note that, while they prove results for stronger local constraints than the simple d-relaxation of neighbour sum distinguishing edge colorings, Theorems 8, 9 and 10 give bounds for $\chi'^{d}_{\sum}(G)$ higher than the one proposed in Conjecture 3.

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