Algorithmic Results on Grundy Total Domination and Grundy Double Domination

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Abstract

A sequence S of vertices is called a *total dominating sequence* of a graph G, if every vertex in the sequence dominates a vertex which was not dominated by the previous vertices in S. Given a graph G, the Grundy Total Domination (GTD) problem is to find a total dominating sequence of maximum length. We refine some previously known results on bipartite graphs by showing that the GTD problem is NP-complete in perfect elimination bipartite graphs and give a linear-time algorithm for this problem in chain graphs. A sequence S is a *double dominating sequence* if every vertex $v_i \in S$ dominates at least one vertex that has not been dominated by at least two vertices preceding v_i. The Grundy Double Domination (GDD) problem is defined in a similar way. We prove that it is NP-complete for split graphs and present an algorithm that solves this problem for threshold graphs efficiently.

1 Introduction

The domination game, introduced in [5] and surveyed in a recent book [4], motivates to study dominating sequences. Brešar, Gologranc, Milanič, Rall and Rizzi in [2] introduced dominating sequences and Grundy domination number of a graph. An additional motivation comes from online expansion of a network, which needs to be backed up by online expansion of a dominating set, where Grundy domination presents the worst-case scenario and gives the largest possible size of the resulting dominating set at the end of the process.

Brešar, Henning and Rall introduced the total version of Grundy domination [3]. Given a graph G with no isolated vertices, let $S = (v_1, \ldots, v_k)$ be a longest possible sequence of vertices such that each v_i is adjacent to a vertex x to which none of the vertices in $\{v_1, \ldots, v_{i-1}\}$ is adjacent. Such a sequence is called a *Grundy total dominating sequence* and its length k is the Grundy total domination number (GTDN) of G. The GTD problem is NP-complete in bipartite graphs and split graphs [3, 6]. On the positive side, determining the Grundy total domination number is linear in forests and cographs and polynomial in bipartite distance-hereditary graphs and in the class of P_4-tidy graphs [6].

In a recent paper [7], Haynes and Hedetniemi proposed several additional variations of vertex sequences that could be interesting, and gave a few initial results on some of these concepts. In particular, they presented Grundy double domination. Size of the Grundy double dominating sequence presents the worst-case scenario when double dominating set is built on-line and the structure of the graph is not known in advance. We investigated the computational complexity of this variation in some graph classes.

The paper is organized as follows. In the next section, we establish basic notations used in the paper. In Section 3, we give the results stating that the GTD problem is an NP-complete problem even when restricted to perfect elimination bipartite graphs. In contrast, we present a linear-time algorithm for determining the Grundy total domination number of a chain graph, a subclass of perfect elimination bipartite graphs. Section 4 deals with double dominating sequences. We prove that the GDD problem is NP-complete in split graphs and for the class of threshold graphs (which is a subclass of split graphs) we present a linear-time algorithm.

2 Notation and preliminaries

Given a graph G, the *neighborhood* of a vertex x is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, while the *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. Vertices u and v in a graph G are *closed twins* (respectively, *open twins*) if $N_G[u] = N_G[v]$ (respectively, $N_G(u) = N_G(v)$). We may omit the indices in the above definitions if the graph G is understood from the context.

Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of G. If $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m)$, $n, m \geq 0$, are two sequences, then the *concatenation* of S_1 and S_2 is the sequence $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$. A sequence $S = (v_1, \ldots, v_k)$ is an *open neighborhood sequence* if

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{1}$$

holds for every $i \in \{2, \ldots, k\}$. If, in addition, $\widehat{S}$ is a total dominating set of G, then we call S a *total dominating sequence* of G. The maximum length of a total dominating sequence in G is the *Grundy total domination number* of G and is denoted by $\gamma_{gr}^t(G)$. The corresponding sequence is a *Grundy total dominating sequence* of G. These concepts were introduced and studied by Brešar, Henning and Rall in [3].

Sequence S is said to be a *double neighborhood sequence* if every vertex $v_i \in S$ dominates at least one vertex that has not been dominated by at least two vertices preceding v_i in S. If $\widehat{S}$ is a double dominating set of G, then we call S a *double dominating sequence* of G. The maximum length of a double dominating sequence is the *Grundy double domination number* $\gamma_{\text{gr}}^{\times 2}(G)$ of G and the corresponding sequence is called *Grundy double dominating sequence* of G. (We note that the original definition of Haynes and Hedetniemi is slightly different, but it results in the same value of the Grundy double domination number in any graph with no isolated vertices.)

3 Total Dominating Sequences

3.1 NP-completeness in perfect elimination bipartite graphs

Theorem 1. Grundy Total Domination Problem *is NP-complete even when restricted to perfect elimination bipartite graphs.*

Let $G = (X, Y, E)$ be a bipartite graph. An edge $e = xy$ is said to be a *bisimplicial edge* if $G[N(x) \cup N(y)]$ is a complete bipartite subgraph of G. Let $\sigma = (x_1y_1, x_2y_2, \ldots, x_ky_k)$ be an ordering of pairwise non-adjacent edges of G. Denote $S_j = \{x_1, x_2, \ldots, x_j\} \cup \{y_1, y_2, \ldots, y_j\}$ and let $S_0 = \emptyset$. The ordering $\sigma = (x_1y_1, x_2y_2, \ldots, x_ky_k)$ is a *perfect edge elimination ordering* of G if $x_{j+1}y_{j+1}$ is a bisimplicial edge in $G[(X \cup Y) \setminus S_j]$ for every $j \in \{0, 1, \ldots, k-1\}$ and $G[(X \cup Y) \setminus S_k]$ has no edge. A graph for which there exists a perfect edge elimination ordering is a *perfect elimination bipartite graph.*

Grundy total domination number problem is known to be NP-complete for bipartite graphs [3]. To prove the NP-completeness of GTD problem for perfect elimination bipartite graphs, we give a polynomial time reduction from GTD problem for bipartite graphs to GTD problem for perfect elimination bipartite graphs.

3.2 Algorithm for chain graphs

A graph G is a *chain graph* if and only if it is bipartite and for each color class the neighborhoods of the nodes in that color class can be ordered linearly with respect to inclusion. In this subsection, $G = (X, Y, E)$ denotes a chain graph.

We say that vertices u and v are in relation R if they are open twins. Clearly, R is an equivalence relation on $V(G)$, so it provides a partition of the vertex set of G. Let $P_X = \{X_1, X_2, \ldots, X_{k_1}\}$ and $P_Y = \{Y_1, Y_2, \ldots, Y_{k_2}\}$ denote the parts of the corresponding partition, which lie in X, resp. Y. Note that $k_1 = k_2 (= k)$ holds for chain graphs. We prove the following result.

Theorem 2. *Let $G = (X \cup Y, E)$ be a chain graph with no isolated vertices, and let P_X and P_Y be the parts obtained for X and Y, respectively, from the relation R. Then $\gamma_{gr}^t(G) = 2k$.*

Above theorem suggests a simple linear-time algorithm to compute a Grundy total dominating sequence of a chain graph. We must choose a vertex from each of the parts obtained from the relation R.

4 Double Dominating Sequences

4.1 NP-completeness in split graphs

In this subsection, we prove the following result.

Theorem 3. Grundy Double Domination Problem *is NP-complete even when restricted to split graphs.*

A graph G is a *split graph* if $V(G)$ can be partitioned into into two sets I and K, where I is an independent set and K is a clique. We may assume that a partition is done in such a way that $\alpha(G) = |I|$, which implies that every vertex in K has a neighbor in I. The partition $V(G) = [I, K]$ is a *split partition* of $V(G)$.

A connection between dominating sequences with covering sequences in hypergraphs was established in the seminal paper on Grundy domination [2]. Recall that given a hypergraph $H = (X, \mathcal{E})$ with no isolated vertices, an *edge cover* of H is a set of hyperedges from $\mathcal{E}$ that cover all vertices of X. Consider a sequence $\mathcal{C} = (C_1, \ldots, C_r)$, where $C_i \in \mathcal{E}$. If in each step i, $i \in [r]$, C_i covers a vertex not covered by C_j, for all $j < i$, $\mathcal{C}$ is called a *legal (hyperedge) sequence* of H. If $\mathcal{C} = (C_1, \ldots, C_r)$ is a legal sequence and the set $\widehat{\mathcal{C}} = \{C_1, \ldots, C_r\}$ is an edge cover of H, then $\mathcal{C}$ is an *edge covering sequence*. The maximum length r of an edge covering sequence of H is the *Grundy covering number* $\rho_{gr}(H)$ of H. Given a hypergraph, the Edge Covering Problem in hypergraphs is to find an edge covering sequence of maximum length.

Edge covering problem for hypergraphs is shown to be NP-complete in [2]. To prove the NP-completeness of GDD problem for split graphs, we give a polynomial time reduction from edge covering problem for hypergraphs to GDD problem for split graphs.

4.2 Efficient algorithm for threshold graphs

A *threshold graph* is a graph that can be constructed from the one-vertex graph by repeated applications of the following two operations:
1) Addition of a single isolated vertex to the graph. 2) Addition of a single dominating vertex to the graph, that is, a single vertex that is adjacent to all other vertices.

The set of vertices of G that were added to G by operation 1 is denoted by I, while D denotes the set of vertices of G added by operation 2.

Algorithm 1: Grundy double dominating sequence of a threshold graph

Input: A connected threshold graph $G = (I, D, E)$ along with an ordering $(y_1, \ldots, y_\ell)$ of the vertices of D and an ordering $(x_1, \ldots, x_k)$ of the vertices of I.

Output: A Grundy double dominating sequence S of G.

$S = (x_1, x_2, \ldots, x_k, y_1)$;
$i = 1$;
while $i \leq \ell$ **do**
    if $N(y_i) \cap I = N(y_{i+1}) \cap I$ **then**
        $i = i + 1$;
    else
        $S = S \oplus y_i$;
        $i = i + 1$;
Output S.

Suppose that the ordering $(y_1, \ldots, y_\ell)$ of vertices of D in which they appear in G according to the definition of threshold graphs is given. This implies that $N(y_1) \cap I \subseteq N(y_2) \cap I \subseteq \cdots \subseteq N(y_\ell) \cap I$ holds true. Due to this property, it can be observed that Algorithm 1 correctly outputs a Grundy double dominating sequence of G. So, we state the following result.

Theorem 4. *Algorithm 1 returns a Grundy double dominating sequence of a threshold graph G.*

Several natural problems remain open. For instance, the computational complexity of GTD problem in chordal bipartite graphs looks intriguing. Also, what is the computational complexity of determining the Grundy double domination number in strongly chordal split graphs (respectively cographs)?

References

[1] C. Berge, *Hypergraphs*, North-Holland, 1989, Amsterdam.

[2] B. Brešar, T. Gologranc, M. Milanič, D. F. Rall, R. Rizzi, Dominating sequences in graphs. *Discrete Math.* **336** (2014), 22–36.

[3] B. Brešar, M.A. Henning, D.F. Rall, Total dominating sequences in graphs, Discrete Math. 339 (2016) 1665–1676.

[4] B. Brešar, M. A. Henning, S. Klavžar, D. F. Rall, *Domination games played on graphs*, SpringerBriefs in Mathematics, Springer, Cham, 2021.

[5] B. Brešar, S. Klavžar, and D. F. Rall, Domination game and an imagination strategy. *SIAM J. Discrete Math.* **24** (2010), 979–991.

[6] B. Brešar, T. Kos, G. Nasini, P. Torres, Total dominating sequences in trees, split graphs, and under modular decomposition, Discrete Optim. 28 (2018) 16–30.

[7] T.W. Haynes, S.T. Hedetniemi, Vertex sequences in graphs, Discrete Math. Lett. 6 (2021) 19–31.