The Ramsey number of square of paths

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Abstract

The problem of finding explicit values of the Ramsey number $R(H,H)$ has been studied extensively, but the explicit value of $R(H,H)$ is known only for a few graphs H. Among the most relevant results in the area, Gerencsér and Gyárfás proved in 1967 that $R(P_{2n},P_{2n}) = 3n-1$, where P_{2n} is the path on $2n$ vertices.

We denote by P_{3n}^2 the square of the path on $3n$ vertices, which is the graph over $3n$ vertices obtained from P_{3n} by adding edges between vertices of distance 2. We prove that $R(P_{3n}^2, P_{3n}^2) = 9n-3$ for n large enough.

1 Introduction

1.1 General introduction

The study of unavoidable regularities has a long history in mathematics and has given interesting results in multiple areas. One of the first (and better known) examples is a lemma used to prove the Bolzano-Weierstrass theorem, and it states that every sequence in $\mathbb{R}$ admits a monotone subsequence. This lemma is a good example of a wide array of results that try to study what kind of regular substructures cannot be avoided in large structures.

The first results of this kind in graph theory are due to Ramsey. In his seminal paper [5], Ramsey showed that for any graph H it is possible to find a monochromatic copy of H in any two-colouring of K_n, provided n is large enough. This result started an active and influential area of graph theory which is now called Ramsey Theory.

Definition 1. *Let H be a graph. We denote by $R(H,H)$ the Ramsey number of H, which is the smallest $n \in \mathbb{N}$ such that any $\{red, blue\}$-edge colouring of K_n admits a monochromatic copy of H.*

In his 1930 paper [5], Ramsey showed that the value $R(H,H)$ is well defined for any graph H. Determining or approximating the value of $R(H,H)$ for any given H has been the driving question of Ramsey theory since then.

One of the first families of graph for which the Ramsey number was determined is the family of paths. By path P_n we mean the graph over the vertex set $\{1,\dots,n\}$ and with edge set $\{12, 23, \dots, (n-1)n\}$. The result, due to Gerencsér and Gyárfás [4], reads as follows.

Theorem 2 (Gerencsér and Gyárfás, [4])**.** *Let $n \geq 2$ be a natural number. Then $R(P_{2n},P_{2n}) = 3n-1$.*

More recently, Chvátal, Rödl, Szemerédi and Trotter proved that if a graph H has bounded degree, then its Ramsey number is linear in the number of vertices of H.

Theorem 3 (Chvátal, Rödl, Szemerédi and Trotter, [3])**.** *Let H be a graph over n vertices and with maximum degree Δ. The Ramsey number $R(H,H)$ is bounded above by $c_\Delta \cdot n$ for some constant c_Δ depending only on Δ.*

However, this result is still very far from giving us more precise estimates for $R(H,H)$.

1.2 Power of paths

For $k, n \in \mathbb{N}^+$, we denote by P_n^k the k-th power of the path P_n, which is the graph obtained from P_n by adding an edge between any two vertices at distance at most k in the path P_n.

There are at least two reasons why studying the Ramsey number of power of paths is an important question in Ramsey theory. Firstly, it is a natural next step in the strengthening of the result of Gerencsér and Gyárfás. Secondly, powers of paths are of particular importance in the study of Ramsey problems because they are related to a measure of complexity of graphs (the bandwidth) that has been proved to be relevant in the area.

More in detail, we say that a graph H over n vertices has bandwidth k if k is the smallest integer such that H is a subgraph of P_n^k. A result by Allen, Brightwell and Skokan [2] shows a better upper bound for the Ramsey number of graphs with bounded maximum degree if in addition we assume that the graph has sublinear bandwidth.

Theorem 4 (Allen, Brightwell and Skokan, [2])**.** *For any Δ positive integer, there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds for any $n \geq n_0$. Let H be a graph over n vertices with maximum degree at most Δ and with bandwidth at most ϵn, then we have that $R(H, H) \leq 2(\chi(H) + 2)n$.*

The proof of this theorem relies on good estimates of the value of $R(P_{(k+1)n}^k, P_{(k+1)n}^k)$ and an improvement in the approximation of the Ramsey number for the square of paths is likely to lead to better upper bounds for the Ramsey number of graphs with sublinear bandwidth. In particular, Allen, Brightwell and Skokan conjectured the following:

Conjecture 5 (Allen, Brightwell and Skokan, [2])**.** *For any Δ positive integer, there exist $n_0 \in \mathbb{N}$ and $c, \epsilon > 0$ such that the following holds for any $n \geq n_0$. Let H be a graph over n vertices with maximum degree at most Δ and with bandwidth at most ϵn, then we have that $R(H, H) \leq (\chi(H) + c)n$.*

For the nature of the proof of Theorem 4, it seems that determining the value of $R(P_{(k+1)n}^k, P_{(k+1)n}^k)$ would be of a big step forward in proving Conjecture 5.

The aim of this paper is to prove the following result.

Theorem 6. *There exists an $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have*

$$R(P_{3n}^2, P_{3n}^2) = 9n - 3.$$

Let us point out that the n_0 of this theorem is given us by the Regularity Lemma, and we did no effort to try to minimise n_0. Even if this result answers a natural question in the Ramsey theory setting and it might be of help in improving the result of Theorem 4, additional study will be required to extend Theorem 6 to higher powers of k and to determine the value of $R(P_{(k+1)n}^k, P_{(k+1)n}^k)$ for other values of k.

2 Lower bound

In order to prove Theorem 6, we first show that there exists a {red, blue}-edge colouring of K_{9n-4} without monochromatic copied of P^2_{3n}. The construction follows the recipe drawn in Figure 1.

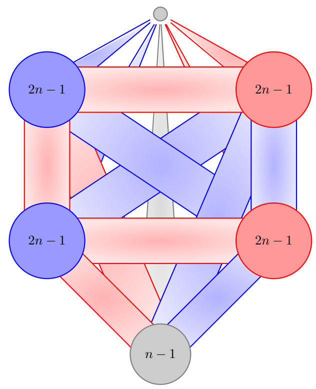

Figure 1: Our extremal colouring

We partition our $9n-4$ vertices in six sets. The sets B_1, B_2, R_1, R_2 of size $2n-1$, the set Z of size $n-1$ and an additional single vertex r. We colour all the edges in $B_1, B_2, (B_1 \cup B_2, \{r\}), (R_1, R_2, Z)$ by blue and all the edges in $R_1, R_2, (R_1 \cup R_2, \{r\}), (B_1, B_2, Z)$ by red. We then arbitrarily colour the rest of the edges.

It is evident that in our construction there is no monochromatic P^2_{3n}

3 Proof strategy for the upper bound

At the base of our strategy for the upper bound of $R(P^2_{3n}, P^2_{3n})$ there are the regularity method of Szemerédi [6] and an embedding lemma due to Allen, Böttcher and Hladký [1].

We show that any two-edge-colouring of a large clique not containing a monochromatic copy of P^2_{3n} must have a very specific structure. Which is, any such colouring must be similar in structure to the colouring in Figure 1. Some of the details follow.

Theorem 7 (Szemerédi, [6])**.** *For every $\epsilon > 0$ there exist natural numbers M and n_0 such that for any $n \geq n_0$ and any two-colouring G of K_n we can partition the vertex set of G in at most M sets $V_0, \dots, V_m$ such that $m \geq \frac{1}{\epsilon}$ and $|V_0| \leq \epsilon n$ and $|V_1| = \ldots = |V_m|$. Moreover, all but at most $\epsilon\binom{m}{2}$ of the pairs (V_i, V_j) are ϵ-regular in both colours.*

Here, by (V_i, V_j) being ϵ-regular we mean that whenever $A \subseteq V_i$ and $B \subseteq V_j$ are such that $|A| \geq \epsilon |V_i|$ and $|B| \geq \epsilon |V_j|$, the density of edges (both in blue and in red) between A and B is the same (up to an error ϵ) of the density of the same colour between V_i and V_j.

Therefore, given n sufficiently large and a two edge colouring of K_n, we can partition the vertex set of K_n in a bounded number of subsets such that between most pairs of subsets we see some strong regularity property. In particular, we can build a support graph R, called an ϵ-reduced graph for G, over the parts $V_1, \dots, V_m$ such that we have the edge V_iV_j if and only if (V_i, V_j) is ϵ-regular in both colours. We can colour each edge V_iV_j of the majority colour in the set of edges $E(V_i, V_j)$. Notice that R is an almost complete two-edge-coloured graph.

The use of Szemerédi regularity lemma has been used to embed substructures in large graphs, and it is of fundamental importance here because it allows us to apply a result introduced by Allen, Böttcher and Hladký [1].

We first need a definition.

Definition 8. *Let R be a $\{red, blue\}$-edge-coloured graph. Let T and T' be monochromatic (wlog blue) triangles. We say that T and T' are triangle-connected if there exists a sequence of blue triangles $T = T_0, \ldots, T_\ell = T'$ such that for every $i = 0, \ldots, \ell - 1$ we have that T_i and T_{i+1} share an edge.*

A triangle factor is a set of vertex disjoint triangles. It is natural to define as monochromatic triangle-connected triangle factor a set of pairwise vertex disjoint monochromatic triangles of the same colour that are pairwise triangle connected.

The following embedding lemma allows us to reduce the problem of finding a monochromatic copy of P_{3n}^2 in a two-colouring of K_n to the problem of finding a monochromatic triangle-connected triangle factor in the reduced graph R.

Theorem 9 (Allen, Böttcher and Hladký, [1])**.** *For all positive $\delta, \lambda < 1$ there exists $\epsilon > 0$ and $M, n_0 \in \mathbb{N}$ such that whenever $n > n_0$ the following holds. Let G be a two-colouring of K_n, and let R be an ϵ-reduced graph of G with $|R| = m \leq M$ vertices. If R contains a monochromatic triangle-connected triangle factor over $3(1 + \delta)\lambda m$ vertices, we can find a monochromatic copy of $P_{3\lambda n}^2$ in G.*

We can show that whenever R is an almost complete two-coloured graph over m vertices, either R contains a triangle-connected triangle factor of the right size or the colouring of R is close to the lower bound construction. A careful analysis of the possible extremal structure finishes the proof of the upper bound.

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