Graphs of large treewidth don't have the edge-Erdős-Pósa property

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Abstract

We show that for every subcubic graph H of treewidth at least 1000, the set of graphs that contains H as a minor does not have the edge-Erdős-Pósa property.

1 Introduction

Menger's theorem provides a strong duality between packing and covering for paths: In every graph G, there are either k disjoint paths between predefined sets $A, B \subseteq V(G)$, or there is a set $X \subseteq V(G)$ of size at most k such that $G - X$ contains no A–B path. Relaxed versions of this result exist for many sets of graphs, and we call this duality the *Erdős-Pósa property.* In this talk, we focus on the edge variant: A graph H has the *edge-Erdős-Pósa property* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph G and every integer k, there are k edge-disjoint graphs in G each containing a minor isomorphic to H or there is an edge set $X \subseteq E(G)$ of size at most $f(k)$ meeting all subgraphs in G containing a minor isomorphic to H.

The vertex-Erdős-Pósa property seems to be well understood: Robertson and Seymour [2] proved that a graph H has the vertex-Erdős-Pósa property if and only if H is planar. While the edge-Erdős-Pósa property is still false for all non-planar graphs (see for example [3]), the situation is much more mysterious for planar graphs. Some simple planar graphs such as long cycles, K_4 and Θ_t still have the edge-Erdős-Pósa property, while some others, for example subcubic trees of large pathwidth, do not. For most planar graphs, it is unknown whether the edge-Erdős-Pósa property holds or not.

We partially fill this gap by proving that every subcubic graph of large treewidth does not have the edge-Erdős-Pósa property. Note that while it was known that large walls do not have the edge-Erdős-Pósa property (claimed without proof in [1]), this does not imply our main result as, unlike the vertex-Erdős-Pósa property, the edge variant is not known to be closed under taking minors.

Theorem 1. *Subcubic graphs of treewidth at least* 1000 *do not have the edge-Erdős-Pósa property.*

To prove Theorem 1, we only use treewidth to deduce that the subcubic graph contains a large wall, for which we use the linear bound provided by Grigoriev [4]. So in fact, we show the following lemma:

Lemma 1. *Subcubic graphs that contain a wall of size* 100×100 *do not have the edge-Erdős-Pósa property.*

There is room for improvement in the theorem. Requiring the graph H to be subcubic simplifies the argument considerably but does not seem to be essential. We expect the theorem to hold for all graphs of large treewidth, whatever the maximum degree. The treewidth/wall size is certainly not best possible. While our proof uses a wall of size 100×100, we believe that with a more careful but somewhat tedious analysis the wall size could be dropped to 30×30. Considering that walls of size 6×4 (but not graphs containing such walls) do not have the edge-Erdős-Pósa property (unpublished result), even 30×30 is unlikely to be close to best possible.

2 Construction

There is only one known tool to prove that a graph H that satisfies the vertex-Erdős-Pósa property does not have the edge-Erdős-Pósa property: The *Heinlein Wall*, named after its discoverer [1], shown at size 5 in Figure 1. We skip a formal definition.

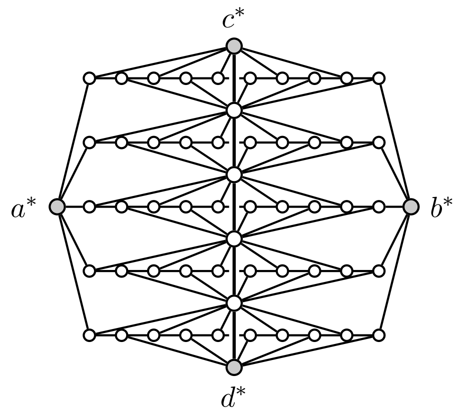

Figure 1: A Heinlein Wall of size 5.

For vertices a^*, b^*, c^*, d^*, an *(a^*–b^*, c^*–d^*) linkage* is the vertex-disjoint union of an a^*–b^* path with a c^*–d^* path. A Heinlein wall has three essential characteristics:

- Its threewidth is small. Therefore, it cannot contain any graph H of large treewidth as a minor.
- It does not containy any two edge-disjoint (a^*–b^*, c^*–d^*) linkages.
- At sufficient size, no small edge set covers all (a^*–b^*, c^*–d^*) linkages.

How does a Heinlein Wall help in proving the main result? Given a size r for a hypothetical edge set X that meets all H-subdivisions, we construct, based on a large Heinlein Wall, a graph G with two key features: No edge set of size at most r will meet all subdivisions of H; and every subdivision of H will induce an (a^*–b^*, c^*–d^*) linkage in the Heinlein Wall. As there cannot be two such linkages without a common edge, we have then shown that H cannot have the edge-Erdős-Pósa property.

Since H has large treewidth, it contains a wall M of size at least 100×100. In M, we pick two (subdivided) edges e_1 and e_2 that are far away from each other. We denote the endvertices of e_1 by a, b and the endvertices of e_2 by c, d.

Given a positive integer r, we define G as follows (see Figure 2):

- start with a copy of $H - \{e_1, e_2\}$, where we denote the copy of a vertex v of H by v^*;
- replace every edge u^*v^* in the copy of $H - \{e_1, e_2\}$ by $2r$ internally disjoint u^*–v^* paths of length 2; and
- add a Heinlein wall W of size $2r$, where the terminals a^*, b^* are identified with the endvertices of e_1, and where the terminals c^*, d^* are identified with the endvertices of e_2.

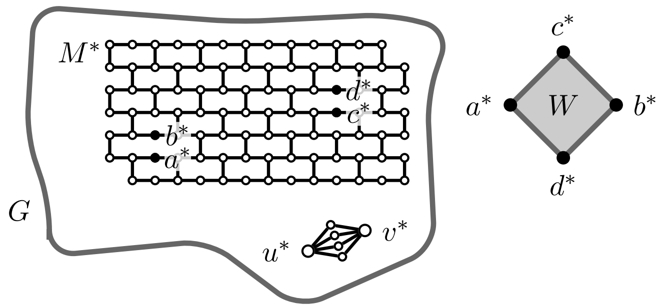

Figure 2: The graph G for which we prove that it contains no two edge-disjoint embeddings of H.

We extend the mapping $V(H) \to V(G)$ defined by $v \mapsto v^*$ to sets of vertices: For a vertex set $S \subseteq V(H)$, we set $S^* = \{v^* : v \in S\}$.

We can easily embed H in G by embedding every v on v^* and by picking one of the $2r$ paths in G for the embedding of every edge uv in $H - \{e_1, e_2\}$. For e_1 and e_2, we can use an (a^*–b^*, c^*–d^*) linkage in W. As there are $2r$ internally disjoint paths and sufficient edges in W to form many different linkages, this still works if we delete r edges in G, which implies that there is no covering of size r for H in G. The hard part is to prove that every embedding of H in G must use an (a^*–b^*, c^*–d^*) linkage.

3 Results

Let Φ be an arbitrary embedding of H in G. In order to get some control on what is mapped where by Φ, we concentrate on a set of "central" vertices which are well connected to a large wall. Let

$$B = \{v \in H : v \text{ has a 3-fan to branch vertices of degree 3 of a 10-wall } \tilde{M}\}.$$

A 3-fan from a vertex v to a set S is the union of 3 paths from v to S which are disjoint except for their first vertex v. A 10-wall is a wall of size at least 10×10. Note that B contains all branch vertices of M: Indeed, every branch vertex of degree 3 in M is connected to its three adjacent branch vertices in M. Those paths form the desired 3-fan. On top of that, B has the following properties:

Lemma 2. *$\Phi(B) \subset B^* \cup V(W^0)$.*

Lemma 3. *$|B^* \setminus \Phi(B)| \leq 40$.*

While we cannot force $\Phi(v) = v^*$ for any vertex, Lemma 2 shows that every $v \in B$ is at least mapped on some $w^* \in B^*$ with similar properties, unless $\Phi(v)$ vanishes into the Heinlein wall. Lemma 3 in turn yields that the latter possibility is very limited, that is, all but a few vertices of B are mapped on B^*. This includes the branch vertices of M^*. Combining both Lemmas, we conclude that all but a few branch vertices of M^* are in $\Phi(B)$. For the rest of our proof, we will give a short outline.

Our next step is to "repair" the paths in between the branch vertices of $M^* \cap \Phi(B)$, that is, we prove that $\Phi(H)$ contains a large wall M' that uses the same branch vertices as M^*, with only a few (at most 40) rows and columns missing.

Finally, we observe that $\Phi(H)$ must use some parts of the Heinlein Wall W as there is too few space outside of it. If $\Phi(H) \cap W$ contains an (a^*–b^*, c^*–d^*) linkage, we are done. Otherwise, we show that $\Phi(H) \cap W$ must contain a path that connects some other terminals, e.g. an a^*–c^* path P. However, a^* and c^* are both contained in (or at least connected to, which again requires proof) the repaired wall $M' \subset \Phi(H)$, and they are very distant to each other in M'. When adding the a^*–c^* path P to an arbitrary planar drawing of M', P must cross several edges of M', which is a contradiction to the planarity of $\Phi(H)$. We conclude that $\Phi(H) \cap W$ contains an (a^*–b^*, c^*–d^*) linkage, which proves Theorem 1.

References

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