Avoiding large squares in trees and planar graphs

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Abstract

The Thue number $\pi(G)$ of a graph G is the minimum number of colors needed to color G without creating a square on a path of G. For a graph class $\mathcal{C}$, $\pi(\mathcal{C})$ is the supremum of $\pi(G)$ over the graphs $G \in \mathcal{C}$. The Thue number has been investigated for famous minor-closed classes: $\pi(tree) = 4$, $7 \leqslant \pi(outerplanar) \leqslant 12$, and $11 \leqslant \pi(planar) \leqslant 768$. Following a suggestion of Grytczuk, we consider the generalized parameters $\pi_k(\mathcal{C})$ such that only squares of period at least k must be avoided. Thus, $\pi(\mathcal{C}) = \pi_1(\mathcal{C})$. We show that $\pi_5(tree) = 2$, $\pi_2(tree) = 3$, and $\pi_k(planar) \geqslant 11$ for every fixed k.

1 Introduction

A coloring of a graph G is non-repetitive if the sequence induced by the colors of any path of G is not a square. The Thue number $\pi(G)$ of G is the minimum number of colors needed in a non-repetitive coloring G. Recall that the period of a square uu is $|u|$.

For a graph class $\mathcal{C}$, $\pi(\mathcal{C})$ is the supremum of $\pi(G)$ over the graphs $G \in \mathcal{C}$. Let tw_k denote the class of graphs with treewidth at most k.

Theorem 1.

- $\pi(path) = 3$ *[12]*
- $\pi(tree) = 4$ *[7]*
- $7 \leqslant \pi(outerplanar) \leqslant 12$ *[2]*
- $\pi(tw_k) \leqslant 4^k$ *[9]*
- $11 \leqslant \pi(planar \cap tw_3) \leqslant \pi(planar) \leqslant 768$ *[5, 4]*

Other types of coloring, namely proper coloring and star coloring [1], are defined by forbidding only squares of period 1 and squares of period 1 and 2, respectively. The corresponding chromatic numbers χ and χ_s thus satisfy $\chi(\mathcal{C}) \leqslant \chi_s(\mathcal{C}) \leqslant \pi(\mathcal{C})$ for every graph class $\mathcal{C}$.

This paper investigates another variation of non-repetitive coloring, suggested by Grytczuk, such that only large enough squares are forbidden. The parameter $\pi_k(G)$ is the minimum number of colors needed to color G such that no squares of period at least k appears in G. We similarly define $\pi_k(\mathcal{C})$ for a graph class $\mathcal{C}$, so that $\pi(\mathcal{C}) = \pi_1(\mathcal{C}) \geqslant \pi_2(\mathcal{C}) \geqslant \pi_3(\mathcal{C}) \geqslant \cdots$

The case of words, i.e. infinite paths, is already settled.

Theorem 2.

- $\pi_k(path) = 3$ *for* $1 \leqslant k \leqslant 2$ *[12]*
- $\pi_k(path) = 2$ *for* $k \geqslant 3$ *[6]*

We settle the case of trees in Section 2.

Theorem 3.

- $\pi_1(tree) = 4$
- $\pi_k(tree) = 3$ *for* $2 \leqslant k \leqslant 4$
- $\pi_k(tree) = 2$ *for* $k \geqslant 5$

We also obtain a lower bound for planar graphs in Section 3.

Theorem 4. *for every fixed* k*,* $\pi_k(planar \cap tw_3) \geqslant 11$*.*

This disproves a conjecture of Grytczuk [8] that $\pi_k(planar) = 4$ for some k.

For any given graph G, $\pi_k(G)$ is a non-increasing function of k that eventually reaches 1. Actually, it reaches 1 for a value k when G contains no path of length $2k$. It is thus natural to ask what are the classes of graphs $\mathcal{C}$ for which there exists k such that $\pi_k(\mathcal{C}) \leqslant 2$. We show that for the class $\mathcal{C}_t$ of graphs with blocks of size at most t, $\pi_k(\mathcal{C}_t)$ eventually reaches 2.

Theorem 5. *For any integer* $t \geqslant 2$ *and graph* G *with no block of order more than* t*,* $\pi_k(G) \leqslant 2$ *for some sufficiently large* k*.*

In contrast, the class of cactus graphs is one of the simplest classes of graphs with blocks of arbitrarily large order, and we show that for this class π_k never reaches 2.

Theorem 6. *For any integer* k*, there exists a cactus graph* G *such that* $\pi_k(G) > 2$*.*

2 Trees

Colorings of trees have been considered [11] that minimize the critical exponent of repetitions. To avoid large squares, and for the same reasons as in [11], we can assume without loss of generality that our colored tree is rooted and that all the vertices at the same distance to the root have the same color. So we only need to describe the word w lying on one branch of the tree. We adopt the counter-intuitive convention that the reading direction of w goes towards the root. Then every factor fs of w with $|s| = 1$ should be such that fsf^R (where f^R is the reverse of f) avoids the forbidden large squares. Let w_3 be any infinite $\left(\frac{7}{4}^+\right)$-free ternary word.

We obtain w by taking the image of any $\left(\frac{7}{4}^+\right)$-free ternary word by the following morphisms.

We use the 12-uniform morphism g_2 to prove $\pi_2(tree) \leqslant 3$ and the 21-uniform morphism g_5 to prove $\pi_5(tree) \leqslant 2$:

$$g_2(0) = \texttt{011220012201} \qquad g_5(0) = \texttt{001101110001010110010}$$
$$g_2(1) = \texttt{122001120012} \qquad g_5(1) = \texttt{001101110001001110101}$$
$$g_2(2) = \texttt{200112201120} \qquad g_5(2) = \texttt{001101110001001101010}$$

A word u is *d-directed* if for every factor f of u of length d, the word f^R is not a factor of u. To prove that a word is d-directed, it suffices to check its factors of length d. A word is (β^+, n)-free if it contains no repetition with exponent strictly greater than β and period at least n. To prove the (β^+, n)-freeness, we use the method described in [10]. This way, we obtain the following.

- $g_2(w_3)$ is 3-directed and $\left(\frac{19}{10}^+, 2\right)$-free.
- $g_5(w_3)$ is 20-directed and $\left(\frac{83}{42}^+, 5\right)$-free.

Consider $g_2(w_3)$. For contradiction, suppose that $g_2(w_3)$ contains a factor fs with $|s| = 1$ such that the word fsf^R contains a square of period $p \geqslant 20$. Since fsf^R is a palindrome, we can assume that the center of the square is on the left of s. Since fs is $\left(\frac{19}{10}^+, 2\right)$-free, fs must contain (as a suffix) a prefix of this square of length at least $p+1$ and at most $\frac{19}{10}p$. So sf^R must contain (as a prefix) a suffix x of this square of length at most p and at least $\frac{1}{10}p + 1 \geqslant 3$. Because of the square, x appears both in fs and sf^R. Notice that $(fs)^R = sf^R$, so that fs contains both x and x^R. This is a contradiction since $|x| \geqslant 3$ and fs is 3-directed. Finally, an exhaustive computer check shows that the words fsf^R contain no square of period p with $2 \leqslant p \leqslant 19$.

The proof for $g_5(w_3)$ is similar.

3 Planar graphs

We start with helpful results on paths and outerplanar graphs.

Lemma 1. *Let k be a fixed integer and let P be a path. In every proper coloring of P avoiding squares of period at least k, every subpath of P with $4k$ vertices contains at least 3 colors.*

Proof. A proper 2-coloring of the path of $4k$ vertices contains the square $(01)^{2k}$ of period $2k$. So at least 3 colors are needed to avoid squares of period at least k. □

Lemma 2. *For every fixed k, there exists an outerplanar graph that admits no proper 5-coloring avoiding squares of period at least k.*

Proof. Our outerplanar graph G has a root vertex and vertices at distance i from the root are said to be on level i. A vertex on level $i+1$ has exactly one neighbor on level i. The neighborhood on level $i+1$ of vertex on level i induces a very long path. Finally, G contains $6k$ levels.

For contradiction, suppose that G has a proper 5-coloring avoiding squares of period at least k using the colors $\{0, 1, 2, 3, 4\}$. Without loss of generality, the root (on level 0) is colored 0. Without loss of generality, the very long path on level 1 contains two non-intersecting occurrences of a long factor of the form $w1$. Now we consider the very long path on level 2 adjacent to the suffix letter 1 of the rightmost occurrence of $w1$. It does not contain color 0, since otherwise we would have the long square $w10w10$ such that the first 0 is the root and the second 0 is on level 2. So it must be colored with the remaining colors $\{2, 3, 4\}$. By Lemma 1, each of the three colors in $\{2, 3, 4\}$ are recurrent in our very long path. In particular, it contains two non-intersecting occurrences of a long factor of the form $z2$. On level 3, below the suffix letter 2 of the rightmost occurrence of $z2$, the very long path does not contain color 2, since otherwise we would have the long square $z21z21$. So this very long path on level 3 must be colored with letters $\{0, 3, 4\}$.

This line of reasonning leads to the existence of a downward path from the root such that the vertex on level i is colored $i \pmod 3$: By induction, we consider the very long path on level i adjacent to the suffix letter $p = (i-1) \pmod 3$ of the rightmost occurrence of a long factor of xp on level $i-1$. Also by induction, the grandparent of this very long path is colored $g = (i-2) \pmod 3$. So this very long path contain neither color p and nor color g, since otherwise we would have the long square $xpgxpg$. Thus it must be colored with the three colors in $\{0,1,2,3,4\}\setminus\{p,g\} = \{i \pmod 3), 3, 4\}$, which implies that color $i \pmod 3$ appears on level i.

So G contains the long square $(012)^{2k}$. $\square$

The proof of Theorem 4 builds on a similar approach. We also consider an arbitrary large graph of our class, a planar 3-tree obtained from K_4 by iteratively adding a vertex of degree 3 in every face, a sufficiently large number of times. As above, we use the fact that for each vertex, its neighborhood contains a graph with particular coloring properties. Here, this graph is the outerplanar graph witnessing Lemma 2.

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