List coloring with separation of the complete graph

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Abstract

We consider the problem of computing, for a graph G and integers a, b, the *separation number* of G which is the largest integer c such that there exists a list assignment of G with the properties that $|L(x) \cap L(y)| \leq c$ for any edge xy, $|L(x)| = a$ for any vertex x, and G is (L, b)-colorable, i.e., from each list $L(x)$, one can choose a b-element subset in such a way that adjacent vertices receive disjoint subsets. We concentrate on the complete graph and prove exact values and bounds for its separation number.

1 Introduction

Let a, b, c be integers and let G be a graph. A a-list assignment L of G is a function which associates to each vertex a set of a integers. The list assignment L is *c-separating* if for any $uv \in E(G)$, $|L(u) \cap L(v)| \leq c$. The graph G is (a, b, c)*-choosable* if for any c-separating a-list assignment L, there exists an (L, b)-coloring of G, i.e. a coloring function φ on the vertices of G that assigns to each vertex v a subset of b elements from $L(v)$ in such a way that $\varphi(u) \cap \varphi(v) = \emptyset$ for any $uv \in E(G)$.

This type of restricted list coloring problem, called choosability with separation, has been introduced by Kratochvíl, Tuza and Voigt [10]. Notice that Kratochvíl et al. [10, 11] defined (a, b, c)-choosability a bit differently, requiring for a c-separating a-list assignment L that the lists of two adjacent vertices u and v satisfy $|L(u) \cap L(v)| \leq a - c$. Among the first results on the topic, a complexity dichotomy was presented [10] and general properties given [11]. Since then, a number of papers has considered choosability with separation of planar graphs, mainly for the case $b = 1$ [1, 2, 3, 4, 5, 12]. While the fact that planar graphs are $(4, 1, 2)$-choosable was proved very recently [13], a still open question is whether all planar graphs are $(3, 1, 1)$-choosable or not. Other recent papers concern balanced complete multipartite graphs and k-uniform hypergraphs (for the case $b = 1$) [8], and bipartite graphs (for the case $b = c = 1$) [7].

In this work, we concentrate on choosability with separation of complete graphs. As a (a, b, c)-choosable graph is also (a, b, c')-choosable for any $c' < c$, our aim is to determine, for given a, b, $a \geq b$, the largest c such that G is (a, b, c)-choosable. We define the parameter $\mathsf{sep}(G, a, b)$ that we call the *(list) separation number* of G as

$$\mathsf{sep}(G, a, b) = \max\{c, G \text{ is } (a, b, c)\text{-choosable}\}.$$

This parameter is well defined since we have, for any graph G and $a \geq b$, $0 \leq \mathsf{sep}(G, a, b) \leq a$.

Determining the separation number of a graph proves to be a difficult problem, even for simple graphs such as complete graphs. The separation number of the cycle was determined in [9].

The following Hall-type condition that we call the *amplitude condition* is necessary for a graph G to be (L, b)-colorable:

$$\forall H \subset G, \sum_{k \in C} \alpha(H, L, k) \geq b|V(H)|,$$

where $C = \bigcup_{v \in V(H)} L(v)$ and $\alpha(H, L, k)$ is the independence number of the subgraph of H induced by the vertices containing k in their color list. Notice that H can be restricted to be a connected induced subgraph of G. As shown by Cropper et al. [6] (in the more general context of weighted list coloring), this condition is also sufficient when the graph is a complete graph or a path (or some other specific graphs).

For a list assignment L on a graph G of order n with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and for $S \subset [1, n]$, we write $\Sigma_S(L) = \sum_{k \in C} \alpha(H, L, k)$, where H is the subgraph of G induced by $\{x_i, i \in S\}$.

Remark that if G is a complete graph, then $\alpha(H, L, k) = 1$ for any k. Hence the amplitude condition for K_n becomes

$$\forall S \subset [1, n], \Sigma_S(L) = | \cup_{i \in S} L(x_i)| \geq b|S| \tag{1}$$

In order to simplify the computations, we partition the lists of colors of the vertices and show some properties in the next section.

2 Algebraic tools on proper intersections

Let $n \geq 1$ be an integer and $[n] = [1, n]$. For sets of elements $A_1, A_2, \ldots, A_n$ and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_i\} \subset [n]$, we define the *proper intersection* $I_{\mathsf{p}}(S_i)$ of $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_i}$ as

$$I_{\mathsf{p}}(S_i) = I_{\mathsf{p}}(\alpha_1, \alpha_2, \ldots, \alpha_i) = \left(\bigcap_{k=1}^{i} A_{\alpha_k} \right) \setminus \left(\bigcup_{\beta \in [n] \setminus S_i} A_\beta \right)$$

Then the classical intersection can be described in terms of proper intersections (proofs are simple but omited due to space constraints):

Property 1. *For any integer i, $1 \leq i \leq n$ and $S \subset [n]$,*

$$\bigcap_{\alpha \in S} A_\alpha = \bigcup_{S' \subset [n] \setminus S} I_{\mathsf{p}}(S \cup S')$$

Property 2. *For any $S, S' \subset [n]$ such that $S \neq S'$, we have $I_{\mathsf{p}}(S) \cap I_{\mathsf{p}}(S') = \emptyset$.*

We thus have the following improved *à-la-Poincaré* crible in which we only do additions (compared to Poincaré's crible where additions and substractions alternate):

Theorem 1.

$$\bigcup_{i=1}^{n} A_i = \bigcup_{S \subset [n], S \neq \emptyset} I_{\mathsf{p}}(S).$$

Corollary 1.

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{S \subset [n], S \neq \emptyset} |I_{\mathsf{p}}(S)|.$$

We will use proper intersections on list assignments of K_n. Let $V(K_n) = \{v_1, \ldots, v_n\}$. For a c-separating a-list assignment L of K_n, we let $A_i = L(v_i)$. Then the separation condition can be rewritten as

$$\forall i, j \in [n], i \neq j, \sum_{S \subset [n], i, j \in S} |I_{\mathsf{p}}(S)| \leq c \tag{2}$$

Consequently, from Equations 1 and 2, we obtain an IPL-formulation of the problem of finding smallest counter examples (c-separating a-list assignments L of K_n for which no (L, b)-coloring exists): For $S \subset [n]$, $S \neq \emptyset$, we consider the variable $x_S = |I_{\mathsf{p}}(S)|$. Then the goal is to minimize c subject to the constraints:

$$\forall i \in [n], \sum_{S \subset [n], i \in S} x_S = a, \text{ and } \forall i, j \in [n], i \neq j, \sum_{S \subset [n], i, j \in S} x_S \leq c, \text{ and } \sum_{S \subset [n]} x_S < nb.$$

We used IPL-solvers to help us finding counter-examples.

3 Separation number of K_n

For K_2, it is easily seen that $\mathsf{sep}(K_2, a, b) = a - b$ if $a \leq 2b$ and $\mathsf{sep}(K_2, a, b) = a$ if $a \geq 2b$. For $K_3 = C_3$, the separation number follows from results on the cycle from [9]:

Corollary 2.

$$\mathsf{sep}(K_3, a, b) = \begin{cases} a - b, & b \leq a < 2b \\ 2a - 3b, & 2b \leq a < 3b \\ a, & a \geq 3b. \end{cases}$$

For K_4, we are able to prove that:

Proposition 1.

$$\mathsf{sep}(K_4, a, b) = \begin{cases} \lfloor \frac{2(a-b)}{3} \rfloor, & b \leq a < 2b \\ \lceil \frac{4a-6b-1}{3} \rceil, & 2b \leq a < 3b \\ 2a - 4b, & 3b \leq a < 4b \\ a, & a \geq 4b. \end{cases}$$

For arbitrary values of n, we have the two following results:

Proposition 2. *For any $n \geq 3$ and , a, b such that $b \leq a \leq 2b$, we have $\mathsf{sep}(K_n, a, b) = \lfloor \frac{2(a-b)}{n-1} \rfloor$.*

Proof sketch. Let a, b such that $b \leq a \leq 2b$ and let $c = \lfloor \frac{2(a-b)}{n-1} \rfloor$. Consider a c-separating a-list assignment L of K_n, for $n \geq 3$. Remark that we have $(n-1)c \leq 2(a-b) < a$.

The idea of the proof is first to obtain a lower bound for the amplitude $\Sigma_S(L)$ and show that the amplitude condition is satisfied for any $S \subset [n]$. By the separation condition, the amplitude of L on vertices from $S \subset [n]$, $|S| = i$, satisfies $\Sigma_S(L) \geq a + (a-c) + \ldots + (a - (i-1)c) = ia - \frac{1}{2}i(i-1)c$. Therefore, as $c \leq 2(a-b)/(n-1)$, we obtain

$$\Sigma_S(L) \geq ia - \frac{1}{2}i(i-1)\frac{2(a-b)}{n-1} = ia\frac{n-i}{n-1} + ib\frac{i-1}{n-1} \geq ib.$$

Second, we construct counter-examples to show that there exist $c+1$-separating a-list assignments L' that do not satisfy the amplitude condition, hence for which no (L', b)-coloring exists. In the general case, the list assignment L is constructed by setting for any $i, j \in [n], i \neq j$, $|I_{\mathsf{p}}(i)| = a - (n-1)c$ and $|I_{\mathsf{p}}(i, j)| = c$, the other proper intersections being empty. $\square$

Proposition 3. *For any $n \geq 3$ and , a, b such that $(n-1)b \leq a \leq nb$, we have* $\mathsf{sep}(K_n, a, b) = 2a - nb$.

Proof sketch. Similar to the one of Proposition 2, noting that $a \leq \frac{n-1}{n-2}c$ in this case. □

For other values of a, partial results and computations lead us to conjecture the following:

Conjecture 1. *for any $n \geq 4, a, b, p$ with $2 < p \leq n-2$ and $pb \leq a < (p+1)b$, we have*

$$\mathsf{sep}(K_n, a, b) = \left\lceil \frac{2pa - p(p+1)b}{n-1} \right\rceil + \epsilon,$$

with $\epsilon \in \{-1, 0\}$.

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