The Perfect Matching-Cut problem in bipartite graphs with diameter three

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Abstract

Perfect Matching-Cut is the problem of deciding whether a connected graph has a perfect matching that contains an edge-cut. Deciding whether a graph has a Perfect Matching-Cut is NP-complete even for the class of bipartite graphs with diameter four. The case of bipartite graphs with diameter one is trivial since only K_2 has a Perfect Matching-Cut. We show that it is polynomial to decide whether a bipartite graph with diameter three has a Perfect Matching-Cut.

1 Introduction

The Matching-Cut problem consists of finding a matching that is an edge-cut. Chvátal [2] proved that the problem is NP-complete for graphs with maximum degree four and polynomially solvable for graphs with maximum degree three. Bonsma [1] showed that it remains NP-complete for planar graphs with maximum degree four and gave polynomial algorithms for some subclasses of graphs. Le and Randerath [6] showed that the Matching-Cut problem is NP-complete for bipartite graphs.

We address the Perfect Matching-Cut problem where the matching involved in the matching-cut is contained in a perfect matching. To our knowledge, the only reference to this problem is from Diwan [3] which he called *Disconnected* 2*-Factors*.

We showed that the Perfect Matching-Cut problem is NP-complete for the 5-regular bipartite graphs, for the graphs with diameter three and for the bipartite graphs with diameter d, for any fixed $d \geq 4$, for planar graphs with degrees three or four, and for planar graphs with girth five.

Here we give the proof that the Perfect Matching-Cut problem is polynomial for bipartite graphs with diameter three.

2 Notations and preliminaries

For a graph $G = (V, E)$, let $\delta(G)$ denote its minimum degree. For $v \in V$, $N(v)$ is its neighborhood, $d(v)$ its degree. The diameter of G is the maximum length of a shortest path. For $M \subseteq E$ we write $G - M$ the partial graph $G = (V, E \setminus M)$. A set $M \subseteq E$ is a *matching* when no edges of M share a vertex. A matching M is *perfect* when every vertex is incident to one edge of M. A *cut* in G is a partition $V = X \cup (V \setminus X)$ with $X, V \setminus X \neq \emptyset$. The set of all edges in G having an endvertex in X and the other endvertex in $V \setminus X$, also written $E(X, V \setminus X)$, is called the *edge-cut* of the cut. A *bridge* is an edge-cut with exactly one edge. A *matching-cut* is an edge-cut that is a matching. A *perfect matching-cut* is a perfect matching that contains a matching-cut. When a connected graph or subgraph cannot be disconnected by a matching it is called *immune*, when it cannot be disconnected by a perfect matching it is called *perfectly immune*.

We give some easy observations:

1. K_3 and $K_{2,3}$ are immune.

2. Assume that G has a perfect matching-cut M with $E(X, V \setminus X) \subset M$ and let H be an immune subgraph of G. Then either $V(H) \subset X$ or $V(H) \subset V \setminus X$.

3. Assume that G has a perfect matching-cut M with $E(X, V \setminus X) \subset M$. If v has two neighbors in X then $v \in X$.

4. If M is a perfect matching that contains a bridge then M is a perfect matching-cut.

5. When a graph G with $\delta(G) = 1$ has a perfect matching then G has a perfect matching-cut.

3 Bipartite graphs with diameter three

We prove that the Perfect Matching-Cut is polynomial for the bipartite graphs with diameter three. For our proof we use the following characterization given in [5].

A) Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then $diam(G) \leq 3$ if and only if $N(a) \cap N(b) \neq \emptyset$ for any pair of distinct vertices a, b in the same side of the bipartition.

We also use the following facts. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and M be a perfect matching-cut with (X, Y) its corresponding partition.

1. Let $uv \in M$ with $u \in X, v \in Y$. Then $N(u) \setminus \{v\} \subseteq X$ and $N(v) \setminus \{u\} \subseteq Y$.

2. If a vertex v has two neighbors in X, respectively in Y, then $v \in X$, respectively $v \in Y$.

3. If a vertex v has a neighbor in $x \in X$ and another neighbor in $y \in Y$, such that $xy \in M$, then M cannot be a perfect matching-cut.

We are ready to show the following.

Theorem 1. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $diam(G) \leq 3$. Deciding if G has a perfect matching-cut can be done in polynomial time.*

Proof. We assume that G has a perfect matching, thus $|V_1| = |V_2|$. If uv is a bridge and there exists M a perfect matching-cut such that $uv \in M$ then G has a perfect matching-cut. Now we can assume that $\delta(G) \geq 2$ and if M is a perfect matching-cut of G then $|E(X, Y)| \geq 2$.

We try to build M a perfect matching-cut with a partition (X, Y). We guess two edges ab, cd with $a, d \in V_1$, $b, c \in V_2$ such that $\{ab, cd\} \subseteq M$. When a perfect matching-cut M is found the algorithm stops, otherwise we try another pair of edges.

We show that when G has a perfect matching-cut, then there exists M a perfect matching-cut such that $\{a, c\} \subseteq X$, $\{b, d\} \subseteq Y$. Let $a \in X$, $b \in Y$. For contradiction we assume that $V_2 \subseteq Y$. Since $d(a) \geq 2$, by Fact 2 we have $a \in Y$, a contradiction.

Hence we put $a, c \in X$, $b, d \in Y$. Note that there are $\mathcal{O}(|V|^2)$ such combinations.

Note that G being bipartite we have $N(a) \cap N(b) = N(a) \cap N(c) = N(b) \cap N(d) = N(c) \cap N(d) = \emptyset$. Moreover, by Fact 3 if there exists $v \in N(a) \cap N(d)$ or $v \in N(b) \cap N(c)$ then M cannot exist. Hence we have $N(a) \cap N(d) = N(b) \cap N(c) = \emptyset$.

We define the following sets of vertices:

- $A = N(a) \setminus \{b, c\}$, $B = N(b) \setminus \{a, d\}$, $C = N(c) \setminus \{a, d\}$, $D = N(d) \setminus \{b, c\}$;
- $S = \{v \in V_1 \mid v \notin N(b) \cup N(c),\ N(v) \cap A \neq \emptyset,\ N(v) \cap D \neq \emptyset\}$;
- $T = \{v \in V_2 \mid v \notin N(a) \cup N(d),\ N(v) \cap B \neq \emptyset,\ N(v) \cap C \neq \emptyset\}$.

We show that $A, B, C, D, S, T, \{a, b, c, d\}$ is a partition of $V_1 \cup V_2$. The subsets are pairwise disjoint. By contradiction, we assume that there exists $v \in V_1$ that is not in one of the previous subsets. By Fact A) v and a have a common neighbor w. Since $N(a) \subseteq A \cup \{b, c\}$ we have a contradiction. The case $v \in V_2$ is the same.

By Fact 1 we have $A \cup C \subseteq X$ and $B \cup D \subseteq Y$. Let $v \in A$. By Fact 2, if v has two neighbors in B then $v \in Y$, so M cannot exist. The situation is the same when a vertex of B has two neighbors in A, a vertex of C has two neighbors in D, a vertex of D has two neighbors in C. If v has exactly one neighbor $w \in B$ then $vw \in M$ and by Fact 1 all its neighbors are put in X, and all the neighbors of w are put in Y. We do the same for the vertices of B, C, D. By Fact 2 when $v \in S$ has two neighbors in X, resp. Y, then $v \in X$, resp. $v \in Y$. We do in a same way for $v \in T$. If a vertex is in both X and Y then M cannot exist and we stop. By Fact 2 if a vertex in X, resp. Y, has two neighbors in Y, resp. X, then M cannot exist.

Let $S' = \{v \in S \mid v \notin X \cup Y\}$ and $T' = \{v \in T \mid v \notin X \cup Y\}$. By above and since $\delta(G) \geq 2$, each vertex $v \in S'$ has exactly one neighbor $v_a \in A$ and one neighbor $v_d \in D$, and each vertex $v \in T'$ has exactly one neighbor $v_b \in B$ and one neighbor $v_c \in C$. Let $A' = \{v_a \in A \mid vv_a \in E,\ v \in S'\}$, $D' = \{v_d \in D \mid vv_d \in E,\ v \in S'\}$, $B' = \{v_b \in B \mid vv_b \in E,\ v \in T'\}$, $C' = \{v_c \in C \mid vv_c \in E,\ v \in T'\}$. Note that from Fact 2, for every pair $v_a \in A'$, $v_d \in D'$ we have $N(v_a) \cap N(v_d) \subseteq S'$. By symmetry, for every pair $v_b \in B'$, $v_d \in D'$ we have $N(v_b) \cap N(v_c) \subseteq T'$.

For every $v \in S'$, resp. $v \in T'$, for M to exist we have either $vv_a \in M$ or $vv_d \in M$, resp. $vv_b \in M$ or $vv_c \in M$. Hence every edge $st, s \in S', t \in T'$ is such that $st \notin M$ and the two vertices s, t will be assigned to a same subset X or Y.

Let $|S'| = \mu$, $|A'| = \alpha$, $|D'| = \delta$. W.l.o.g we assume that $\alpha \geq \delta$ (the case $\alpha \leq \delta$ being symmetric). By Fact A) and since each vertex of S' has exactly one neighbor in A' and one neighbor in D', we have $\mu \geq \alpha\delta$. For M to exist we need $\mu \leq \alpha + \delta$. Thus $\alpha + \delta \geq \alpha\delta$, which is possible only for $\alpha = \delta = 2$ or $\delta = 1$, $\alpha \geq 1$.

Let $\alpha = \delta = 2$. We denote $A' = \{v_a^1, v_a^2\}$, $D' = \{v_d^1, v_d^2\}$, $S' = \{v_1, v_2, v_3, v_4\}$. Then $G' = G[A' \cup D' \cup S']$ consists of the four paths $v_a^1 - v_1 - v_d^1$, $v_a^1 - v_2 - v_d^2$, $v_a^2 - v_3 - v_d^1$, $v_a^2 - v_4 - v_d^2$. There exist two perfect matchings of G', that are, $M_a = \{v_a^1 v_1, v_d^2 v_2, v_d^1 v_3, v_a^2 v_4\}$, $M_d = \{v_a^1 v_2, v_d^2 v_4, v_d^1 v_1, v_a^2 v_3\}$.

Let $\delta = 1$. We have $\alpha \leq \mu \leq \alpha + 1$. We denote $A' = \{v_a^1, \ldots, v_a^\alpha\}$, $D' = \{v_d\}$, $S' = \{v_1, \ldots, v_\mu\}$ and we assume that $v_a^i v_i \in E, 1 \leq i \leq \mu$. We denote $G' = G[A' \cup D' \cup S']$.

First $\mu = \alpha$. Then G' consists of α paths $v_a^1 - v_1 - v_d, \ldots, v_a^\alpha - v_\alpha - v_d$. Note that G' has no perfect matching but recall that for each $v_i \in S'$, either $v_i v_a^i \in M$ or $v_i v_d \in M$. So in G' there exists $\alpha + 1$ matchings that disconnect A' from D'. These matchings are $M_0 = \{v_a^1 v_1, \ldots, v_a^\alpha v_\alpha\}$ and $M_i = \{v_i v_d\} \cup \{v_a^j v_j,\ 1 \leq j \leq \alpha,\ j \neq i\}$.

Second $\mu = \alpha+1$. Then G' consists of the two paths $v_a^1 - v_1 - v_d$, $v_a^1 - v_2 - v_d$ and the $\alpha-1$ paths $v_a^2 - v_3 - v_d, \ldots, v_a^\alpha - v_{\alpha+1} - v_d$. There exists exactly two (perfect) matchings of G' that disconnect A' from D', that are $\bar{M}_1 = \{v_a^1 v_1, v_d v_2, v_a^2 v_3, \ldots, v_a^\alpha v_{\alpha+1}\}$ and $\bar{M}_2 = \{v_a^1 v_2, v_d v_1, v_a^2 v_3, \ldots, v_a^\alpha v_{\alpha+1}\}$.

By symmetry, to $G'' = G[B' \cup C' \cup T']$ correspond the following matchings: either $M'_0 = \{{v'}_a^1 {v'}_1, \ldots, {v'}_a^{\alpha'} {v'}_{\alpha'}\}$ and $M'_i = \{v'_i v'_d\} \cup \{{v'}_a^j v'_j, 1 \leq j \leq \alpha', j \neq i\}$ or $\bar{M}'_1 = \{{v'}_a^1 v'_1, v'_d v'_2, {v'}_a^2 v'_3, \ldots, {v'}_a^{\alpha'} - v'_{\alpha'+1}\}$ and $\bar{M}'_2 = \{{v'}_a^1 v'_2, v'_d v'_1, {v'}_a^2 v'_3, \ldots, {v'}_a^{\alpha'} - v'_{\alpha'+1}\}$. Hence there are $\mathcal{O}(|E|^2)$ combinations between the matchings of G' and G''.

For each combination we test if $E(X,Y)$ is a matching-cut. If not then M with $E(X,Y) \subseteq M$ cannot exist. Otherwise, let $X' \subseteq X$ such that $N(X') \cap Y = \emptyset$ and $Y' \subseteq Y$ such that $N(Y') \cap X = \emptyset$. We check if $G[X' \cup Y']$ has a perfect matching. If not, M with $E(X,Y) \subseteq M$ cannot exist, else we have M a perfect matching-cut of G.

We estimate the running time of our algorithm as follows. From [4], we know that computing a perfect matching in a bipartite graph takes $\mathcal{O}(|V|^{\frac{5}{2}})$. To check if there exists a perfect matching that contains a bridge can be done in time $\mathcal{O}(|V|^{\frac{5}{2}})$. Now, there are $\mathcal{O}(|V|^2)$ pairs of edges ab, cd. Given a pair ab, cd, one can verify that the running time until the next pair is $\mathcal{O}(|V|^{\frac{5}{2}})$. Hence the complexity of the algorithm is $\mathcal{O}(|V|^{\frac{9}{2}})$. $\square$

Remark: The cliques are the graphs with diameter one. Hence K_2 is the sole graph of diameter one that has a perfect matching-cut.

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