Homomorphisms of (n, m)-graphs with respect to generalized switch

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Abstract

An (n, m)-graph has n different types of arcs and m different types of edges. A homomorphism of an (n, m)-graph G to an (n, m)-graph H is a vertex mapping that preserves adjacency types and directions. Notice that, in an (n, m)-graph a vertex can possibly have $(2n + m)$ different types of neighbors. In this article, we study homomorphisms of (n, m)-graphs while an Abelian group acts on the set of different types of neighbors of a vertex.

1 Introduction

A *graph homomorphism*, that is, an edge-preserving vertex mapping of a graph G to a graph H, also known as an H-coloring of G, was introduced as a generalization of coloring [4]. It allows us to unify certain important constraint satisfaction problems, especially related to scheduling and frequency assignments, which are otherwise expressed as various coloring and labeling problems on graphs. Thus the notion of graph homomorphism manages to capture a wide range of important applications in an uniform setup.

The research on graph homomorphisms have evolved in the following three major directions. (i) The study of various application motivated optimization problems modeled using graph homomorphisms. These problems usually involve in finding an H having certain prescribed properties such that every member of a graph family $\mathcal{F}$ is H-colorable. (ii) The study of the algorithmic aspects of the H-coloring problem, including characterizing its dichotomy, and finding approximation or parametrized algorithms for the hard problems. (iii) The study of the algebraic structures that gets induced by the notion of graph homomorphisms, namely, quasiorder (and partial order), lattice, and category. Unsurprisingly, these three areas of research have interdependencies and connections.

The notion of graph homomorphisms, initially introduced for undirected and directed graphs, later got extended to 2-edge-colored graphs [1], k-edge-colored graphs [1] and (n, m)-colored mixed graphs [8]. These graphs, due to their various types of adjacencies, manages to capture complex relational structures and are useful for mathematical modeling. For instance, the query analysis problem in graph database, the databases that are now popularly used to handle highly interrelated data networks (such as, social networks like Facebook, Twitter, etc.), is modeled on homomorphisms of (n, m)-colored mixed graphs.

Moreover, researchers have further extended the study by exploring the effect of switch operation on homomorphisms. Notably, homomorphisms of signed graphs, which is essentially obtained by observing the effect of the switch operation on homomorphisms of 2-edge-colored graphs, has gained immense popularity [7] in recent times due to its strong connection with the graph minor theory. Also, graph homomorphism with respect to some other switch-like operations, such as, push operation on oriented graphs [2], cyclic switch on k-edge-colored graphs [3], and switching (n, m)-colored mixed graphs with respect to Abelian groups of special type (which does not allow switching an edge to an arc or vice versa) [5] to list a few, has been recently studied.

Naturally, all the three major directions of research listed above in context of graph homomorphism is also explored for the above mentioned extensions and variants. However, in comparison, the global algebraic structure is a less explored branch for the extensions.

In this article, we introduce a generalized switch operation on (n, m)-colored mixed graphs and study some of its algebraic aspects. The results proved here will be valid for all known graph homomorphism variants, to the best of our knowledge.

2 Homomorphisms of (n, m)-graphs and generalized switch

Throughout this article, standard graph theoretic, algebraic and category theory definitions, notion and terminology is followed and all the graphs considered here have underlying graph as simple graph.

Nešetřil and Raspaud [8] introduced the concept of colored mixed graphs in 2000 as a generalization to the study of oriented and k-edge-colored graphs. An (n, m)-*graph* G is a graph with vertex set $V(G)$, arc set $A(G)$ and edge set $E(G)$, where each arc is colored with one of the n colors from $\{1, 2, \cdots, n\}$ and each edge is colored with one of the m colors from $\{n+1, n+2, \cdots, n+m\}$. In particular, if there is an arc of color i from u to v, we say that v is an i-neighbor of u, or equivalently, u is a $-i$-neighbor of v. Furthermore, if there is an edge of color j between u and v, then we say that u (resp., v) is a j-neighbor of v (resp., u).

Let $\Gamma \subseteq S_{2n+m}$ be a subgroup, where S_{2n+m} is the permutation group on $A_{n,m} = \{\pm 1, \cdots, \pm n, n+1, \cdots, n+m\}$. To Γ-*switch* a vertex v of an (n, m)-graph is to change its incident arcs and edges in such a way that its t-neighbors become $\sigma(t)$-neighbors, for some $\sigma \in \Gamma$ and for all $t \in A_{n,m}$. For an element $\sigma \in S_{2n+m}$, a σ-switch of a vertex is defined similarly. An (n, m)-graph G' obtained by a sequence of Γ-switches performed on the vertices of G is a Γ-*equivalent* graph of G.

A Γ-*homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that there exists a Γ-equivalent graph G' of G satisfying the following: if u is a t-neighbor of v in G', then $f(u)$ is a t-neighbor of $f(v)$ in H. We denote this by $G \xrightarrow{\Gamma} H$. A Γ-*isomorphism* of G to H is a bijective Γ-homomorphism whose inverse is also a Γ-homomorphism. We denote this by $G \equiv_{\Gamma} H$.

Observe that, if $\Gamma = \langle e \rangle$ is the singleton group with the identity element e, then our Γ-homomorphism definition becomes the same as the colored homomorphism of (n, m)-graphs defined by Nešetřil and Raspaud [8].

Let u, v be any two vertices of any (n, m)-graph G. A *switch-commutative* group $\Gamma \subseteq S_{2n+m}$ is such that for any $\sigma, \sigma' \in \Gamma$, by performing σ-switch on u and σ'-switch on v, the adjacency between u and v changes in the same way irrespective of the order of the switches. Observe that the special type of Γ-switch operations studied in [6] are all where Γ is a commutative-switch group. On the other hand, there are infinitely many examples of commutative-switch groups which are not considered in [6].

A *consistent* group $\Gamma \subseteq S_{2n+m}$ is such that each orbit $A_{n,m}$ due to the action of Γ on it contains $-i$ if and only if it contains i for $i \in \{1, 2, \cdots, n\}$.

We will restrict ourselves to Abelian subgroups Γ of S_{2n+m} unless otherwise stated. A study when Γ is non-Abelian is explored in [5].

3 Basic algebraic properties

Let $\Gamma \subseteq S_{2n+m}$ be a switch-commutative group and let G be an (n,m)-graph with set of vertices $\{v_1, v_2, \cdots, v_k\}$. Let G^* be the (n,m)-graph having vertices of the type v_i^{σ} where $i \in \{1, 2, \cdots, k\}$ and $\sigma \in \Gamma$. Also a vertex v_i^{σ} is a t-neighbor of $v_j^{\sigma'}$ in G^* if and only if v_i is a t-neighbor of v_j in G where $i, j \in \{1, 2, \cdots, k\}$ and $\sigma, \sigma' \in \Gamma$. The Γ-*switched graph* $\rho_\Gamma(G)$ of G is the (n,m)-graph obtained from G^* by performing a σ-switch on v_i^{σ} for all $i \in \{1, 2, \cdots, k\}$ and $\sigma \in \Gamma$. This Γ-switch graph helps build a bridge between $\langle e \rangle$-homomorphism and Γ-homomorphism of two (n,m)-graphs.

Proposition 1. *Let G, H be (n,m)-graphs. We have $G \xrightarrow{\Gamma} H$ if and only if $G \xrightarrow{\langle e \rangle} \rho_\Gamma(H)$ where Γ is a switch-commutative group.*

Theorem 2. *Let G, H be (n,m)-graphs. $G \equiv_\Gamma H$ if and only if $\rho_\Gamma(G) \equiv_{\langle e \rangle} \rho_\Gamma(H)$ where Γ is a switch-commutative group.*

The next result follows from the fundamental theorem of finite Abelian groups.

Theorem 3. *Let Γ_1 be a switch-commutative subgroup of S_{2n+m}. Let $\Gamma_2 \subseteq \Gamma_1$. If $p^2 \nmid |\Gamma_1|$ for any prime p, then $\rho_{\Gamma_1}(G) \equiv_{\langle e \rangle} \rho_{\Gamma_1/\Gamma_2}(\rho_{\Gamma_2}(G))$.*

A Γ-*core* of an (n,m)-graph G is a subgraph H of G such that $G \xrightarrow{\Gamma} H$, whereas H does not admit a Γ-homomorphism to any of its proper subgraphs.

Theorem 4. *The core of an (n,m)-graph G is unique up to Γ-isomorphism.*

Due to the above theorem, it is possible to define the Γ-core of G and we denote it by $core_\Gamma(G)$. Notice that, this is the analogue of the fundamental algebraic concept of core in the study of graph homomorphism.

4 Categorical products

It is possible to consider the category of (n,m)-graphs with respect to Γ-homomorphisms, taking the set of (n,m)-graphs as objects and their Γ-homomorphisms as morphisms. The existence of categorical product and co-product not only shows that the category is rich, but also implies that the lattice of (n,m)-graphs induced by Γ-homomorphisms is a distributive lattice with the categorical products and co-products playing the roles of join and meet, respectively. Moreover, as categorical product was useful in proving the density theorem [4] for undirected and directed graphs, one may hope to use it to prove an analogue in this context.

Let G, H be two (n,m)-graphs and let $\Gamma \subseteq S_{2n+m}$ be an Abelian group. Then $G \times_{\langle e \rangle} H$ denotes the (n,m)-graph on set of vertices $V(G) \times V(H)$ where (u,v) is a t-neighbor of (u',v') in $G \times_{\langle e \rangle} H$ if and only if u is a t-neighbor of u' in G and v is a t-neighbor of v' in H. Moreover, the (n,m)-graph $G \times_\Gamma H$ is the subgraph of $\rho_\Gamma(G) \times_{\langle e \rangle} \rho_\Gamma(H)$ induced by the set of vertices

$$X = \{(u^e, v^\sigma) : (u,v) \in V(G) \times V(H) \text{ and } \sigma \in \Gamma\}.$$

Theorem 5. *The categorical product of (n,m)-graphs G and H with respect to Γ-homomorphism exists and is Γ-isomorphic to $G \times_\Gamma H$ where Γ is a switch-commutative group.*

Let $G + H$ denotes the disjoint union of the (n,m)-graphs G and H.

Theorem 6. *The categorical co-product of (n,m)-graphs G and H with respect to Γ-homomorphism exists and is Γ-isomorphic to $G + H$.*

Theorem 7. *For any (n,m)-graphs G, H, K we have $G \times_\Gamma H \equiv_\Gamma H \times_\Gamma G$, $G \times_\Gamma (H \times_\Gamma K) \equiv_\Gamma (G \times_\Gamma H) \times_\Gamma K$, and $G \times_\Gamma (H + K) \equiv_\Gamma (G \times_\Gamma H) + (G \times_\Gamma K)$ where Γ is a switch-commutative group.*

5 Chromatic number

We know that the ordinary chromatic number of a simple graph G can be expressed as the minimum $|V(H)|$ such that G admits a homomorphism to H. The analogue of this definition is a popular way for defining chromatic number of other types of graphs, namely, oriented graphs, k-edge-colored graphs, (n,m)-graphs, signed graphs, push graphs, etc. Here also, we follow the same.

The Γ-*chromatic number* of an (n,m)-graph is given by $\chi_{\Gamma:n,m}(G) = \min\{|V(H)| : G \xrightarrow{\Gamma} H\}$. Moreover, for a family $\mathcal{F}$ of (n,m)-graphs, the Γ-chromatic number is given by $\chi_{\Gamma:n,m}(\mathcal{F}) = \max\{\chi_{\Gamma:n,m}(G) : G \in \mathcal{F}\}$.

Proposition 8. *Let $\Gamma \subseteq S_{2n+m}$ be a consistent group, G be a (n,m)-graph and G' be a Γ-equivalent graph of G. If a vertex u is a t-neighbor of v in G, then u must be a $\sigma(t)$ neighbor of v in G' for some $\sigma \in \Gamma$.*

Theorem 9. *Let $\mathcal{F}$ be the family of (n,m)-forests and let t be the number of orbits obtained with respect to Γ acting on $A_{n,m}$. Then $\chi_{\Gamma:n,m}(\mathcal{F}) \leq t+1$ when t is odd, and $\chi_{\Gamma:n,m}(\mathcal{F}) \leq t+2$ when t is even. Moreover, equality holds if Γ is consistent.*

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