Pósa-type results for Berge-hypergraphs

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Abstract

A Berge-cycle of length k in a hypergraph $\mathcal{H}$ is a sequence of distinct vertices and hyperedges $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ such that $v_i, v_{i+1} \in h_i$ for all $i \in [k]$, indices taken modulo k. Füredi, Kostochka and Luo recently gave sharp Dirac-type minimum degree conditions that force non-uniform hypergraphs to have a Hamiltonian Berge-cycles. We give a sharp Pósa-type lower bound for r-uniform and non-uniform hypergraphs that force Hamiltonian Berge-cycles.

1 Introduction

The study of Hamiltonian-cycles is one of the essential topics of Graph Theory. In the present paper, we study sufficient degree conditions for a hypergraph to be Hamiltonian, in both uniform and non-uniform cases. We call a hypergraph Hamiltonian if there is a Berge-cycle containing all of the vertices of the hypergraph as defining vertices. Note that it is natural to follow the definition of Berge for cycles in hypergraphs. There is a one-to-one correspondence between Berge-cycles of the hypergraph and cycles in the incidence bipartite graph of the hypergraph. In particular, for a hypergraph $\mathcal{H}$ consider the incidence bipartite graph $G(A, B)$, where the vertices from A in G represent the vertices of $\mathcal{H}$ and the vertices from B represent the hyperedges of $\mathcal{H}$. A vertex a from A is adjacent with a vertex b from B in G if and only if the corresponding vertex of a in $\mathcal{H}$ is contained in the corresponding hyperedge of b in $\mathcal{H}$. There is a one-to-one correspondence between cycles in G and Berge-cycles of $\mathcal{H}$. The corresponding cycle in G of a Hamiltonian Berge-cycle in $\mathcal{H}$ is a cycle containing all vertices of the set A. Note that, incidence bipartite graphs of a simple hypergraphs have an extra condition: the neighborhoods of any two vertices representing different hyperedges of the hypergraph are distinct. While the topic of searching for cycles covering a color class for the class of all bipartite graphs is an interesting and active topic of research see [11, 12, 17, 15, 14], in this paper, we study simple hypergraphs only.

1.1 Hamiltonicity for hypergraphs

In order to describe the problem and existing results, first we need to introduce some standard notions. Let $\mathcal{H}$ be a hypergraph. We denote the vertex set of $\mathcal{H}$ by $V(\mathcal{H})$ and we denote the set of hyperedges of $\mathcal{H}$ by $E(\mathcal{H})$, where $E(\mathcal{H}) \subseteq \wp(V(\mathcal{H}))$. We say $\mathcal{H}$ is r-uniform if every hyperedge has size r. For a vertex v, $v \in V(\mathcal{H})$, the degree of the vertex is number of hyperedges incident with it, and it is denoted by $d_{\mathcal{H}}(v)$. If the host hypergraph is obvious from the context we will use $d(v)$ instead of $d_{\mathcal{H}}(v)$. The neighborhood of a vertex v is a set of vertices for which there exists a hyperedge incident with both vertices. We denote the neighborhood of a vertex v by $N_{\mathcal{H}}(v)$. In particular, we have $N_{\mathcal{H}}(v) = \{u \in V(\mathcal{H}) \setminus \{v\} : \{u, v\} \subseteq h, h \in \mathcal{H}\}$. The closed neighborhood of v is denoted by $N_{\mathcal{H}}[v]$, where $N_{\mathcal{H}}[v] = N_{\mathcal{H}}(v) \cup \{v\}$. For a hypergraph $\mathcal{H}$ and sub-hypergraph $\mathcal{H}'$ the hypergraph on the same vertex set as $\mathcal{H}$ and with hyperedges $E(\mathcal{H}) \setminus E(\mathcal{H}')$ is denoted by $\mathcal{H} \setminus \mathcal{H}'$. For a set S and integer r let us denote the set of all subsets of S of size r by $\binom{S}{r}$. Let us introduce the notions of Berge-paths [1], the notion of Berge-cycles is analogues.

Definition 1. *A* Berge-path *of length t is an alternating sequence of $t+1$ distinct vertices and t distinct hyperedges of the hypergraph, $v_1, e_1, v_2, e_2, v_3, \ldots, e_t, v_{t+1}$ such that $v_i, v_{i+1} \in e_i$, for $i \in [t]$.*

The vertices $v_1, v_2, \dots, v_{t+1}$ *are called* defining vertices *and the hyperedges* $e_1, e_2, \dots, e_t$ *are called* defining hyperedges *of the Berge-path.*

Long Berge-cycles are well-studied for hypergraphs. Turán-type questions for uniform hypergraphs without long Berge-cycles are settled in [5, 6, 9, 18]. Bermond, Germa, Heydemann, and Sotteau [2] found a Dirac-type condition forcing long Berge-cycles in uniform hypergraphs. Recently Coulson and Perarnau [3] found a Dirac-type condition for a hypergraph forcing Hamiltonicity. Füredi, Kostochka, and Luo [7] generalized Dirac's theorem [4] to non-uniform hypergraphs. For uniform linear hypergraphs Jiang, and Ma [13] settled a conjecture of Verstraëte. In a recent work Kostochka, Luo and McCourt proved a version of Dirac's theorem [4] for r-uniform hypergraphs. Here we state the theorem of Pósa which is a generalization of Dirac's theorem.

Theorem 2 (Pósa [19])**.** *Let G be an n vertex graph. Let $n \geq 3$ and the degree sequence of G be $d_1 \leq d_2 \leq \dots \leq d_n$. If for all $k < \frac{n}{2}$ the inequality $k < d_k$ holds then G is Hamiltonian.*

In this work we prove a theorem analogous to Pósa's theorem 2 with similar methods to those used in [8, 10]. We say an integer sequence $(d_1, d_2, \dots, d_n)$ is r-Hamiltonian if every r-uniform hypergraph with this degree sequence is Hamiltonian. We say an integer sequence $(d_1, d_2, \dots, d_n)$ is $\mathbb{N}$-Hamiltonian if every non-uniform hypergraph with the given degree sequence is Hamiltonian.

Theorem 3. *An integer sequence $(d_1, d_2, \dots, d_n)$ such that $d_1 \leq d_2 \leq \dots \leq d_n$, $n > 2r$ and $r \geq 3$ is r-Hamiltonian if the following conditions hold*

$$d_i > i \text{ for } 1 \leq i < r, \tag{1}$$

$$d_i > \binom{i}{r-1} \text{ for } r \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{2}$$

$$d_{\frac{n-2}{2}} > \binom{\frac{n-2}{2}}{r-1} + 1 \text{ if } n \text{ is even.} \tag{3}$$

One may interpret Theorem 3 as an analogue of Theorem 2 for r-uniform hypergraphs. Moreover, we prove analogue of Theorem 2 for non-uniform hypergraphs.

Theorem 4. *An integer sequence $(d_1, d_2, \dots, d_n)$ such that $d_1 \leq d_2 \leq \dots \leq d_n$ and $n > 40$ is $\mathbb{N}$-Hamiltonian if the following conditions hold:*

$$d_i > 2^i \text{ for } 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{4}$$

$$d_{\frac{n-2}{2}} > 2^{\frac{n-2}{2}} + 1 \text{ if } n \text{ is even.} \tag{5}$$

In the following subsection, we show that the conditions of Theorem 3 and Theorem 4 are sharp.

1.2 Examples showing the sharpness of conditions

In this subsection, we show that it is impossible to strengthen Theorem 3 or Theorem 4 by changing a condition for a given i. We start with Theorem 3, and we show it is impossible to strengthen it by changing a condition for some fixed $i = k$, where $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Example 1 shows the sharpness of Condition 1 for all k, where $1 \leq k < r$.

Example 1. *For integers n, r and k, such that $n > 2r > 2k > 0$, let $\mathcal{H}_k^1$ be an n-vertex, r-uniform hypergraph. Let us partition the vertex set of $\mathcal{H}_k^1$, into two disjoint sets V_1 and V_2 of sizes k and $n-k$. The hyperedge set of $\mathcal{H}_k^1$ contains all hyperedges from $\binom{V_2}{r}$ and k distinct hyperedges all containing V_1 as a proper subset.*

Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the degree sequence of $\mathcal{H}_k^1$. We have $d_1 = d_2 = \cdots = d_k = k$ and $\binom{n-k-1}{r-1} \leq d_{k+1} \leq d_{k+2} \leq \cdots \leq d_n$. Observe that, for the degree sequence of hypergraph $\mathcal{H}_k^1$ all conditions hold from Theorem 3 except one, Condition 1 for $i = k$. In particular $d_k = k$ instead of the required $d_k > k$. Clearly $\mathcal{H}_k^1$ is non-Hamiltonian, since the vertices from V_1 are incident with only k hyperedges, therefore there is no Berge-cycle in $\mathcal{H}_k^1$ containing all vertices of V_1 longer than $|V_1| = k$, hence $\mathcal{H}_k^1$ is not Hamiltonian since $n > 2k$.

Example 2 shows the sharpness of Condition 2 for all k, $r \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. The idea of this construction is to construct a hypergraph with a special vertex set of size k adjacent with only k vertices.

Example 2. *For integers n, r and k, such that $3 \leq r \leq k < \frac{n}{2}$, let $\mathcal{H}_k^2$ be an n-vertex, r-uniform hypergraph. Let us partition the vertex set of $\mathcal{H}_k^2$, into three disjoint sets V_1, V_2 and V_3 of sizes $|V_1| = k$, $|V_2| = k$ and $|V_3| = n - 2k$. The hyperedges of $\mathcal{H}_k^2$ are*

$$E(\mathcal{H}_k^2) = \left\{ h \in \binom{V(\mathcal{H}_k^2)}{r} : (h \subset V_1 \cup V_2 \text{ and } |h \cap V_1| = 1) \text{ or } h \in V_2 \cup V_3 \right\}.$$

The degree sequence of $\mathcal{H}_k^2$ is $d_1 = d_2 = \cdots = d_k = \binom{k}{r-1}$, $d_{k+1} = \cdots = d_{n-k} = \binom{n-k-1}{r-1}$ and $d_{n-k+1} = \cdots = d_n = \binom{n-k-1}{r-1} + k\binom{k-1}{r-2}$. Observe that, for the degree sequence of the hypergraph $\mathcal{H}_k^2$ all conditions hold from Theorem 3 except one, Condition 2 for $i = k$. In particular we have $d_k = \binom{k}{r-1}$ instead of required $d_k > \binom{k}{r-1}$. Here we show that $\mathcal{H}_k^2$ is non-Hamiltonian. The vertices from V_1 are pairwise non-adjacent, and the number of vertices adjacent to V_1 is $k = |V_1|$, therefore there is no Berge-cycle in $\mathcal{H}_k^2$ containing all vertices of V_1 longer than $2|V_1| = 2k$, hence $\mathcal{H}_k^2$ is not Hamiltonian since $n > 2k$.

The next example shows the sharpness of Condition 3. The idea is very similar to the previous example.

Example 3. *For integers n and r, such that $2|n$, $3 \leq r < \frac{n}{2}$, let $\mathcal{H}^3$ be an n vertex, r-uniform hypergraph. Let us partition the vertex set of $\mathcal{H}^3$, into two disjoint sets V_1 and V_2 of sizes $|V_1| = \frac{n}{2}+1$ and $|V_2| = \frac{n}{2} - 1$. Let us fix a subset of V_1 of size r and denote it by h'. The hyperedge set of $\mathcal{H}^3$ is*

$$E(\mathcal{H}^3) = \left\{ h \in \binom{V(\mathcal{H}^3)}{r} : (h \subset V_1 \cup V_2 \text{ and } |h \cap V_1| \leq 1) \text{ or } h = h' \right\}.$$

The degree sequence of $\mathcal{H}^3$ is $d_1 = d_2 = \cdots = d_{\frac{n}{2}+1-r} = \binom{\frac{n}{2}-1}{r-1}$, $d_{\frac{n}{2}+2-r} = \cdots = d_{\frac{n}{2}+1} = \binom{\frac{n}{2}-1}{r-1} + 1$ and $d_{\frac{n}{2}+2} = \cdots = d_n = \binom{\frac{n}{2}-2}{r-1} + (\frac{n}{2}+1)\binom{\frac{n}{2}-2}{r-2}$. As one can observe all conditions hold for hypergraph $\mathcal{H}^3$ of Theorem 3 except Condition 3. Here we show that $\mathcal{H}^3$ is non-Hamiltonian and the number of vertices in V_1 is $\frac{n}{2}+1$. Therefore if there is a Hamiltonian Berge-cycle, then there should be at least two pairs of consecutive vertices from V_1 on the cycle. This is not possible since the number of hyperedges incident with at least two vertices of V_1 is just one.

These three examples show the sharpness of each condition of Theorem 3. Since the examples showing the sharpness of conditions of Theorem 4 are very similar we will skip them. Please see the proofs of Theorem 3 and Theorem 4 in manuscript [20].

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