Obstruction Set Constructibility of Minor-Closed Graph Classes[1]

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Abstract

We introduce a new operation between graph classes and we prove the existence of an explicitly constructive bound on the size of the obstruction set of the resulting class, i.e., the set of minor-minimal graphs not belonging to the class. We define the binary operation $\blacktriangleright$ between minor-closed graph classes as follows. Given a graph G and a set $X \subseteq V(G)$, the torso of X in G is the graph obtained from G after removing all vertices not in X and making adjacent each pair of non-adjacent vertices in X that are connected by a path whose internal vertex are not in X. Given two minor-closed graph classes $\mathcal{B}$ and $\mathcal{G}$, we set $\mathcal{B} \blacktriangleright \mathcal{G}$ to be the class of all graphs G for which there is some set $X \subseteq V(G)$ such that the torso of X in G belongs to $\mathcal{B}$ and the graph obtained from G after the removal of X belongs to $\mathcal{G}$. We prove that if the obstruction set of $\mathcal{B}$ contains at least one planar graph, then the size of the obstructions of $\mathcal{B} \blacktriangleright \mathcal{G}$ is bounded by a computable function of the size of the obstructions of $\mathcal{B}$ and $\mathcal{G}$.

1 Introduction

A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of vertex removals, edge removals, and edge contractions. Given a set of graphs $\mathcal{F}$, we denote by $\mathsf{excl}(\mathcal{F})$ the set containing every graph that excludes all graphs in $\mathcal{F}$ as minors. A graph class $\mathcal{G}$ is *minor-closed* if every minor of a graph in $\mathcal{G}$ is also a member of $\mathcal{G}$. The *(minor) obstrucion set* of a graph class $\mathcal{G}$ is the set $\mathsf{obs}(\mathcal{G})$ of all minor-mininal graphs that are not contained in $\mathcal{G}$. When $\mathcal{G}$ is minor-closed, the set $\mathsf{obs}(\mathcal{G})$ *completely* characterizes $\mathcal{G}$, as $\mathcal{G} = \mathsf{excl}(\mathsf{obs}(\mathcal{G}))$. By definition, no two elements of $\mathsf{obs}(\mathcal{G})$ are comparable with respect to the minor relation, and by Robertson-Seymour theorem [12] we know that there is no infinite set of graphs where every pair of graphs is non-comparable by the minor relation. This implies that for every graph class $\mathcal{G}$, the set $\mathsf{obs}(\mathcal{G})$ is always finite. Unfortunately, while we know that $\mathsf{obs}(\mathcal{G})$ is finite, there is no general way to construct this set, given some (finite) description of $\mathcal{G}$ [7] (see also [9, 10]). Therefore, we may resort to a case study of proving bounds on the size of $\mathsf{obs}(\mathcal{G})$ for particular instantiations of $\mathcal{G}$ (see [2, 11, 13, 1, 4, 5]). Having such a bound on $\mathsf{obs}(\mathcal{G})$, one may use the finite description of $\mathcal{G}$ in order to identify all obstructions of $\mathsf{obs}(\mathcal{G})$, by exhaustive search. As an attempt to enlarge the constructibility horizon of Robertson-Seymour theorem, researchers have considered several mechanisms to build minor-closed graphs classes from simpler ones. An interesting problem is whether it is possible to *construct* the obstruction of the new class given the obstructions of the simpler ones. To detect the widest possible set of operations between graphs classes that maintains this constructibility is an interesting challenge. We proceed with some definitions.

[1] The results of this extended abstract are based on `https://arxiv.org/abs/2111.02755`.

Basic definitions on graphs. For $S \subseteq V(G)$, we set $G[S]$ to be the graph induced by the vertices in S and use the shortcut $G \setminus S$ to denote $G[V(G) \setminus S]$. Given a class of graphs $\mathcal{G}$ we define $\mathbf{s}_{\mathsf{obs}}(\mathcal{G}) = \max\{|V(H)| \mid H \in \mathsf{obs}(\mathcal{G})\}$. Given a graph G, we denote by $\mathsf{cc}(G)$ the set of all connected components of G. For a graph G and a set $X \subseteq V(G)$, the *torso* of X in G is the graph $\mathsf{torso}(G, X)$ obtained by contracting each $C \in \mathsf{cc}(G \setminus X)$ to a vertex v_C, adding all edges between the neighborhood of each v_C, and finally removing all vertices v_C.

Constructive operations. Given an integer $r \geq 1$, a *graph class operation of arity* r is any function $\mathfrak{f} : (2^{\mathcal{G}_{\mathsf{all}}})^r \to 2^{\mathcal{G}_{\mathsf{all}}}$, where $2^{\mathcal{G}_{\mathsf{all}}}$ is the powerset of the class $\mathcal{G}_{\mathsf{all}}$ of all graphs. Such an operation $\mathfrak{f}$ is *minor-invariant* if whenever $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are minor-closed graph classes, then so is $\mathfrak{f}(\mathcal{G}_1, \ldots, \mathcal{G}_r)$. We say that a minor-invariant graph class operation $\mathfrak{f}$ is *explicitly constructive* if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that if $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are minor-closed graph classes, then $\mathbf{s}_{\mathsf{obs}}(\mathfrak{f}(\mathcal{G}_1, \ldots, \mathcal{G}_r)) \leq f(\max\{\mathbf{s}_{\mathsf{obs}}(\mathcal{G}_i) \mid i \in \{1, \ldots, r\}\})$.

It is easy to verify that the intersection operation $\cap$ is explicitly constructive. The case of the union operation $\cup$ is more difficult and has been studied by Adler, Grohe, and Kreutzer [2] – see also [11] where the notion of *intertwines* has been introduced. It has also been proved by Bulian and Dawar [3] that the operation $^{\mathsf{c}}$ defined as $\mathcal{G}^{\mathsf{c}} = \{G \mid \forall C \in \mathsf{cc}(G), C \in \mathcal{G}\}$ is also explicitly constructive. The same was proven recently in [6] for the operation $^{\mathsf{b}}$ defined as $\mathcal{G}^{\mathsf{b}} = \{G \mid \forall C \in \mathsf{bc}(G), C \in \mathcal{G}\}$, where $\mathsf{bc}(G)$ is the set of all blocks of G. We enlarge this set of operations by defining the graph class operation $\blacktriangleright$ as follows:

$$\mathcal{B} \blacktriangleright \mathcal{G} = \{G \mid \exists X \subseteq V(G), \mathsf{torso}(G, X) \in \mathcal{B} \ \wedge \ G \setminus X \in \mathcal{G}\}.$$

It is easy to prove that $\blacktriangleright$ is minor-invariant. Here we should stress that, alternatively, one might define $\mathcal{B} \blacktriangleright \mathcal{G}$ by replacing torso by torso^+, where $\mathsf{torso}^+(G, X)$ is defined as $\mathsf{torso}(G, X)$ with the difference that now we do not remove the contracted vertices v_C in the end. We can prove the following.

Theorem 1. *The operation $\blacktriangleright$ is explicitly constructive when restricted to $\mathcal{B}$'s where $\mathsf{obs}(\mathcal{B})$ contains some planar graph.*

2 Concluding remarks

As discussed above, the operation $\blacktriangleright$ can be seen as a way to create minor-closed classes by "composing together" simpler ones. The only previously known result about the constructibility of $\blacktriangleright$ follows from [2, 13] (see also [8]) for the case where $|X| \leq 1$, that is, when $\mathsf{obs}(\mathcal{B}) = \{K_2\}$. Theorem 1 extends this for every obstruction set containing some planar graph. It is an interesting question whether $\blacktriangleright$ remains explicitly constructive if we drop the planarity condition on $\mathsf{obs}(\mathcal{B})$. Finally, we wish to clarify that the upper bounds emerging from our proof are immense. It would be certainly desirable to give an estimation of an upper bound on $\mathbf{s}_{\mathsf{obs}}(\mathcal{B} \blacktriangleright \mathcal{G})$ as a "reasonable" function of $\mathbf{s}_{\mathsf{obs}}(\mathcal{B})$ and $\mathbf{s}_{\mathsf{obs}}(\mathcal{G})$.

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