Interval vertex coloring

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Abstract

A vertex k-coloring (not necessarily proper) is an open (resp. close) interval k-coloring if for every vertex v the set of colors used on the open (resp. close) neighborhood of v forms an interval of integers. For a given graph G the largest k for which there exists an open (resp. close) interval k-coloring is called the open (resp. close) interval chromatic number of G, and is denoted by $\chi_{io}(G)$ (resp. $\chi_{ic}(G)$). We present the exact values of open and close interval chromatic numbers for trees, cycles, complete graphs and wheels.

1 Introduction

The study of interval vertex coloring of graphs is based on a relatively studied problem of edge interval coloring. There was a lot of attention given to the interval edge coloring (the existence of a proper interval edge coloring of a given graph or finding the value or bounds for the corresponding chromatic index - see e.g. [1], [2], [3]), but the vertex version of this coloring was (as far as we know) not investigated.

In the case of interval vertex coloring of graphs we considered two cases: either we require that the set of colors used on the neighbors of each vertex forms an interval of integers, or the set of colors used on the neighbors together with the color of vertex itself forms an interval of integers. In the former case the corresponding coloring is called an *open interval vertex coloring*, and in the latter case it is called a *close interval vertex coloring*. We distinguish between proper and improper versions of both, open and close, interval vertex colorings. Hence, there are four types of interval vertex coloring in total.

In case of proper open and close interval coloring, already the problem determining whether a given coloring exists is intriguing. On the other hand, when considering improper versions of these colorings, each graph admits an open and close interval vertex coloring with one color. Therefore in these cases we want to maximize the number of colors used for the coloring of vertices.

The maximum number of colors for which there exists a (proper) open interval vertex coloring of a graph G is denoted by $\chi_{io}(G)$ (or $\chi_{io}^P(G)$, respectively). Similarly, the maximum number of colors for which there exists a (proper) close interval vertex coloring of G is denoted by $\chi_{ic}(G)$ (or $\chi_{ic}^P(G)$, respectively).

2 Interval vertex coloring of simple classes of graphs

For improper open and close interval vertex colorings, we have the following results for some special classes of graphs.

Theorem 1. *If P_n is a path on n vertices, then $\chi_{io}(P_n) = \chi_{ic}(P_n) = n$.*

Theorem 2. *Let C_n be a cycle on n vertices. Then*

$$\chi_{io}(C_n) = \begin{cases} k+1, & \text{if } n = 2k+1, \\ 2k+2, & \text{if } n = 4k \text{ or } n = 4k+2. \end{cases}$$

Theorem 3. *If C_n is a cycle on n vertices, $n \geq 4$, then $\chi_{ic}(C_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Theorem 4. *For a complete graph K_n, $n \geq 4$, $\chi_{io}(K_n) = \lfloor \frac{n+1}{2} \rfloor$ and $\chi_{ic}(K_n) = n$.*

Theorem 5. *Let W_n be a wheel graph on $n+1$ vertices. Then*

$$\chi_{io}(W_n) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{if } n \in \{4,5,6\}, \\ 5, & \text{if } n \geq 7. \end{cases} \qquad \chi_{ic}(W_n) = \begin{cases} 4, & \text{if } n \in \{3,4,5\}, \\ 5, & \text{if } n \in \{6,7,8\}, \\ 6, & \text{if } n \in \{9,10,11\}, \\ 7, & \text{if } n \geq 12. \end{cases}$$

3 Interval vertex coloring of trees

First we focus on the open interval vertex coloring of trees. It can be shown that not every graph is proper open interval vertex colorable.

Theorem 6. *A graph G admits a proper open interval vertex coloring if and only if G is bipartite. Moreover, $\chi_{io}(G) = \chi_{io}^P(G)$ in this case.*

As trees are bipartite graphs, it follows that they are colorable and our aim is to find the exact value of the open interval chromatic number of a given tree. A *caterpillar* is a tree in which all vertices are within distance 1 of a central path. A caterpillar H is called a *caterpillar subtree* of a tree T, if $\deg_H(x) \geq 2$ implies $\deg_H(x) = \deg_T(x)$ for each vertex x. We used caterpillar subtrees for determining the exact value of the open interval chromatic number of a tree.

Theorem 7. *Let T be a tree with parts X and Y. Let H and H' be two (not necessarily distinct or disjoint) caterpillar subtrees of T such that $|X_H|$ and $|Y_{H'}|$ are maximal. Then*

$$\chi_{io}(T) = |V(H) \cap X| + |V(H') \cap Y|.$$

In case of close interval vertex colorings, as every proper close interval vertex coloring of a graph G is a close interval vertex coloring of G, we have $\chi_{ic}^P(G) \leq \chi_{ic}(G)$. There are graphs where these two invariants are distinct (see Figure 1 for an example of such a graph). Moreover, there are graphs which do not admit proper close interval vertex coloring (see Figure 2).

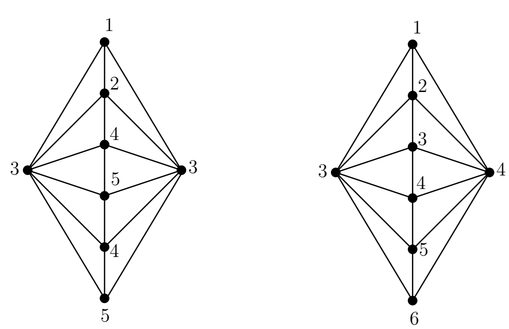

Figure 1: Graph G with $\chi_{ic}^P(G) = 5$ and $\chi_{ic}(G) = 6$.

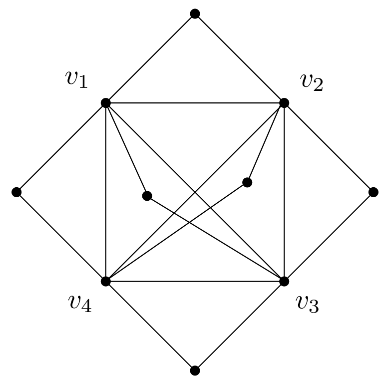

Figure 2: Graph for which there is no proper close interval vertex coloring.

For determining the close interval chromatic number of trees, we constructed an algorithm which computes the exact value of $\chi_{ic}(T)$. For an arbitrary maximal path P (see horizontal path on five vertices in Figure 3), we start with color 1 at one end of P. Next we continue in a "greedy" way − each yet uncolored vertex on P we color with the maximal possible color which guarantees the interval condition for already colored vertices on P and on pendant subtrees of these colored vertices.

We denote as $w(P)$ the color assigned to the other end-vertex of P. In our example $w(P) = 8$ but there is another maximal path Q with $w(Q) = 10$. It can be proved the following:

Theorem 8. *Let T be a tree. Then $\chi_{ic}(T) = \max_P w(P)$, where the maximum is computed through all maximal paths P of T.*

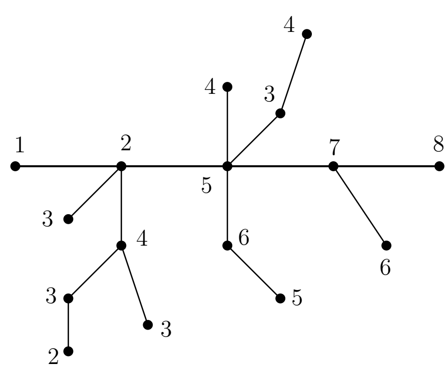

Figure 3: Example of computing $w(P)$ for the horizontal path P.

References

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