A generalization of Brussels sprout

Zin Mar Myint — Indian Institute of Technology Dharwad, India
Soura Sena Das — Indian Statistical Institue, Kolkata, India
Soumen Nandi — Institute of Engineering and Management, Kolkata, India
Sagnik Sen — Indian Institute of Technology Dharwad, India
Éric Sopena — Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, F-33400, Talence, France

Abstract

The Brussels sprout is a two-player pen and paper game which has connections with the structural properties of planar graphs. We generalize the game for all hereditary graph classes and study it for the family of forests, graphs on orientable and non-orientable surfaces of given genus ≥ 0, sparse planar graphs, etc. In the process, we also introduce a new game called Circular sprout and study it as a tool to solve problems on Brussels sprout.

1 Introduction

In 1967, Conway and Paterson [1] introduced the two-player pen and paper game called Sprout. The game starts with n spots (vertices) on a paper and two players place their moves alternately. A valid move consists of connecting any spot to itself or to another spot with a curve (edge) and then placing a new spot on the curve drawn (subdivision). There are two conditions that needs to be maintained during a move: the curve should not cross itself or any other curve and a spot can have at most three lines incident to it (degree three). The first player that cannot make a move, loses. The restrictions make it clear that the structure, thus obtained, remain planar throughout the game.

Since it is a finite game with no possibility for a draw, there must exist a winning strategy for either Player 1 or Player 2 based on the initial number of spots. It was conjectured in [2] that the first player has a winning strategy if and only if the number of spots, when divided by six, leaves a remainder of three, four, or five. To date, it has been possible to verify (mostly using computers) the correctness of the conjecture when the initial number of spots is $n \in \{1, 2, \cdots, 44, 46, 47, 53\}$(see [2]).

Conway later introduced an extension to Sprout, called Brussels sprout, possibly as a potential way to approach the study of Sprout. We look into its generalised version, where instead of crosses we have variable open tips for each spot. A set of graphs $\mathcal{F}$ is called a *hereditary class* if every graph isomorphic to an induced subgraph of a graph in $\mathcal{F}$ belongs to $\mathcal{F}$. We also restrict the intermediate steps to hereditary graph families.

Given a hereditary graph family $\mathcal{F}$, the game n-Brussels sprout for $\mathcal{F}$ with parameters $(t_1, t_2, \cdots, t_n)$, denoting it as $BS_n(\mathcal{F} : t_1, t_2, \ldots, t_n)$, as follows. The game $BS_n(\mathcal{F} : t_1, t_2, \ldots, t_n)$ starts with n spots, having $t_1, t_2, \cdots, t_n$ open tips, respectively. A valid move consists of joining two open tips with a curve followed by drawing a crossbar on the curve to create two new open tips. The graph obtained by considering the spots and intersections of a curve and a crossbar as vertices, and the curves joining two such vertices as edges, must remain inside the family $\mathcal{F}$ at all times. The first player unable to provide a valid move, loses. We will follow standard graph notation according to West [3] throughout this article unless otherwise stated.

In this article, Section 2 studies the game for forests, and surfaces. In Section 3, we study sparse planar graphs. In Section 4, we introduce and study a new game Circular sprout and conclude the article in Section 5.

2 Forests and graphs on surfaces

To begin the study, let us first consider the family of forests.

Theorem 1. *Let $\mathcal{F}_t$ be the family of forests. Then $BS_n(\mathcal{F}_t : t_1, t_2, ..., t_n)$ ends after exactly $n-1$ moves.*

Next we will move our attention to the family $\mathcal{O}_k$ of graphs that can be drawn on orientable surfaces of genus k without crossings.

Theorem 2. *Let $\mathcal{O}_k$ be the family of graphs that can be drawn on orientable surfaces of genus k without crossings. Then the only possible numbers of moves until the game $BS_n(\mathcal{O}_k : t_1, t_2, ..., t_n)$ terminates are*

$$(n-2) + 2j + \sum_{i=1}^{n} t_i,$$

where $j = 0, 1, \cdots, k$.

Proof. Suppose the game ends after x moves and let the resultant graph after the end of the game be G. Thus, $|V(G)| = n + x$ and $|E(G)| = 2x$ as we start with n vertices, 0 edges, and include exactly one vertex and two edges in each move.

Furthermore, we observe that the game cannot end if a particular face contains two or more open tips, while the last move involved in creating a particular face of G will ensure at least one open tip inside the face. Thus, the number of open tips is equal to the number of faces of G, that is, $|F(G)| = \sum_{i=1}^{n} t_i$.

Let j be least number for which G can be embedded on $\mathcal{O}_j$. Thus, G will satisfy the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2 - 2j$ for orientable surfaces. Hence we are done by replacing the values of $|V(G)|$, $|E(G)|$ and $|F(G)|$ in the above formula, and then by solving x. □

Recall that the orientable surface with genus 0 is nothing but the sphere, and thus, the family $\mathcal{O}_0$ of graphs are nothing but planar graphs.

Corollary 3. *The game $BS_0(\mathcal{O}_0 : t_1, t_2, \cdots, t_n)$ will end exactly after $(n-2) + \sum_{i=1}^{n} t_i$ moves and the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

When $k \geq 1$, even though the number of moves after which the game may end is not a constant, there is a fixed winner.

Corollary 4. *In the game $BS_n(\mathcal{O}_k : t_1, t_2, \cdots, t_n)$ for $k \geq 1$, the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

On a similar vein, we also study the family $\mathcal{N}_k$ of graphs that can be drawn on non-orientable surfaces of genus k without crossings.

Theorem 5. *Let $\mathcal{N}_k$ be the family of graphs that can be drawn on non-orientable surfaces of genus k without crossings. Then the only possible numbers of moves until the game $BS_n(\mathcal{N}_k : t_1, t_2, ..., t_n)$ terminates are*

$$(n-2) + j + \sum_{i=1}^{n} t_i,$$

where $j = 0, 1, \cdots, k$.

Recall that the non-orientable surface with genus 0 is the projective plane, and thus, the graphs of $\mathcal{N}_0$ are the projective planar graphs.

Corollary 6. *The game $BS_0(\mathcal{N}_0 : t_1, t_2, \cdots, t_n)$ will end exactly after $(n-2) + \sum_{i=1}^{n} t_i$ moves and the first player will win if and only if $n + \sum_{i=1}^{n} t_i$ is odd.*

However, for higher genus, unlike in the case of orientable surfaces, we do not have a fixed winner.

Question 1. *Which player has a winning strategy for the game $BS_n(\mathcal{N}_k : t_1, t_2, \cdots, t_n)$ when $k \geq 1$?*

3 Sparse planar graphs

In this section, let us focus on the family $\mathcal{P}_g$ of planar graphs with girth at least g.

Theorem 7. *Let $\mathcal{P}_g$ be the family of planar graphs having girth at least g. Then the game $BS_n(\mathcal{P}_g : t_1, t_2, \cdots, t_n)$ game ends exactly after $(n-1)$ moves for all $g \geq 2n+1$.*

Next, we focus particularly on the family of triangle-free planar graphs, that is, $\mathcal{P}_4$. We find upper and lower bounds of the number of moves after which the game $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ ends.

Theorem 8. *The maximum and the minimum number of moves after which the game $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ ends is $(n-2) + \sum_{i=1}^{n} t_i$ and $(4+n)$, respectively, where $n \geq 2$ and $t_i \geq 3$.*

A natural question we can ask here is whether there is a play of $BS_n(\mathcal{P}_4 : t_1, t_2, ..., t_n)$ that ends after $(n-2) + \sum_{i=1}^{n} t_i$ moves and one that ends after $(4+n)$ moves. In the next result, we will see that indeed for $n = 2$, such plays exist when the ratio of p and q is at most two.

Theorem 9. *There exists plays of $BS_2(\mathcal{P}_4 : p, q)$ which ends after $(p+q)$ and 6 moves, respectively, for $p \leq q \leq 2p$.*

From the above theorem, we can notice that the game $BS_2(\mathcal{P}_4 : p, q)$ does not have a clear winner, and depending on the play, either Player 1 or Player 2 can win. Therefore, studying which player has a winning strategy makes sense. However, we will use a potential general technique to attack this problem in the next section, which will also contain a proof of the following.

Theorem 10. *There exists a winning strategy for Player 2 in the game $BS_2(\mathcal{P}_4 : p, q)$ for all $p, q \geq 3$.*

4 The Circular sprout game and the Proof of Theorem 10

While studying the game $BS_2(\mathcal{P}_4 : p, q)$, we encountered another similar game which we found to be interesting. Let us define this new game independently, and in a generalized form, even though in this article we will study only a specific restriction of it which will help us in proving Theorem 10.

This new game, named the n-Circular sprout game for the family $\mathcal{F}$ with parameters $(t_1, t_2, \cdots, t_n)$, is denoted by $CS_n(\mathcal{F} : t_1, t_2, \cdots, t_n)$, The initial set up of this game consists of n spots $v_1, v_2, \cdots, v_n$ arranged in a clockwise order on the perimeter of a circle with v_i having t_i open tips coming out in the interior of the circle. The rest of the rules of the game is the same as Brussels sprout with the following added constraint: the curves drawn by the players must be entirely contained in the interior of the circle.

Next let us observe how this game is related to $BS_2(\mathcal{P}_4 : p, q)$.

Lemma 1. *The game obtained after two moves in the game $BS_2(\mathcal{P}_4 : p, q)$ can be expressed as the sum of $CS_4(\mathcal{P}_4 : i-2, 1, j-2, 1)$ and $CS_4(\mathcal{P}_4 : p-i, 1, q-j, 1)$ for some $i \in \{2, 3, \cdots, p\}$ and $j \in \{2, 3, \cdots, q\}$.*

Note that all the games discussed here are two player finite impartial games, and thus their *nimber* [4] can be calculated which is enough to decide who has a winning strategy. Let $\eta[X]$ denote the nimber of the game X.

Theorem 11. *For all $0 \le p \le q$, we have*

$$\eta[CS_4(\mathcal{P}_4 : p, 1, q, 1)] = \begin{cases} 1 & \textit{if } p = q, \\ \frac{4}{5}(p+q-i) + 2\lfloor \frac{i}{4} \rfloor & \textit{if } p < q < 2p - \lfloor \frac{|p-2|}{2} \rfloor \textit{ where } i \equiv p + q (mod\ 5) \\ 2p & \textit{if } q \ge 2p - \lfloor \frac{|p-2|}{2} \rfloor. \end{cases}$$

where $1, 2, 3, 4, 5$ are the representative of the integers modulo 5.

The proof of the above theorem is lengthy and novel and is a major contribution of this article.

Corollary 12. *The nimber of the game $BS_2(\mathcal{P}_4 : p, q)$ is 0 for all $p, q \ge 3$.*

Proof of Theorem 10. As $\eta[BS_2(\mathcal{P}_4 : p, q)] = 0$ for all $p, q \ge 3$, the second player must have a winning strategy. □

5 Conclusions

We studied two variants of the combinatorial game Sprout, namely, the previously known Brussels sprout–albeit a generalization introduced by us, and the Circular sprout– a related game introduced as a tool to study a special case of Brussels sprout. We ended up discovering that Circular sprout is interesting on its own, and we propose to study it in general. Moreover, we also think that Circular sprout is potentially a tool to attack the long standing Sprout Conjecture.

We have omitted several proofs and other details due to space constraints. Interested readers are encouraged to find the detailed proofs in: `https://homepages.iitdh.ac.in/~sen/Zin_ICGT.pdf`.

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