Waypoint routing on bounded treewidth graphs[1]

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Abstract

In the Waypoint Routing Problem one is given an undirected capacitated and weighted graph G, a source-destination pair $s, t \in V(G)$, and a set $W \subseteq V(G)$ of *waypoints*. The task is to find a walk that starts at the source vertex s, visits, in any order, all waypoints, ends at the destination vertex t, respects edge capacities, that is, traverses each edge at most as many times as its capacity, and minimizes the cost computed as the sum of costs of traversed edges with multiplicities.

We study the problem for graphs of bounded treewidth, prove that the problem is fixed-parameter tractable with respect to this parameter, and present a new algorithm for the problem working in $2^{\mathcal{O}(\mathrm{tw})} \cdot n$ time, significantly improving upon the previously known XP algorithms.

To complement our algorithmic results, we show that the running time of the algorithms is optimal for the problem under the Exponential Time Hypothesis (ETH). Finally, we show that, under reasonable theoretical assumptions, the problem does not admit a polynomial kernel with respect to the treewidth of the input graph.

1 Introduction

We study the Waypoint Routing Problem, which can be formally defined as follows:

Waypoint Routing Problem (WRP)

Input:	An undirected, simple, capacitated and weighted graph $G = (V, E, \kappa, \omega)$, where $\kappa\colon E \to \mathbb{N}$ and $\omega\colon E \to \mathbb{N}$, a source-destination pair $s, t \in V$, and a set $W \subseteq V$ of *waypoints*.
Task:	Find a s,t-walk R that visits, in any order, all waypoints $w \in W$, traverses each edge at most $\kappa(e)$ times, and the cost of R, computed as the sum of weights of all edges in a solution with multiplicities, is minimal over all satisfying walks.

The problem is motivated by modern networking systems that connect distributed network functions, often composed of middleboxes, possibly virtualized, such as service chaining, hybrid software-defined networks, or segment routing. Moreover, WRP is also interesting from the theoretical point of view as it is a natural generalization of the famous Traveling Salesperson Problem and Subset TSP.

The study of (this variant of) the Waypoint Routing Problem was initiated by Amiri et al. [1]. Inspired by the result of Rost et al. [7], who showed that many natural network topologies such as backbone transit and wide area networks have small treewidth (i.e., they are in certain sense similar to trees), Amiri et al. [1] analyzed the parameterized complexity of the problem with respect to the treewidth (denoted tw) of the input graph. They presented a dynamic programming algorithm with running time $n^{\mathcal{O}(\mathrm{tw}^2)}$.

[1]The algorithmic part of this work has been already published in [8]. Kernelization results are novel.

2 Preliminaries

We assume that the reader is familiar with basic notions of graph theory. A language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed finite alphabet, is called *parameterized problem* and $(x, k) \subseteq \Sigma^* \times \mathbb{N}$, where x is called *input* and k is a *parameter*, is an *instance* of the problem L. A parameterized problem L is called *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathbb{A}$ that correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$, where $f\colon \mathbb{N} \to \mathbb{N}$ is a computable function and $c \in \mathbb{N}$ is a constant.

A *tree decomposition* of a graph $G = (V, E)$ is a pair $\mathcal{T} = (T, \beta)$ where T is a tree and $\beta\colon V(T) \to 2^V$ is a function that associates with each node $t \in V(T)$ a vertex subset $B_t \subseteq V(G)$, called a *bag*, such that a) for each edge $\{u, v\} \in E$, there is a node t of T such that $\{u, v\} \in B_t$, and b) for each $v \in V$ the nodes t such that $v \in B_t$ induce a connected subtree of T. The *width* $w(\mathcal{T})$ of a tree decomposition $\mathcal{T}$ is $\max_{t \in V(T)} \{|B_t| - 1\}$. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width of a decomposition over all tree decompositions of G.

3 The Algorithm

We can show that every instance of the Waypoint Routing Problem can be (in polynomial time) turned into the *unified* one, which is a connected multigraph with unary capacities, parallel edges have the same weight, there are at most two parallel edges between any pair of vertices, $s = t$, and $s \in W$. Hence, for the remainder of our paper, we assume that an instance given as an input is unified.

To simplify the description of the algorithm, we will describe the computation in terms of a *nice tree decomposition* [3, p. 168] of the underlying graph G. We will compute partial solutions for subgraphs corresponding to a node in a tree decomposition by combining partial solutions for subgraphs corresponding to its children. To make the definition more intelligible, we introduce the following notation. Let $G = (V, E)$ be a graph, and $\mathcal{T} = (T, \beta)$ be a nice tree decomposition for G rooted in node r. For any node $x \in T$ we will denote by $G_x = (V_x, E_x)$ the subgraph induced by a tree decomposition node x with $V_x = \bigcup_{y \text{ is a descendant of } x} B_y$, and $E_x = \{e \in E \mid e \text{ is introduced in any descendant of } x\}$. We consider all $x \in T$ to be a descendant of itself. Now, we define a partial solution in subgraphs induced by a node of a tree decomposition.

Definition 1. *Let $I = (G, \kappa, \omega, s, t, W)$ be a unified instance of WRP, $\mathcal{T} = (T, \beta)$ be a nice tree decomposition of G, $x \in T$. For every $X \subseteq B_x$ with $(W \cap B_x) \subseteq X$ and every $L \subseteq X$ we call $S = (X, L)$ a* presignature *at x. For a presignature $S = (X, L)$ at x and partition $\mathcal{P}$ of X we call $(X, L, \mathcal{P})$ a* solution signature *at x.*

A subgraph $H \subseteq G_x$ is a partial solution *compatible with solution signature $(X, L, \mathcal{P})$ at x, if all the following conditions are met.*

(i) every already introduced waypoint $w \in W \cap V_x$ is present in $V(H)$,

(ii) $V(H) \cap B_x = X$,

(iii) a vertex $v \in V(H)$ has an odd degree in H if and only if $v \in L$, and

(iv) for every connected component C of H we have $V(C) \cap B_x \neq \emptyset$ and $V(C) \cap B_x$ is in $\mathcal{P}$.

The signature of a solution allows us to recognize partial solutions that are equivalent from a global perspective. Our dynamic programming algorithm then works in a bottom-up manner (from leaf nodes to root node) and computes for every signature a minimal weight partial solution.

After completion of this procedure, we ask for the weight computed for signature $(\{s\}, \emptyset, \{\{s\}\})$ at the root r. It is easy to verify that a subgraph of $G = G_r$ is compatible with this signature if and only if it is connected, even, and contains all waypoints. This approach finds an optimal solution, but the running time of this procedure is suboptimal, since we have to store the weight of an optimal partial solution for every signature and the number of partitions is large. To improve the running time of our algorithm, we use the framework of Boadlaender et al. [2]. This allows us, instead of considering all possible partitions, to limit ourselves to *representative sets* of weighted paritions that contains all the needed information. This gives us the main result.

Theorem 1. *There exists an algorithm that, given an instance $(G, \kappa, \omega, s, t, W)$ of the* Waypoint Routing Problem*, solves it in $2^{\mathcal{O}(\mathrm{tw})} \cdot n$ time.*

4 ETH Lower Bound

The Exponential Time Hypothesis (ETH for short) introduced by Impagliazzo and Paturi [5] states that there is a constant $\delta_3 > 0$ such that there is no algorithm for 3-SAT with running time $2^{\delta_3 n} m^{\mathcal{O}(1)}$, where n is the number of variables and m is the total length of the input formula. To prove the optimality of our algorithm, we will use the following ETH implication.

Theorem 2 (Impagliazzo, Paturi, and Zane [6])**.** *Unless ETH fails,* Hamiltonian Cycle *admits no algorithm working in $2^{o(n+m)}$ time, where n and m are the number of vertices and edges of the input graph, respectively.*

Using Theorem 2, we are able to easily prove the lower bound on the running time of any algorithm solving the Waypoint Routing Problem and, in particular, show that, unless ETH fails, the running time of our algorithm from Theorem 1 is optimal.

Theorem 3. *Unless ETH fails, there is no algorithm for the* Waypoint Routing Problem *working in $2^{o(n+m)}$ time and, in particular, none working in $2^{o(\mathrm{tw}(G))} \cdot n^{\mathcal{O}(1)}$ time, where n, m, and $\mathrm{tw}(G)$ are the number of vertices, edges, and the treewidth of the input graph, respectively.*

5 No Polynomial Kernel with Respect to Treewidth

Kernelization is a formal approach to study preprocessing soubroutines that, given an instance of a computational problem, produces, in polynomial time, an equivalent instance of smaller size. Using this approach, we can quickly eliminate the "easy parts" of the problem and focus on the "hard" *kernel* of the problem. The ultimate goal of this preprocessing phase is to produce an equivalent instance whose size is polynomial (or even linear) with respect to the assumed parameter. By showing that some problem admits a polynomial kernel, we automatically show that this problem is fixed-parameter tractable. However, not all fixed-parameter tractable problems admit a polynomial kernel, which is also the case in this work. For a more comprehensive introduction to kernelization, we refer the reader to the monograph of Fomin et al. [4].

Theorem 4. *There is no polynomial kernel for the* Waypoint Routing Problem *with respect to the fractioning number of the input graph, unless the polynomial hierarchy collapses.*

To prove Theorem 4, we show that Hamiltonian Path AND-cross-composes to WRP parameterized by the fractioning number of the input graph. It is not hard to see that every graph with a bounded fractioning number has a bounded treewidth. Therefore, Theorem 4 directly implies our final result, showing that it is unlikely that the Waypoint Routing Problem admits a polynomial kernel with respect to the treewidth of the input graph.

Corollary 1. *There is no polynomial kernel for the* Waypoint Routing Problem *with respect to the treewidth of the input graph, unless the polynomial hierarchy collapses.*

6 Open Problems

An interesting open problem is to determine the complexity of the problem in directed graphs with underlying undirected graphs of a small treewidth. Although the correspondence with degree-constrained submultigraphs via Eulerian trails is still valid in directed graphs, it is no longer true that each edge is traversed at most twice in an optimal walk. It is easy to find instances in which a particular edge must be traversed as many times as $n-1$, where n is the number of vertices, in any feasible walk. Furthermore, while in the final submultigraph the indegree of each vertex must be equal to its outdegree, for a partial solution the difference between these two degrees can be arbitrarily large. This makes the problem more challenging in directed graphs.

Acknowledgements

This work was supported by the OP VVV MEYS funded project CZ.02.1.01/0.0/0.0/16_019/0000765 “Research Center for Informatics” and by the Grant Agency of the Czech Technical University in Prague, grant No. SGS20/208/OHK3/3T/18.

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