A Note on Generalized Majority Colorings

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Abstract

A *majority coloring* of a directed graph is a vertex coloring in which each vertex has the same color as at most half of its out-neighbors. In this note we simplify some proof techniques and generalize previously known results on various variants of majority coloring. In particular, our unified and simple approach gives the best known results for:

- directed and undirected graphs,
- $\frac{1}{k}$-majority colorings (each vertex has the same color as at most $\frac{1}{k}$ of its out-neighbors),
- weighted edges,
- list colorings (choosability),
- on-line list colorings (paintability),
- non-uniform list lengths,
- *ranked* colors.

1 Introduction

Let $D = (V, E)$ be a directed graph. Let $d^+(v)$ denote the number of out-neighbors of a vertex v. A coloring c of the vertices of D is called a *majority coloring* if for every vertex v the number of its out-neighbors in color $c(v)$ is at most $\frac{1}{2}d^+(v)$. This concept was studied by Kreutzer, Oum, Seymour, van der Zypen, and Wood [11]. It is proved there, among other results, that every directed graph is majority 4-colorable. It is conjectured that 3 colors are sufficient for majority coloring of any directed graph and this would be the best possible.

This problem was initially considered for undirected graphs, where the majority condition states that every vertex needs at least as many neighbors in a different color than its own. Lovász [12] proved that every finite graph is majority 2-colorable (although the theorem was not stated explicitly).

We can naturally generalize the majority coloring constraint for weighted directed graphs. Let σ be a vertex weighting that assigns real weight $0 \leq \sigma(v) \leq 1$ to every vertex v, and τ be an edge weighting that assigns non-negative real weight $\tau(e)$ to every edge e. Coloring c is a (σ, τ)-*majority coloring* if for every vertex v we have that at most $\sigma(v)$-fraction of τ-weighted out-edges of v are monochromatic, i.e.:

$$\forall_{v \in V} \frac{\sum_{e \in E, e = v \to w, c(v) = c(w)} \tau(e)}{\sum_{e \in E, e = v \to w} \tau(e)} \leq \sigma(v).$$

When σ assigns value s uniformly for every vertex v, or τ assigns value t uniformly for every edge e, we can say that a (σ, τ)-majority coloring is an (s, t)-majority coloring. Observe that the regular majority coloring corresponds to $(\frac{1}{2}, 1)$-majority coloring.

Suppose that each vertex v of a directed graph D is assigned with a list of colors $L(v)$. Then D is (σ, τ)-majority colorable from these lists if there is a (σ, τ)-majority coloring c of D with $c(v) \in L(v)$ for every vertex v. If D is (σ, τ)-majority colorable from any lists of size k, then we say that D is (σ, τ)-majority k-choosable. Anholcer, Bosek and Grytczuk [3] showed that every directed graph is $(\frac{1}{2}, 1)$-majority 4-choosable. Their techniques can also give that for every integer $k \geq 1$, every directed graph is $(\frac{1}{k}, 1)$-majority k^2-choosable. Later, Girão, Kittipassorn and Popielarz [8], and independently Knox and Šámal [10] showed that every directed graph is $(\frac{1}{k}, 1)$-majority $2k$-choosable. It is possible that every directed graph is $(\frac{1}{k}, 1)$-majority $(2k-1)$-choosable. This would be the best possible and it is unknown if it holds even for $k = 2$. Some evidence supporting this conjecture is given by Anastos, Lamaison, Steiner and Szabó [2]. In the case of undirected graphs, the proof of Lovász [12] mentioned above extends easily on the list version of the problem (i.e., every finite graph is $(\frac{1}{2}, 1)$-majority 2-choosable).

The problems of majority coloring and majority list coloring were also considered for infinite graphs and directed graphs. The famous *Unfriendly Partition Conjecture* by Cowan and Emerson ([7], see [1]) states that every countable graph is majority 2-colorable. It was proved for graphs with finitely many vertices of infinite degree by Aharoni, Milner and Prikry [1], for rayless graphs by Bruhn, Diestel, Georgakopoulos, and Sprüssel [6] and for graphs not containing an infinite clique subdivision by Berger [5]. On the other hand, Shelah and Milner [13] showed that every infinite graph is majority 3-colorable and that there are uncountable graphs for which 3 colors are necessary. Anholcer, Bosek and Grytczuk [4] proved that every countable graph is $(\frac{1}{2}, 1)$-majority 4-choosable) and that the same list size suffices for each countable directed graph. Recently Haslegrave [9] improved the result for graphs, showing that every countable graph is $(\frac{1}{2}, 1)$-majority 3-choosable) and that the same length of the list is enough also for countable directed acyclic graphs.

A natural generalization of list colorings is the concept of *paintability* (also called on-line list coloring). Let λ be a function that assigns a positive integer $\lambda(v)$ to every vertex v. The (σ, τ)-majority λ-painting game on D is a game played in rounds by two players: Lister and Painter. The i-th round starts with Lister presenting a subset X_i of vertices of D. Then, Painter selects a subset $Y_i \subseteq X_i$ so that all vertices in Y_i can receive the same color in (σ, τ)-majority coloring of D, i.e.:

$$\forall_{v \in Y_i} \frac{\sum_{e \in E, e = v \to w, w \in Y_i} \tau(e)}{\sum_{e \in E, e = v \to w, w \in V} \tau(e)} \leq \sigma(v),$$

and assigns color i to all the vertices in Y_i. Painter wins the game if after some round all the vertices are colored, i.e. $\bigcup_i Y_i = V$. Lister wins the game if after some round each vertex v was presented at least $\lambda(v)$ many times, and not all the vertices are colored, i.e.:

$$\begin{aligned} &\forall_{v \in V} \left|\{i : v \in X_i\}\right| \geq \lambda(v), \text{ and} \\ &\exists_{v \in V} v \notin \textstyle\bigcup_i Y_i. \end{aligned}$$

We say that D is (σ, τ)-majority λ-paintable if Painter has a winning strategy in the corresponding painting game on D. When λ assigns value k uniformly to every vertex v, then a winning strategy for Painter constructs a (σ, τ)-majority coloring from any lists of size k and implies (σ, τ)-majority k-choosability.

2 Results

Our main contribution is the following lemma (of *kernel* flavor) that allows for a construction of easy, yet effective, strategies for Painter in (σ, τ)-majority λ-painting games on undirected graphs.

Intuitively, the lemma gives a good Painter response Y to any Lister move X. For every vertex v in X we have that: either v is in Y (and gets colored instantly), or many (a reasonable fraction of the weighted edges) out-neighbors of v are in Y (and this can happen only limited number of times).

Lemma 1. *Let $G=(V,E)$ be an undirected graph, σ be a vertex weighting that assigns real weight $0 \le \sigma(v) \le 1$ to every vertex v, and τ be an edge weighting that assigns non-negative real weight $\tau(e)$ to every edge e. For every subset $X \subseteq V$ of vertices there exists a subset $Y \subseteq X$ such that for every vertex $v \in X$ we have:*

$$v \in Y \iff \frac{\sum_{e\in E, e=vw, w\in Y} \tau(e)}{\sum_{e\in E, e=vw, w\in V} \tau(e)} \le \sigma(v).$$

Applying Lemma 1 directly in every round of the painting game gives the following.

Corollary 1 (Undirected Paintability)**.** *Every undirected graph with any non-negative edge weighting τ is $(\frac{1}{k},\tau)$-majority k-paintable.*

Lemma 1 combined with some of the ideas from Girão, Kittipassorn and Popielarz [8], and Knox and Šámal [10] allows to derive the following strengthening of their results.

Corollary 2 (Directed Paintability)**.** *Every directed graph with any non-negative edge weighting τ is $(\frac{1}{k},\tau)$-majority $2k$-paintable.*

We can also easily improve on the ideas from Anholcer, Bosek and Grytczuk [3]:

Corollary 3 (Ranked Colors)**.** *Let G be an undirected graph with any non-negative edge weighting τ. Suppose that each vertex v is assigned with a list $L(v)$ of colors. Suppose further that for each vertex v, each color x in $L(v)$ is assigned a real number $r_v(x)$, the* rank *of color x in $L(v)$. Assume that for every vertex v, the color ranks $r_v(x)$ satisfy the following condition:*

$$\sum_{x\in L(v)} r_v(x) \ge \sum_{e\in E, e=vw} \tau(e).$$

Then there is a vertex coloring of G from lists $L(v)$ satisfying the following constraint: If x is a color assigned to v, then the sum of weights of edges connecting v to a neighbor in color x is at most $r_v(x)$.

Corollary 4 (Non-uniform List Lengths)**.** *Let $\lambda(v)$ be a positive integer for each vertex v of an undirected graph G. Set $\sigma(v)=\frac{1}{\lambda(v)}$. For any non-negative edge weighting τ, G is (σ,τ)-majority λ-paintable.*

Both Corollary 3 and 4 have their directed analogues with an additional multiplicative factor of 2 (as in Corollary 2 compared to Corollary 1).

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