The smallest 5-chromatic tournament[1]

Thomas Bellitto — Sorbonne Université, CNRS, LIP6, F-75005 Paris, France
Nicolas Bousquet — CNRS, Univ. Lyon, Université Lyon 1, LIRIS UMR 5205, F-69621, Lyon, France
Adam Kabela — Faculty of Applied Sciences, University of West Bohemia, Pilsen, Czech Republic
Théo Pierron — Univ. Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France

Abstract

A dicoloring of a directed graph, as introduced by Neumann-Lara, is a partition of the graph in sets of vertices that induce graphs with no directed cycles. A directed graph is said to be oriented if there exists at most one arc between each pair of vertices. While it is obvious that the complete directed graph with n vertices is the smallest n-chromatic graph, finding the smallest oriented graph that requires n colors is a very interesting and challenging problem. It is already known that the smallest 2-, 3- and 4-chromatic oriented graphs have respectively 3, 7 and 11 vertices. However, the question of the smallest oriented graph of dichromatic number 5 has been open since 1994, when Neumann-Lara conjectured that the answer was 17. We solve the problem by proving that it actually has size 19.

Proper coloring of undirected graphs lies among the most studied problems in graph theory. It asks to color vertices while giving different colors to adjacent ones. In [11], Neumann-Lara introduced a generalization of this problem to directed graphs. When walking in an undirected graph, an undirected edge between two vertices u and v can be used both to go from u to v and from v to u, while in a directed graph, an arc uv can only lead from u to v. As such, undirected graphs can be seen as a special case of directed graphs where for every arc uv, there also exists an arc vu (such graphs are called symmetric directed graphs). Neumann-Lara's generalized coloring of directed graphs requires that there exists no monochromatic closed walk in the graph *i.e.* no walk that starts and ends on the same vertex and only uses vertices of the same color. Like in the undirected case, a pair of arcs uv and vu enforce that u and v receive different colors since they form a closed walk. Such a pair or arc is called a **digon**. However, if there is no arc vu, u and v may receive the same color even if there is an arc uv, as long as there is no monochromatic walk from v to u. The smallest number of colors required for properly coloring a directed graph is called its dichromatic number is sometimes denoted $\overrightarrow{\chi}$. While there exist other generalizations of coloring to directed graphs (for example based on graph homomorphisms, see [4]), Neumann-Lara's is the best-known and has received evergrowing attention since its introduction.

Since digons are similar to undirected edges, their impact on colorability of graphs has already been studied very extensively. Hence, many papers on directed coloring forbid them and only focus on oriented graphs *i.e.* graphs where there can be at most one arc between two given vertices. One can illustrate the difference with the following: for any integer k, the smallest directed graph of dichromatic number k is easily constructed as the complete graph on k vertices (complete means

[1]The first author was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme Grant Agreement 714704. The third and fourth authors were supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University. The third author was also supported by the project GA20-09525S of the Czech Science Foundation. The second and fourth authors were supported by ANR project GrR (ANR-18-CE40-0032).

that for every pair of vertices u, v, there is both an arc uv and an arc vu). However, the question becomes much harder when digons are forbidden. Actually, determining the size of the smallest oriented graph of dichromatic number k was raised by Neumann-Lara in 1982 [12].

Many other related questions have also been studied in the literature. For instance, Bang-Jensen *et al.* establish some structural results about **k-critical directed graphs** in [2], *i.e.* directed graphs with dichromatic number k that are minimal by inclusion. The average degree of such graphs was also source of attention in recent years. In [8, 10], the authors provide some bounds on the smallest possible value of this parameter among all k-critical directed graphs on n vertices. Note that the question is easily answered without the dependency in n since each vertex of a k-critical graph needs to have in- and out-degree at least $k-1$ and this value is reached by complete graphs on k vertices. But here again, the question becomes much harder when digons are forbidden: the smallest average degree of oriented k-critical graphs is still open even for $k = 3$ [1].

These works are also reminiscent of numerous works in the undirected case that look for the smallest graph of chromatic number k that does not contain any complete subgraph of size c. The problem has been especially well-studied for triangle-free graphs (the case $c = 3$), since this is the smallest value of c that makes the problem non-trivial. In [3], Chvátal proved that the smallest triangle-free k-chromatic graph has size 11 for $k = 4$ and Jensen and Royle proved in [9] through a computer search that it has size 22 for $k = 5$. The question is still open for $k = 6$ where Goedgebeur proved in [7] that it is between 32 and 40. For directed graphs, forbidding cliques of size $c = 2$ corresponds to considering oriented graphs, and actually yields again Neumann-Lara's question.

Observe that adding arcs to a graph cannot decrease its dichromatic number. Therefore, for every oriented graph, one can construct another one with same dichromatic number where every pair of vertices is linked by exactly one arc. Such graphs are called **tournaments**. As such, the search of the smallest oriented graph of dichromatic number k can be restricted to tournaments. A tournament or a vertex set in a tournament is **transitive** if for every arc uv and arc vw, the arc between u and w is from u to w. One can easily observe that coloring a tournament with k colors amounts to partition its vertices into transitive subtournaments. Thus, our question can be rephrased as finding the smallest tournament that cannot be partitioned into $k-1$ transitive sets. This formulation connects this problem to questions that Erdős and Moser raised 20 years before Neumann-Lara's definition of directed coloring in [5].

The question was asymptotically solved since the maximum chromatic number of a tournament on n vertices is $\Theta(\frac{n}{\log n})$ [6, 5]. However, the question of finding the size of such tournaments for small values of k is still mostly open. The smallest tournament of dichromatic number 2 (*i.e.* non-transitive) is the directed cycle of size 3. The constructions for $k = 4, 5$ rely on the so-called Paley tournaments. For every prime integer n of the form $4k + 3$, the **Paley tournaments** on n vertices P_n is the tournament whose vertex set is $\{0, ..., n-1\}$ and containing the arc ij if and only if $i - j$ is a square modulo n. In [12], Neumann-Lara proved that the smallest tournament of dichromatic number 3 has size 7 and that there exists 4 such tournaments, including P_7. He also proved that the smallest tournament of dichromatic number 4 has size 11, is unique and is P_{11}.

In the conclusion of [12], Neumann-Lara discussed future works about the size of the smallest 5-chromatic tournaments. He claimed to know that the answer is between 17 and 19 and conjectured that it is 17. To the best of our knowledge, no proof of these bounds has been published. Note that the next natural candidate, namely the Paley tournament P_{19} is actually 4-colorable.

Moreover, the number of non-isomorphic tournaments on 17, 18 and 19 vertices have respectively 27, 31 and 35 digits [13], generating them up to isomorphism is already a very challenging task and the problem of 5-colorability that we need to solve on each of them is NP-complete. Therefore, it is definitely out of question to solve the problem by bruteforce.

To this day, almost 30 years after the question was raised, the question of the smallest tournament of dichromatic number 5 is still open, despite the efforts of many authors. The question still appears frequently in the literature, see for example [2, 10] for recent examples, where this problem is presented as an open question in conclusion.

We provide a definitive answer to Neumann-Lara's question for $k = 5$. Building on the known structural results on smallest tournaments with fixed chromatic number or avoiding a transitive subtournament of a given size, we disprove Neumann-Lara's conjecture by showing the following.

Theorem 1. *Every 17-vertex tournament is 4-colorable.*

The proof relies on a surprising following intermediate result proved by a computer analysis, and from which we derive our result in a human-readable way.

Theorem 2. *Every 4-chromatic tournament on 12 vertices contains* P_{11}.

Using slightly more involved arguments, we extend the computer search to tournaments on 18 vertices. The analysis still does not provide a 5-chromatic tournament.

Theorem 3. *Every 18-vertex tournament is 4-colorable.*

Finally, we exhibit an example of a 5-chromatic tournament on 19 vertices, which settles 19 as the right answer to Neumann-Lara's question, see Figure 1.

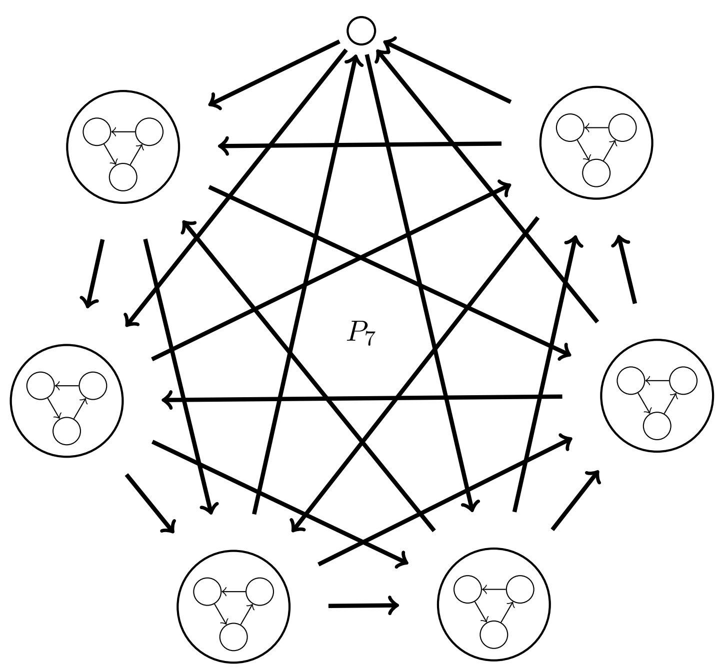

Figure 1: A 5-chromatic graph on 19 vertices. Thick arrows induce P_7 and represent (3 or) 9 arcs with the same direction.

References

[1] Pierre Aboulker, Thomas Bellitto, Frédéric Havet, and Clément Rambaud. Smallest possible average-degree of 3 critical oriented graphs. To appear.

[2] Jørgen Bang-Jensen, Thomas Bellitto, Thomas Schweser, and Michael Stiebitz. Hajós and ore constructions for digraphs. *Electron. J. Comb.*, 27(1):P1.63, 2020.

[3] Václav Chvátal. The smallest triangle-free-4-chromatic regular graph. *Journal of Combinatorial Theory*, 9:93–94, 07 1970.

[4] Bruno Courcelle. The monadic second order logic of graphs VI: on several representations of graphs by relational structures. *Discret. Appl. Math.*, 54(2-3):117–149, 1994.

[5] Paul Erdős and Leo Moser. On the representation of directed graphs as unions of orderings. *Math. Inst. Hung. Acad. Sci*, 9:125–132, 1964.

[6] Paul Erdős. Problems and results in number theory and graph theory. In *Proceedings of the 9th Manitoba Conference on Numerical Mathematics and Computing*, pages 3–21, 1979.

[7] Jan Goedgebeur. On minimal triangle-free 6-chromatic graphs. *J. Graph Theory*, 93(1):34–48, 2020.

[8] Richard Hoshino and Ken-ichi Kawarabayashi. The edge density of critical digraphs. *Comb.*, 35(5):619–631, 2015.

[9] Tommy R. Jensen and Gordon F. Royle. Small graphs with chromatic number 5: A computer search. *J. Graph Theory*, 19(1):107–116, 1995.

[10] Alexandr V. Kostochka and Michael Stiebitz. The minimum number of edges in 4-critical digraphs of given order. *Graphs Comb.*, 36(3):703–718, 2020.

[11] Victor Neumann-Lara. The dichromatic number of a digraph. *J. Comb. Theory, Ser. B*, 33(3):265–270, 1982.

[12] Victor Neumann-Lara. The 3 and 4-dichromatic tournaments of minimum order. *Discret. Math.*, 135(1-3):233–243, 1994.

[13] Neil J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences.