Arc-disjoint out- and in-branchings with the same root in co-bipartite digraphs

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Abstract

A digraph is **semicomplete** if it has no pair of non-adjacent vertices and it is **locally semicomplete** if for every vertex v, the out-neighbors of v induce a semicomplete digraph and the in-neighbors of v induce a semicomplete digraph. A digraph D is **co-bipartite** if its vertex set can be partitioned in two sets W_1, W_2 such that $D\langle W_1\rangle$ and $D\langle W_2\rangle$ are semicomplete digraphs. An **out-branching** B_s^+ **(in-branching** B_s^-**)** of a digraph D is a spanning tree in the underlying graph of D whose edges are oriented in D such that every vertex except one, s, called the root, has in-degree (out-degree) one. The **arc-connectivity** of D is the minimum out-degree of a proper subset of V. Thomassen conjectured that there exists a natural number C such that every digraph of arc-connectivity at least C contains arc-disjoint out- and in-branchings B_s^+, B_t^- for every choice of roots s, t. It has been verified that $C = 2$ is sufficient for locally semicomplete digraphs due to Bang-Jensen and Huang.

Recently, Bang-Jensen et al. showed that every digraph of independence number at most 2 and arc-connectivity at least 2 has a pair of arc-disjoint out- and in-branchings and they raised the following conjecture.

Conjecture: Every 2-arc-strong digraph with independence number 2 has a pair of arc-disjoint out- and in-branchings with the same root s for any choice of $s \in V$.

We found counterexamples for the conjecture even in subclass of co-bipartite digraphs. However, we can characterize the counterexamples. The arc-connectivity and independence number bound in the conjecture is best possible in the following sense: there are infinitely many strong digraphs with independence number 2 and arbitrarily high minimum in- and out-degrees that have no arc-disjoint out- and in-branchings and there are infinitely many digraphs with independence number 3 and arc-connectivity 2 which do not have arc-disjoint out- and in-branchings rooted at some given vertex.

Keywords: Out-branching, in-branching, digraphs of independence number 2, co-bipartite digraphs

1 Introduction

It is a well-known result due to Nash-Williams and Tutte [7, 8] that every $2k$ edge-connected graph has a set of k edge-disjoint spanning trees. For digraphs, Edmonds' branching theorem [6] provides a characterization for the existence of k arc-disjoint out-branchings rooted at the same vertex.

Theorem 1. *(Edmonds' Branching Theorem) [6] A directed multigraph $D = (V, A)$ with a special vertex s has k arc-disjoint out-branchings rooted at s if and only if*

$$d^-(X) \geq k, \ \ \forall\ \emptyset \neq X \subseteq V - s.$$

A natural related problem is to characterize digraphs having an out-branching and an in-branching which are arc-disjoint. Such a pair of branchings are called **a good pair**. Thomassen (see Bang-Jensen [1]) proved that it is NP-complete to decide whether a given digraph has a good pair rooted at the same vertex. This implies that it is NP-complete to decide if a given digraph has a good pair [2]. Therefore, a satisfactory characterization of those digraphs containing a good pair seems unlikely. Thomassen also conjectured that every digraph of sufficiently high arc-connectivity should have such a pair of branchings. Bang-Jensen et al. [2] showed that the conjecture is equivalent to the following:

Conjecture 1. *[2] There is a constant C, such that every digraph with arc-connectivity at least C has a good pair.*

This conjecture has been verified for semicomplete digraphs [1] and for locally semicomplete digraphs [4]. Bang-Jensen et al. also proved the conjecture for semicomplete compositions [3] and for digraphs of independence number two [2]. In both cases arc-connectivity 2 suffices. For general digraphs the conjecture is wide open.

Now we consider the existence of a pair of arc-disjoint branchings B_s^+, B_t^-, where $s, t \in V$ can be chosen arbitrarily. Bang-Jensen [1] gave a complete characterization for tournaments and Bang-Jensen and Yeo [5] verified the problem for semicomplete digraphs. In both cases, arc-connectivity 2 suffices. Bang-Jensen and Huang [4] generalized these two results to 2-arc-strong locally semicomplete digraphs.

In [2], the authors raised the following conjecture.

Conjecture 2. *[2] Every 2-arc-strong digraph with independence number 2 has a good pair with the same root s for any choice of $s \in V$.*

Our main result is the following which implies that the above conjecture is not true even for co-bipartite digraphs.

Theorem 2. *Every 2-arc-strong co-bipartite digraph has a good pair rooted the same root s for any choice of $s \in V$ unless it is belongs to well-defined classes of digraphs.*

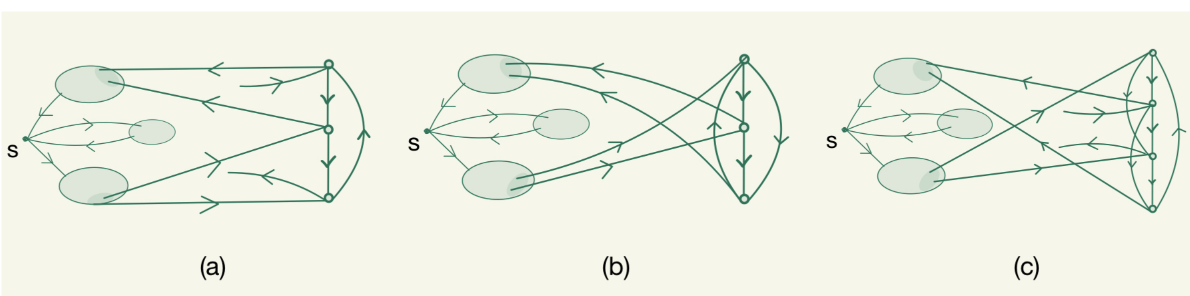

Figure 1: Digraphs without good pairs with root s.

Bang-Jensen [1] gave a good characterization for the existence of arc-disjoint (a, b)- and (c, d)-path given a tournament T and distinct vertices $a, b, c, d \in V(T)$. Motivated by this, we give a characterization for the existence of an out-branching B_u^+ and a (v, w)-path such that they are arc-disjoint and the existence of a pair of arc-disjoint out-branching with root u and in-branching with root v in a given semicomplete digraph. We believe these two results may be of independent interest.

Theorem 3. *Let D be a semicomplete digraph and let u, v, w be three vertices such that D contains an out-branching with root u and a (v, w)-path. Then D has an out-branching with root u which is arc-disjoint from some (v, w)-path unless (D, u, v, w) belongs to four well-defined classes of digraphs.*

Theorem 4. *Let D be a semicomplete digraph and u, v be arbitrary chosen vertices with $u \neq v$. Then D has a good (u, v)-pair if and only if (D, u, v) satisfies none of the following conditions.*

(i) D is isomorphic to one of the digraphs in Figure 2.

(ii) D is not strong and either u is not in the initial component of D or v is not in the terminal component of D.

(iii) D is strong and there exists an edge $e \in A(D)$ such that u is not in the initial component of $D - e$ and v is not in the terminal component of $D - e$.

(iv) D is strong and there exists a partition $V_1, \ldots, V_{2\alpha+3}$ of V for some $\alpha \geq 1$ such that $v \in V_2, u \in V_{2\alpha+2}$ and all arcs between V_i and V_j with $i < j$ from V_i to V_j with the following exceptions. There exists precisely one arc from V_{i+2} to V_i for all $i \in [2\alpha + 1]$ and it goes from the terminal component of $D\langle V_{i+2}\rangle$ to the initial component of $D\langle V_i\rangle$.

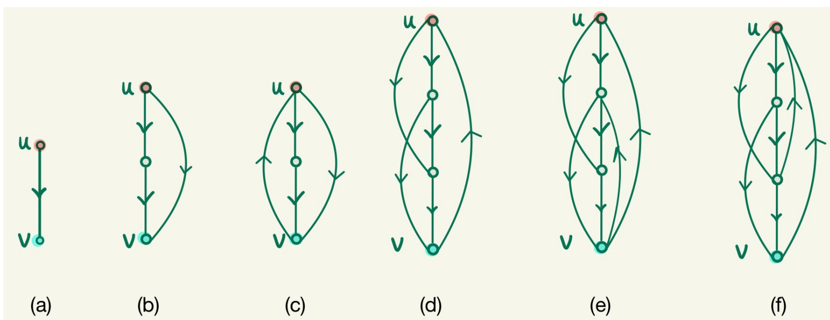

Figure 2: Semicomplete digraphs without good (u, v)-pairs.

2 Sketch proof of Theorem 2

Let $D = (V, A)$ be a co-bipartite digraph with a vertex partition $V = W_1 \cup W_2$ such that $D_i = D\langle W_i\rangle$ is semicomplete. Without loss of generality, we may assume $s \in D_1$.

For the case that D_1 has a good pair $(B^+_{s,D_1}, B^-_{s,D_1})$ in D_1, let $\mathcal{O}$ (resp., $\mathcal{I}$) be a collection of disjoint out-trees (resp., in-trees) such that both $\mathcal{O}$ and $\mathcal{I}$ cover all vertices of D_2 and they are arc-disjoint. Moreover, each root of the out-trees (resp., in-trees) has an in-neighbor (resp., out-neighbor) in D_1. Then a good pair with root s in D can be obtained from $(B^+_{s,D_1}, B^-_{s,D_1})$ and $\mathcal{O}$, $\mathcal{I}$ by adding the arcs between roots and D_1.

For the case that D_1 has no good pair with root s in D_1. Let A, B and C form a partition of $V(D_1) - s$ such that $N^+_{D_1}(s) = A \cup C$ and $N^-_{D_1}(s) = B \cup C$, where C is the set of vertices that form a 2-cycle with s in D_1. We construct a good pair with root s in D as follows. Let H^+ (resp., H^-) be the set of vertices of D_2 that has an in-neighbor (resp., out-neighbor) in D_1.

Step 1: Find two arc-disjoint paths OP and IP from the terminal component of $D\langle A\rangle$ to the initial component of $D\langle B\rangle$.

Step 2: Find a collection of disjoint out-trees (resp., in-trees) in D_2, denoted by $\mathcal{O}$ (resp., $\mathcal{I}$), such that the following properties hold:

(a) For each out-tree in $\mathcal{O}$, only one vertex, i.e., its root, may intersect OP. Moreover, its root either belongs to H^+ or on the path OP.

(b) For each in-tree in $\mathcal{I}$, only one vertex, i.e., its root, may intersect IP. Moreover, its root either belongs to H^- or on the path IP.

(c) The union of the path OP and out-trees in $\mathcal{O}$ is arc-disjoint with the union of the path IP and in-trees in $\mathcal{I}$. And, each of them covers all vertices of D_2.

3 Concluding remarks

Bang-Jensen et al. [2] provide an example to show that arc-connectivity 2 is not sufficient to guarantee that a digraph with independence number 2 has an out-branching rooted at a prescribed vertex s which is arc- disjoint from an in-branching rooted at a prescribed vertex t for every choice of vertices s, t. They believed that if the arc-connectivity in Conjecture 2 increased slightly, then we may obtain such branchings.

Conjecture 3. *[2] Every 3-arc-strong digraph $D = (V, A)$ with independence number 2 has a pair of arc-disjoint branchings B_s^+, B_t^- for every choice of $s, t \in V$.*

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