Heroes in orientations of chordal graphs

Pierre Aboulker — **DIENS, École normale supérieure, CNRS, PSL University, Paris, France**
Guillaume Aubian — **Université de Paris, CNRS, IRIF, F-75006, Paris, France**
Raphael Steiner — **ETH Zürich, Switzerland**

Abstract

We characterize all digraphs H such that orientations of chordal graphs with no induced copy of H have bounded dichromatic number.

1 Introduction

A full version of this extended abstract can be found in [1].

In this paper, digraphs have no multiple arc, no loop, and have at most one arc between any pair of vertices.

A **k-dicolouring** of D is a k-partition $(V_1, \dots, V_k)$ of $V(D)$ such that $D[V_i]$ is acyclic for every $1 \leq i \leq k$. Such a partition is also called an **acyclic colouring** of D. The **dichromatic number** of G, denoted by $\overrightarrow{\chi}(G)$ and introduced by Neumann-Lara in [4], is the smallest integer k such that G admits a k-dicolouring.

A **tournament** is an orientation of a complete graph. A **transitive tournament** is an acyclic tournament and we denote by TT_k the unique acyclic tournament on k vertices. Given two tournaments H_1 and H_2, we denote by $\mathbf{\Delta(1, H_1, H_2)}$ the tournament obtained from pairwise disjoint copies of H_1 and H_2 plus a vertex x, and all arcs from x to H_1, all arcs from H_1 to H_2, and all arcs from H_2 to x. We write $\mathbf{\Delta(1, k, H)}$ for $\Delta(1, TT_k, H)$. For tournaments H_1 and H_2, we denote by $H_1 \Rightarrow H_2$ the digraph obtained from disjoint copies of H_1 and H_2 by adding all arcs from the copy of H_1 to the copy of H_2.

Given two digraphs G and H, we say that G is **H-free** if it does not contain an induced copy of H. Given a class of digraphs $\mathcal{C}$, we say that a digraph H is a **hero in** $\mathcal{C}$ if every H-free digraph of $\mathcal{C}$ has bounded dichromatic number.

In a seminal paper, Berger, Choromansky, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé gave a recursive characterization of all heroes in tournaments, as follows.

Theorem 1 (Berger et al. [3])**.** *A digraph H is a hero in tournaments if and only if :*

- $H = K_1$ *(the one-vertex digraph), or*
- $H = (H_1 \Rightarrow H_2)$*, where H_1 and H_2 are heroes in tournaments, or*
- $H = \Delta(1, k, H')$ *or* $H = \Delta(1, H', k)$*, where $k \geq 1$ and H' is a hero in tournaments.*

Observe that if a class of digraphs $\mathcal{C}$ contains all tournaments, then a hero in $\mathcal{C}$ must be a hero in tournaments. A **chordal graph** is a graph with no induced cycle of length at least 4. An **oriented chordal graph** is an orientation of a chordal graph. A classical theorem of Dirac states that all chordal graphs can be obtained by iteratively gluing some complete graphs along cliques. This implies for undirected graph colouring that chordal graphs are perfect graphs, and thus their chromatic number is determined by the (largest) cliques contained in them. It is then natural to ask whether the same kind of properties hold for the dichromatic number of oriented chordal graphs. In particular, it is a natural problem to characterise the heroes in oriented chordal graphs and to see whether they are the same as for tournaments.

We find surprising answers to the above questions. First, there is very few heroes in oriented chordal graphs and as our main contribution, we completely describe them:

Theorem 2. *A digraph H is a hero in oriented chordal graphs if and only if H is a transitive tournament or isomorphic to $\Delta(1, 1, k)$ for some integer $k \geq 1$.*

Secondly, the constructions used in the proof of the theorem above exhibit oriented chordal graphs with arbitrarily large dichromatic number all whose subtournaments are 2-colourable, showing that in contrast to chromatic number, the dichromatic number of an oriented chordal graph heavily depends on its global structure and not only on the cliques (subtournaments) contained in it.

We denote by $\vec{\mathbf{C}}_{\mathbf{3}}$ the directed cycle on three vertices, also called **directed triangle** (observe that $\vec{C}_3 = \Delta(1, 1, 1)$). Since a hero in oriented chordal graphs must be a hero in tournaments, Theorem 1 easily implies that, to prove Theorem 2, it is enough to prove the following:

- Transitive tournaments and $\Delta(1, 1, k)$ for $k \geq 1$ are heroes in oriented chordal graphs. See Section 2.
- $\Delta(1, 2, 2)$ and $\vec{C}_3 \Rightarrow K_1$ are not heroes in oriented chordal graphs. This is respectively done in Section 3 and 4.

2 $\Delta(1, 1, k)$ and transitive tournaments are heroes in oriented chordal graphs

Theorem 3. *Transitive tournaments and $\Delta(1, 1, k)$ are heroes in oriented chordal graphs.*

Proof. A TT_k-free oriented chordal graph has no clique of size at least $2^{k-1}-1$, hence its underlying graph has chromatic bounded chromatic number.

The **triangle degree** of a vertex x is the maximum size of a collection of directed triangles that pairwise share the common vertex x but no further vertices. To prove that a $\Delta(1, 1, k)$ is a hero in oriented chordal graphs, we prove that every $\Delta(1, 1, k)$-free oriented chordal graph has a vertex of bounded triangle degree, which easily implies the result. □

3 $\Delta(1, 2, 2)$ is not a hero in orientations of chordal graphs

We inductively construct a sequence $(G_k)_{k \in \mathbb{N}}$ of digraphs such that for each $k \geq 1$, the digraph G_k is an orientation of a chordal graph with no copy of $\Delta(1, 2, 2)$ and satisfying $\vec{\chi}(G_k) = k$.

Let G_1 be the digraph on one vertex, and having defined G_k, define G_{k+1} as follows. Start with a copy T of TT_{k+1}, and for each arc $e = uv$ of T, create a distinct copy G_k^e of G_k (vertex-disjoint for different choices of the arc $e \in A(T)$, and all vertex-disjoint from T). Next, for each $e = uv \in A(T)$, we add all the arcs vy and yu for every $y \in V(G_k^e)$.

One can prove that for every $k \geq 1$, G_k is a $\Delta(1, 2, 2)$-free oriented chordal graph with dichromatic number k. Hence, $\Delta(1, 2, 2)$ is not a hero in oriented chordal graphs.

4 $\vec{C_3} \Rightarrow K_1$ is not a hero in orientations of chordal graphs

All along this subsection, we denote by $\mathcal{C}$ the class of $(\vec{C_3} \Rightarrow K_1)$-free oriented chordal graphs. The goal of this subsection is to construct digraphs in $\mathcal{C}$ with arbitrarily large dichromatic number.

In the following, given a k-colouring $c : V(F) \to \{1, \dots, k\}$ of a digraph F, we say that a subdigraph of F is **rainbow** (with respect to c), if its vertices are assigned pairwise distinct colours.

Lemma 1. *Let $G \in \mathcal{C}$ such that $\vec{\chi}(G) = k$. There exists a digraph $F = F(G) \in \mathcal{C}$ with $\vec{\chi}(F) = k$ satisfying the following property: For every k-dicolouring of F, there exists a rainbow transitive tournament of size k contained in F.*

Proof sketch. We prove the lemma by showing the following statement using induction on i (the lemma then follows by setting $F := F^{(k)}$).

($\star$) For every $i \in \{1, \dots, k\}$, there exists a digraph $F^{(i)} \in \mathcal{C}$ such that $\vec{\chi}(F^{(i)}) = k$, and for every k-dicolouring of $F^{(i)}$, there exists a copy of TT_i contained in $F^{(i)}$ which is rainbow.

The statement of ($\star$) is trivially true for $i = 1$, since we may put $F^{(1)} := G$, and in every k-dicolouring of $F^{(1)}$ any single vertex forms a rainbow TT_1.

For the inductive step, let $i \in \{1, \dots, k-1\}$ and suppose we have established the existence of a digraph $F^{(i)} \in \mathcal{C}$ of dichromatic number k such that every k-dicolouring of $F^{(i)}$ contains a rainbow copy of TT_i.

We now construct a digraph $F^{(i+1)}$ from $F^{(i)}$ as follows: Let $\mathcal{X}$ denote the set of all $X \subseteq F^{(i)}$ such that X induces a TT_i in $F^{(i)}$. Now, for every $X \in \mathcal{X}$ create a distinct copy G_X of the digraph G (pairwise vertex-disjoint for different choices of X, and all vertex-disjoint from $F^{(i)}$). Finally, for every $X \in \mathcal{X}$, add all the arcs xy with $x \in X$ and $y \in V(G_X)$. This complete the description of F^{i+1}. One can prove that F^{i+1} has the desired properties. □

Theorem 4. *The digraph $\vec{C_3} \Rightarrow K_1$ is not a hero in oriented chordal graphs.*

Proof. We construct a sequence of digraphs $(G_k)_{k \in \mathbb{N}}$ such that $\vec{\chi}(G_k) = k$ and $G_k \in \mathcal{C}$. Let G_1 be the one-vertex-digraph and, having defined G_k, define G_{k+1} as follows. Let $F_k := F(G_k) \in \mathcal{C}$ be the digraph given by Lemma 1, so $\vec{\chi}(F_k) = k$ and every k-dicolouring of F_k contains a rainbow copy of TT_k.

Let $\mathcal{T}$ denote the set of subdigraphs of F_k that are transitive tournaments. Now, for each transitive subtournament $T \in \mathcal{T}$, add a copy F_k^T of F_k (vertex-disjoint for different choices of T, and all vertex-disjoint from F_k). Next, for every $T \in \mathcal{T}$, add all the arcs xy with $x \in V(T)$ and $y \in V(F_k^T)$. Finally, for every choice of $T \in \mathcal{T}$ and every transitive subtournament T' of F_k^T, add a vertex $x_{T,T'}$ that is seen by every vertex of T' and that sees every vertex of T. One can proof that the G_k's have the desired properties. □

5 Further works

After characterising heroes in oriented chordal graphs, it is natural to ask what are the heroes in orientations of subclasses or superclasses of chordal graphs.

Concerning superclasses of chordal graphs, consider the following construction (already mentioned in [2]). Let G_1 be the graph on 1 vertex, and having defined G_{k-1} inductively, define G_k

as follows: start with three disjoint copies $G^1_{k-1}, G^2_{k-1}, G^3_{k-1}$ of G_{k-1} plus a vertex x, and add all arcs from x to $V(G^1_{k-1})$, all arcs from $V(G^1_{k-1})$ to $V(G^2_{k-1})$, all arcs from $V(G^2_{k-1})$ to $V(G^3_{k-1})$ and finally, all arcs from $V(G^3_{k-1})$ to x. It is then easy to see that $\overrightarrow{\chi}(G_k) = k$ and that the underlying graph of G_k does not contain induced path of length 4. Hence, the underlying graphs of the G_k's are perfect graphs, and even co-graphs, which implies that $\vec{C}_3$ is not a hero in orientation of perfect graphs. So the only possible heroes are transitive tournaments, which are trivially, since transitive tournaments are heroes in any orientations of graphs in $\mathcal{C}$, whenever $\mathcal{C}$ is a χ-bounded class of graphs.

Concerning subclasses of chordal graphs, orientations of interval graphs seems to be an intriguing case. On one hand, we were not able to decide whether or not $\Delta(1, 2, 2)$ or $\vec{C}_3 \Rightarrow K_1$ are heroes in this class, and our attempts have not led us to a strong opinion as to the answer. On the other hand, we can prove the following. A **unit interval graph** is an interval graph that admits an interval representation in which every interval has unit length.

Theorem 5. *Heroes in orientations of unit interval graphs are the same as heroes in tournaments.*

Proof. Since complete graphs are unit interval graphs, the set of heroes in orientations of proper interval graphs is a subset of the set of heroes in tournaments.

We are going to prove the following, which easily implies that every hero in tournaments is a hero in orientation of unit interval graphs.

($\star$) For every integer C, if G is an orientation of a unit interval graph in which every subtournament has dichromatic number at most C, then G is $2C$-dicolourable.

Let G be an orientation of a unit interval graph and C an integer such that every subtournament of G has dichromatic number at most C. Consider an interval representation of G where each interval has length 1 and assume without loss of generality that the endpoints of each interval are not integers. For every integer k, let K_k be the set of vertices of G whose associated interval contains k. So each K_k induces a subtournament of G, and by hypothesis, $G[K_k]$ is C-dicolourable. Moreover, since each interval has length 1 and their extremities are not integers, the K_k's partition the vertices of G and there is no arc between K_i and K_j whenever $|i-j| \geq 2$. Hence, piecing together dicolourings of $G[K_k]$ with colours from $\{1, \dots, C\}$ when k is odd, and from $\{C+1, \dots, 2C\}$ when k is even, results in a $2C$-dicolouring of G. □

References

[1] Pierre Aboulker, Guillaume Aubian, and Raphael Steiner. Heroes in orientations of chordal graphs. *arXiv:2202.09191v2*, 2022.

[2] Pierre Aboulker, Pierre Charbit, and Reza Naserasr. Extension of Gyárfás-Sumner conjecture to digraphs. *Electronic Journal of Combinatorics*, 28, 2021.

[3] Eli Berger, Krzysztof Choromanski, Maria Chudnovsky, Jacob Fox, Martin Loebl, Alex Scott, Paul Seymour, and Stéphan Thomassé. Tournaments and colouring. *Journal of Combinatorial Theory, Series B*, 103:1–20, 2013.

[4] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265 – 270, 1982.