On the hitting/packing ratio of axis-parallel segments

Marco Caoduro — Laboratoire G-SCOP, Univ. Grenoble Alpes, France
Jana Cslovjecsek — EPFL, Switzerland
Michał Pilipczuk — Institute of Informatics, University of Warsaw, Poland
Karol Węgrzycki — Saarland University and Max Planck Institute for Informatics, Saarbrücken, Germany

Abstract

Let $\mathcal{S}$ be a set of segments on the plane. The *hitting number* $\tau(\mathcal{S})$ is the minimum number of points hitting all the segments in $\mathcal{S}$, while the *packing number* $\nu(\mathcal{S})$ is the maximum number of pairwise disjoint segments in $\mathcal{S}$. Clearly, $\tau \geq \nu$. It is straightforward and well-known that for segments lying in at most d directions $\tau \leq d\nu$.

We construct a sequence of sets of axis-parallel segments with a ratio τ/ν that approaches 2 as τ (and ν) grows. This shows that the previous bound is optimal for $d = 2$.

A famous conjecture of Wegner (1965) says that for any set of axis-parallel rectangles, the hitting number is bounded by twice the packing number. Our result implies that the multiplicative constant in the conjecture is tight already for the degenerate case of axis-parallel segments.

1 Introduction

Let $\mathcal{S}$ be a set of segments on the plane. A *hitting set* of $\mathcal{S}$ is a set of points that has non-empty intersection with each segment in $\mathcal{S}$ and a *packing* of $\mathcal{S}$ is a set of pairwise disjoint segments in $\mathcal{S}$. The *hitting number* of $\mathcal{S}$, denoted by $\tau(\mathcal{S})$, is the minimum size of a hitting set of $\mathcal{S}$. The *packing number* of $\mathcal{S}$, denoted by $\nu(\mathcal{S})$, is the maximum size of a packing of $\mathcal{S}$. Both these parameters are $\mathcal{NP}$-complete to compute for sets of segments, even if the segments are axis-parallel [3, 5].

A trivial lower bound of the hitting number is the packing number, $\tau \geq \nu$. For sets of segments lying in at most d directions, it is also easy to prove $\tau \leq d\nu$. Indeed, such a set can be decomposed into d subsets of parallel segments and each of these subsets is a set of intervals for which $\tau = \nu$. This argument gives a bound on the hitting number that could look rough. It is natural to ask if the upper bound can be improved.

In this paper, we study the case $d = 2$. Quite surprisingly we show that for sets of segments lying in at most 2 directions the bound is tight.

Theorem 1. *For any $\epsilon > 0$, there is a set $\mathcal{M}$ of segments lying in at most 2 directions for which*

$$\frac{\tau(\mathcal{M})}{\nu(\mathcal{M})} \geq 2 - \epsilon.$$

In order to prove our result it is enough to focus on the class of axis-parallel rectangles with the additional properties that no three segments meet in a point. We denote this class by $\mathcal{G}_{\text{seg}}$.

We construct a sequence of sets in $\mathcal{G}_{\text{seg}}$ with a ratio τ/ν that approaches 2 as the size of the set grows. More precisely, we prove the following result.

Theorem 2. *For any integer $k \geq 1$, there exists a set $\mathcal{M}_k$ of axis-parallel segments in $\mathcal{G}_{\text{seg}}$ with $n = 4k^2$ vertices, hitting number $\frac{n}{2}$ and packing number*

$$\alpha(\mathcal{M}_k) = \frac{n}{4} + \frac{3}{2}\sqrt{n} - 2.$$

In addition to the previous result, we give a lower bound on the packing number of a set in $\mathcal{G}_{\text{seg}}$.

Theorem 3. *Let $\mathcal{S}$ be a set in $\mathcal{G}_{\text{seg}}$ with n axis-parallel segments. Then the packing number of $\mathcal{S}$ is*

$$\alpha(\mathcal{S}) \geq \frac{n}{4} + \frac{\sqrt{3}}{12}\sqrt{n}.$$

This implies that the packing number of a set in Theorem 2 is the smallest possible up to a factor of $\sqrt{n}$.

Consequences

A set of axis-parallel segments can be seen as a set of degenerate axis-parallel rectangles. With regards to a set of axis-parallel rectangles there is a long-standing conjecture of Wegner concerning the relation between the hitting number τ and the packing number ν.

Conjecture 1 (Wegner [6], 1965)**.** *Let $\mathcal{R}$ be a set of axis-parallel rectangles. Then*

$$\tau(\mathcal{R}) \leq 2\nu(\mathcal{R}) - 1.$$

The best-known upper bound on the hitting number of a set of axis-parallel rectangles is $\tau = \mathcal{O}(\nu \log^2 \log \nu)$ [1]. In particular, no linear bounds are known.

Obtaining lower bounds on the ratio τ/ν is a difficult task. For nearly thirty years after Wegner formulated his conjecture, the largest known ratio remained $3/2$. This was simply achieved by taking five rectangles forming a cycle. Only in 1993 Fon-Der-Flaass and Kostochka [4] presented a set of axis-parallel rectangles with hitting number 5 and packing number 3. Finally, in 2015 Jelínek [1] found a sequence of sets with ratio τ/ν arbitrarily close to 2.

Theorem 1 shows that the multiplicative constant of 2 conjectured by Wegner cannot be improved even for axis-parallel segments. The class of axis-parallel segments is one of the few subclasses of axis-parallel rectangles for which Conjecture 1 is proved to be optimal. Indeed, Wegner's conjecture holds in other subclasses as unit squares [2] and diagonal-pierced rectangles [1], but in any of these cases it is not known if the bound in Conjecture 1 is optimal or not.

2 Construction of the sequence $\{\mathcal{M}_k\}$

In this section, we exhibit a sequence of sets of axis-parallel segments satisfying the requirements of Theorem 2.

For each natural number k we construct a set of $4k^2$ axis-parallel segments $\mathcal{M}_k$. The construction combines k sets with $4k$ segments each into a large set of segments. A *k-box* is a set of $4k$ axis-parallel segments distributed onto k horizontal and k vertical lines each with exactly two segments on it. For every line, the two segments on it intersect in a single point, which we call a *meeting point*. In the construction of a k-box the meeting points are arranged in a diagonal from the top left to the bottom right, see the case $k = 6$ in Figure 1. The *up segments* (resp. *down segments*) of a k-box are the segments lying vertically above (resp. below) a meeting point. Similarly, we define the *left* and *right segments* of a k-box.

To construct $\mathcal{M}_k$, consider a large square and place k different k-boxes $\{\mathcal{B}_i\}_{i=1}^k$ along its diagonal from the bottom left to the top right. Then, extend each segment away from the meeting point until

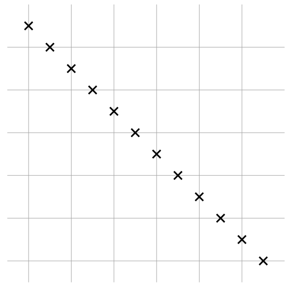

Figure 1: Drawing of a 6-box. The meeting points are represented with crosses.

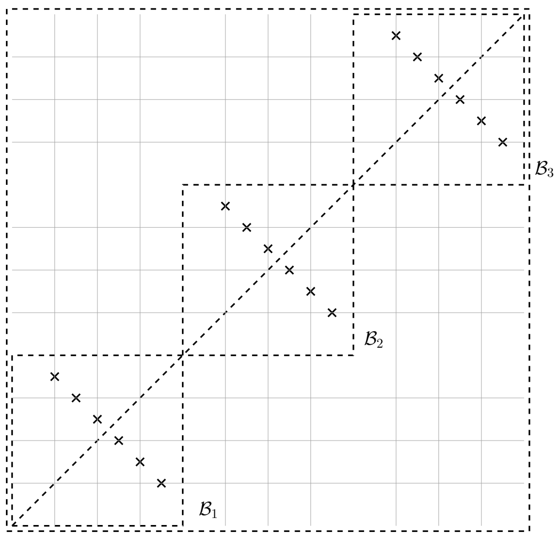

Figure 2: A representation of $\mathcal{M}_3$. The meeting points are represented by crosses. The dashed lines are just the borders of the large square and the k-boxes.

it touches a border of the square, see Figure 2. The construction results in the set $\mathcal{M}_k$ consisting of $4k^2$ segments.

Let us introduce some notations and definitions helping to prove Theorem 2. Let $\mathcal{I}$ be a subset of pairwise disjoint segments in $\mathcal{M}_n$. A k-box $\mathcal{B}_i$ of $\mathcal{M}_n$ is said *interesting* for $\mathcal{I}$ if $\mathcal{B}_i \cap \mathcal{I}$ contains either at least one down segment and one right segment, or at least one up segment and one left segment. A k-box is said *boring* for $\mathcal{I}$ otherwise. Distinguishing between interesting and boring boxes allows more precise estimates for the number of pairwise disjoint elements possible.

In the next two lemmas, we consider $\mathcal{I}$ to be a subset of pairwise disjoint segments in $\mathcal{M}_k$.

Lemma 1. *For any k-box $\mathcal{B}$ in $\mathcal{M}_k$, $|\mathcal{B} \cap \mathcal{I}| \leq 2k$. Moreover, if $\mathcal{B}$ is boring, $|\mathcal{B} \cap \mathcal{I}| \leq k+1$.*

The crucial property of $\mathcal{M}_k$ is that it cannot have too many interesting sets.

Lemma 2. *The set $\mathcal{M}_k$ has at most two interesting boxes.*

Proof of Theorem 2. By construction, the set $\mathcal{M}_k$ consists of $4k^2$ axis-parallel segments and $\mathcal{M}_k$ is in $\mathcal{G}_{\text{seg}}$.

We first compute the hitting number of $\mathcal{M}_k$. Note that a set of pairwise intersecting segments in $\mathcal{M}_k$ is of size at most 2. Thus any hitting set of $\mathcal{M}_k$ is of size at least $\frac{|\mathcal{M}_k|}{2}$. The meeting points of $\mathcal{M}_k$ hit each of its segments, and their number is $2k^2 = \frac{|\mathcal{M}_k|}{2}$. Giving a hitting number of exactly $\theta(G_k) = \frac{n}{2} = 2k^2$.

It remains to prove that $\nu(\mathcal{M}_k) = \frac{n}{4} + \frac{3}{2}\sqrt{n} - 2 = k^2 + 3k - 2$. First, we give a set of pairwise disjoint segments in $\mathcal{M}_k$, corresponding to a packing in $\mathcal{M}_k$. This shows that $\nu(\mathcal{M}_k) \geq k^2 + 3k - 2$. The set of segments in $\mathcal{M}_k$ consists of:

- the left and up segments of $\mathcal{B}_1$
- the right and down segments of $\mathcal{B}_2$
- the right segments and the topmost up segment of B_i for $3 \leq i \leq k$

This is indeed a set of pairwise disjoint segments in $\mathcal{M}_k$ and it contains $2(2k) + (k-2)(k+1) = k^2 + 3k - 2$ segments.

We finally show $\nu(\mathcal{M}_k) \leq k^2 + 3k - 2$. Applying Lemma 1 and Lemma 2 we obtain for any set $\mathcal{I}$ of pairwise disjoint segments of $\mathcal{M}_k$,

$$|\mathcal{I}| = |\mathcal{M}_k \cap \mathcal{I}| = \sum_{i=1}^{k} |\mathcal{B}_i \cap \mathcal{I}| \leq 2(2k) + (k-2)(k+1) = k^2 + 3k - 2.$$

This concludes the proof of Theorem 2. □

References

[1] J. Correa, L. Feuilloley, P. Pérez-Lantero, and J.A. Soto. *Independent and hitting sets of rectangles intersecting a diagonal line: algorithms and complexity.* Discrete Computational Geometry, vol.53, 344-365, 2015.

[2] A. Dumitrescu and M. Jiang. *Piercing translates and homothets of a convex body.* Algorithmica, vol.61, 94-115, 2011.

[3] S.P. Fekete, K. Huang, J.S.B. Mitchell, O. Parekh, and C.A. Phillips. *Geometric Hitting Set for Segments of Few Orientations.* in: Sanità L., Skutella M. (eds) Approximation and Online Algorithms. WAOA 2015, Springer, Cham, 145-157, 2015.

[4] D.G. Fon Der Flaass and A.V. Kostochka. *Covering boxes by points.* Discrete Mathematics, vol.120, 269-275, 1993.

[5] J. Kratochvíl and J. Nešetřil. *INDEPENDENT SET and CLIQUE problems intersection-defined classes of graphs.* Commentationes Mathematicae Universitatis Carolinae, vol.31, 85-93, 1990.

[6] G. Wegner. *Über eine kombinatorisch geometrische frage von hadwiger und debrunner.* Israel J. Math, vol.3, 187-198, 1965.