Necessary induced subgraphs of minimally tough graphs*

Clément Dallard — University of Primorska, Slovenia
Blas Fernández — University of Primorska, Slovenia
Gyula Y. Katona — Budapest University of Technology and Economics, Hungary
Martin Milanič — University of Primorska, Slovenia

Abstract

A graph G is minimally t-tough if G has toughness t and for any edge e of G, the graph $G-e$ is not t-tough. Katona, Soltész, and Varga showed that for every positive rational number t, any graph is an induced subgraph of some minimally t-tough graph. Hence, no induced subgraph can be excluded for the class of minimally t-tough graphs. We consider the opposite point of view and ask which induced subgraphs, if any, must necessarily be present in each minimally t-tough graph. Katona and Varga showed that for all $t \in (1/2, 1]$, every minimally t-tough graph contains a hole, that is, an induced cycle of length at least four. We complement this result by showing that for all finite $t > 1$, every minimally t-tough graph must contain a hole or an induced subgraph isomorphic to the k-sun for some $k \geq 3$. Our approach also shows that for all finite $t > 1$, every minimally t-tough graph containing a universal vertex also contains a hole.

1 Introduction

In 1973, Chvátal defined a (finite, simple and undirected) graph G to be *t-tough*, where t is a real number, if any cutset S of G satisfies $|S|/\omega(G-S) \geq t$, where $\omega(G-S)$ denotes the number of connected components of $G-S$ [5]. The *toughness* of G, denoted by $\tau(G)$, is the largest real number t such that G is t-tough, and is infinite if G is complete. As informally described by Chvátal himself, toughness "measures in a simple way how tightly various pieces of a graph hold together." Toughness aimed at generalizing the notion of Hamiltonicity, since Hamiltonian graphs are 1-tough, and Chvátal conjectured that that there exists a real t_0 such that every t_0-tough graph is Hamiltonian. Chvátal even left open the possibility that any $t_0 > 3/2$ would satisfy the condition; in particular, he asked whether one could take $t_0 = 2$. While this version of the conjecture has been disproved by Bauer, Broersma, and Veldman in 2000 [1], the general conjecture remains open until this day.

A concept closely related to Chvátal's conjecture is that of a *minimally t-tough graph*, which Broersma, Engbers, and Trommel defined in 1999 as a graph that is t-tough but for which the deletion of any edge decreases the toughness [4]. This notion has been studied in a number of subsequent works (see, e.g., [7, 8, 9, 11]). In particular, Katona, Soltész, and Varga showed in [8] that for every positive rational number t, any graph is an induced subgraph of some minimally t-tough graph.

In this paper, we continue the study of minimally t-tough graphs. We provide a necessary and sufficient condition for a non-complete connected graph G to be minimally t-tough, where $t = \tau(G)$ is the toughness of G. We then use this condition to derive that for all finite $t > 1$ every minimally t-tough graph must contain a hole or an induced subgraph isomorphic to the k-sun for some $k \geq 3$. This result complements the one of Katona and Varga stating that for all $t \in (1/2, 1]$, every minimally t-tough graph contains a hole [9]. Hence, this shows that there are no minimally t-tough, strongly chordal graphs, for all $t > 1/2$. Besides, our approach also implies that for all $t > 1$, if a minimally t-tough graph contains a universal vertex, then it contains a hole.

*This work is supported in part by the Slovenian Research Agency (I0-0035, research program P1-0285 and research projects J1-2451, J1-9110, N1-0102, N1-0160, J3-3001, J3-3002, J3-3003, and a Young Researchers Grant).

2 Preliminaries

Definition of minimally tough graphs. A *cutset* in G is a set $S \subseteq V(G)$ such that $G - S$ is disconnected (that is, $\omega(G-S) > 1$). Given a real number $t \geq 0$, a graph G is said to be *t-tough* if $\frac{|S|}{\omega(G-S)} \geq t$ for all cutsets S in G. The *toughness* of G, denoted by $\tau(G)$, is the largest value of $t \geq 0$ such that G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$, where K_n denotes the n-vertex complete graph. Note that a graph is disconnected if and only if its toughness is 0. A graph G is said to be *minimally t-tough* if $\tau(G) = t$ and for all edges $e \in E(G)$, we have $\tau(G - e) < t$. We say that a graph G is *minimally tough* if the deletion of any edge decreases its toughness, that is, if G is minimally $\tau(G)$-tough.

Chordal and strongly chordal graphs. A *hole* in a graph is an induced cycle of length at least four. A graph is *chordal* if it does not contain a hole. A graph is *strongly chordal* if it is chordal and every cycle of length at least six has a chord joining two vertices at an odd distance in the cycle.

We will make use of known characterizations of strongly chordal graphs due to Farber [6]. Given a graph G and a vertex $v \in V(G)$, we denote by $N(v)$ the *neighborhood* of v in G, that is, the set of all vertices in G that are adjacent to v, and by $N[v]$ its *closed neighborhood*, that is, the set $N(v) \cup \{v\}$. A vertex $u \in N[v]$ is a *maximum neighbor* of v if for all $w \in N[v]$, it holds $N[w] \subseteq N[u]$. A vertex is *simplicial* if its closed neighborhood is a clique. A vertex s is a *simple vertex* if its closed neighborhood can be linearly ordered so that for $x, y \in N[s]$, if x is smaller than y with respect to the order, then $N[x] \subseteq N[y]$. This implies that s has a maximum neighbor (the largest neighbor of s with respect to the order) and that s is simplicial. Given an integer $k \geq 3$, a *k-sun* is the graph with vertex set $A \cup B$ such that $A = \{a_1, \ldots, a_k\}$ is a clique, $B = \{b_1, \ldots, b_k\}$ is an independent set, for all $i \in \{1, \ldots, k\}$ vertex b_i is adjacent to a_i and a_{i+1} (indices modulo k), and there are no other edges. A graph G is said to be *sun-free* if for all $k \geq 3$, G does not contain an induced subgraph isomorphic to the k-sun.

Theorem 1 (Farber [6])**.** *For every graph G, the following conditions are equivalent.*

1. *G is strongly chordal.*
2. *G is chordal and sun-free.*
3. *Each induced subgraph of G has a simple vertex.*

Moplexes. A vertex set M in a graph G is a *module* if each vertex $v \in V(G) \setminus M$ is either adjacent to every vertex in M or not adjacent to any vertex in M. A *clique module* is a module that is a clique. Given a graph G and a set $X \subseteq V(G)$, we denote by $N[X]$ the union $\cup_{x \in X} N[x]$ and by $N(X)$ the set $N[X] \setminus X$. A *moplex* in a graph G is an inclusion-maximal clique module $X \subseteq V(G)$ such that $N(X)$ is empty or there exists a component C of the graph $G - N[X]$ such that each vertex in $N(X)$ has a neighbor in C. A vertex that belongs to a moplex is said to be *moplicial*. A classical theorem of Dirac states that every chordal graph with at least two vertices has at least two simplicial vertices. This result was strengthened by Berry and Bordat in [2] using the concept of moplexes, which in the case of chordal graphs consist of simplicial vertices only.

Theorem 2 (Berry and Bordat [2])**.** *Every non-complete graph contains at least two moplexes.*

The following lemma provides a link between simple and moplicial vertices.

Lemma 1. *If s is a simple vertex of a graph G, then s belongs to a moplex.*

3 A characterization of non-minimally tough graphs

For two non-adjacent vertices u, v in a graph G, a set $S \subseteq V(G) \setminus \{u, v\}$ is a *u,v-cutset* in G if u and v are contained in different components of the graph $G - S$. The following theorem characterizes graphs that are not minimally tough.

Theorem 3. *Let G be a connected non-complete graph and $t = \tau(G)$. Then G is not minimally t-tough if and only if G contains an edge $e = uv$ such that the following conditions are met.*

1. *There exist at least $2t + 1$ internally vertex-disjoint u,v-paths in G (including uv).*
2. *Every cutset S in G that is a u,v-cutset in $G - e$ satisfies $|S| \geq (\omega(G - S) + 1) \cdot t$.*

Theorem 3 implies the following sufficient condition for a graph not to be minimally tough.

Proposition 1. *Let $t > 0$ and let G be a graph containing two adjacent vertices u and v such that u and v have at least $2t$ common neighbors, at least t of which have all their neighbors in $N(u) \cup N(v)$. Then G is not minimally t-tough.*

4 Implications for chordal and strongly graphs

Proposition 1 leads to an alternative proof of the following result, which appears already in [9].

Corollary 1. *For all $t \in (1/2, 1]$, there are no minimally t-tough, chordal graphs.*

Corollary 1 can be equivalently stated as follows.

Corollary 2. *For all $t \in (1/2, 1]$, every minimally t-tough graph contains a hole.*

Building upon the notion of maximum neighbor defined in Section 2, we say that vertex v has a *maximum neighboring edge* if there exist two adjacent vertices $u, u' \in N(v)$ such that, for all $w \in N[v]$, it holds $N[w] \subseteq N[u] \cup N[u']$.

Theorem 4. *Let G be a chordal graph. If G contains a moplicial vertex s such that s has a maximum neighbor or s has a maximum neighboring edge, then G is not minimally t-tough, for all finite $t > 1/2$.*

Theorem 4 has several interesting consequences. Together with Theorem 1 and Lemma 1, it implies the following.

Corollary 3. *There are no minimally t-tough, strongly chordal graphs, for all finite $t > 1/2$.*

Corollaries 2 and 3 and the forbidden induced subgraph characterization of strongly chordal graphs given by Theorem 1 imply the following.

Corollary 4. *For all finite $t > 1$, every minimally t-tough graph contains a hole or an induced k-sun for some $k \geq 3$.*

Furthermore, since every interval graph is strongly chordal (see [10]), Corollary 3 directly implies the following.

Corollary 5. *There are no minimally t-tough, interval graphs, for all finite $t > 1/2$.*

It is natural to ask whether the result of Corollary 5 generalizes to the larger class of cocomparability graphs (see [3] for the definition). This is not the case.

Proposition 2. *For every integer $n \geq 2$, there exists a minimally $(n/2)$-tough cobipartite graph.*

Another consequence of Theorem 4 concerns chordal graphs with a *universal vertex*, that is, a vertex adjacent to all other vertices.

Corollary 6. *For all finite $t > 1$, there are no minimally t-tough, chordal graphs with a universal vertex.*

We remark that Theorem 4 is a *proper* common generalization of Corollaries 3 and 6. To illustrate this, we give an example of a graph G for which Theorem 4 can be used to show that G is not minimally t-tough, but neither Corollary 3 nor Corollary 6 gives the same conclusion. Let G be the graph obtained from the 3-sun by fixing a vertex u of degree two, adding a new vertex v and making it adjacent only to u. Then G is a chordal graph in which v is a moplicial vertex with a maximum neighbor u. However, G is not strongly chordal (by Theorem 1) and does not have any universal vertices.

Acknowledgments. We are grateful to Kitti Varga for her helpful suggestions and comments.

References

[1] D. Bauer, H. J. Broersma, and H. J. Veldman, "Not every 2-tough graph is Hamiltonian," in *Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997)*, vol. 99, 2000, pp. 317–321. DOI: 10.1016/S0166-218X(99)00141-9.

[2] A. Berry and J.-P. Bordat, "Separability generalizes Dirac's theorem," *Discrete Appl. Math.*, vol. 84, no. 1-3, pp. 43–53, 1998. DOI: 10.1016/S0166-218X(98)00005-5.

[3] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes: a survey.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999, pp. xii+304.

[4] H. Broersma, E. Engbers, and H. Trommel, "Various results on the toughness of graphs," *Networks*, vol. 33, no. 3, pp. 233–238, 1999. DOI: 10.1002/(SICI)1097-0037(199905)33:3<233::AID-NET9>3.0.CO;2-A.

[5] V. Chvátal, "Tough graphs and Hamiltonian circuits," *Discrete Math.*, vol. 5, pp. 215–228, 1973. DOI: 10.1016/0012-365X(73)90138-6.

[6] M. Farber, "Characterizations of strongly chordal graphs," *Discrete Math.*, vol. 43, no. 2-3, pp. 173–189, 1983. DOI: 10.1016/0012-365X(83)90154-1.

[7] G. Y. Katona, I. Kovács, and K. Varga, "The complexity of recognizing minimally tough graphs," *Discrete Appl. Math.*, vol. 294, pp. 55–84, 2021. DOI: 10.1016/j.dam.2021.01.022.

[8] G. Y. Katona, D. Soltész, and K. Varga, "Properties of minimally t-tough graphs," *Discrete Math.*, vol. 341, no. 1, pp. 221–231, 2018. DOI: 10.1016/j.disc.2017.08.033.

[9] G. Y. Katona and K. Varga, *Minimally toughness in special graph classes*, arXiv:1802.00055 [math.CO], 2018.

[10] C. G. Lekkerkerker and J. C. Boland, "Representation of a finite graph by a set of intervals on the real line," *Fund. Math.*, vol. 51, pp. 45–64, 1962/63. DOI: 10.4064/fm-51-1-45-64.

[11] K. Varga, "Properties of minimally tough graphs," Ph.D. dissertation, Department of Computer Science, Information Theory, Budapest University of Technology, and Economics, 2021.