Towards Stahl's Conjecture: Multi-Colouring of Kneser Graphs

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Abstract

If a graph is n-colourable, then it obviously is n'-colourable for any $n' \geq n$. But the situation is not so clear when we consider *multi-colourings* of graphs. A graph is (n, k)*-colourable* if we can assign each vertex a k-subset of $\{1, 2, \ldots, n\}$, such that adjacent vertices receive disjoint subsets.

We consider the following problem: if a graph is (n, k)-colourable, then for what pairs (n', k') is it also (n', k')-colourable? This question can be translated into a question regarding multi-colourings of Kneser graphs, for which Stahl formulated a conjecture in 1976. We present new results and discuss some observations that lead to simple proofs of some known cases of the conjecture.

1 Introduction and Results

All graphs in this note are finite, undirected and without multiple edges or loops. All colourings of such graphs are vertex colourings. A *proper* colouring of a graph assigns a colour to each vertex such that adjacent vertices receive different colours. A graph G is *n-colourable* if n colours are enough for a proper colouring of G, and the *chromatic number* $\chi(G)$ is the smallest n for which G is n-colourable.

Multi-colouring generalises vertex colouring. It has been studied extensively; see e.g. [8] for background. In a multi-colouring of a graph, each vertex receives a set of colours, and such a colouring is *proper* if adjacent vertices receive disjoint colour sets. A graph G is (n, k)*-colourable* if there is a proper multi-colouring by assigning k-subsets of $[n]$ ($= \{1, 2, \ldots, n\}$) to the vertices of G. And the *k-th multi-chromatic number* $\chi_k(G)$ is the smallest n such that G is (n, k)-colourable.

It this note we consider the following question.

Question 1.
If G is (n, k)-colourable, then for what pairs (n', k') is G also (n', k')-colourable?

We note that the corresponding question for more standard n-colouring is trivial: if G is n-colourable, then it is n'-colourable for all $n' \geq n$. Or, more precise: if $\chi(G) = n$, then G is n'-colourable if and only if $n' \geq n$. Maybe somewhat surprisingly, the question for multi-colouring appears to be much more challenging, and in fact it is mostly open!

Kneser graphs play a central role in the studies of multi-colouring. For $n \geq k \geq 1$, the *Kneser graph* $K(n, k)$ has as vertex set the collection of all k-subsets of $[n]$, and there is an edge between two vertices if and only if the two k-sets are disjoint. We will always assume $n \geq 2k$, as otherwise the graph is edgeless.

It is well known and easy to prove (see e.g. [8]) that a graph G is (n, k)-colourable if and only if there is a homomorphism from G to $K(n, k)$. (A *homomorphism* from a graph G to a graph H is a mapping $\varphi : V(G) \to V(H)$ that preserves edges; i.e. if uv is an edge in G, then $\varphi(u)\varphi(v)$ is an edge in H.)

This means that the following questions are all equivalent to Question 1.

1. *Given n, k, for what n', k' is the Kneser graph $K(n,k)$ also (n',k')-colourable?*
2. *Given n, k, for what n', k' is there a homomorphism from $K(n,k)$ to $K(n',k')$?*
3. *Given n, k, for what n', k' do we have $n' \geq \chi_{k'}(K(n,k))$?*

The last question was studied by Stahl [9], who formulated the following conjecture.

Conjecture 2 (Stahl [9])**.**
If $k' = qk - r$ where $q \geq 1$ and $0 \leq r \leq k-1$, then we have $\chi_{k'}(K(n,k)) = qn - 2r$.

Note that for $k = 1$, the Kneser graph $K(n,1)$ is just the complete graph on n vertices, and hence Stahl's conjecture is trivially true in that case. For $k' = 1$ (hence $q = 1$ and $r = k-1$) we find that Conjecture 1 is true, since $\chi_1(G) = \chi(G)$, and we know $\chi(K(n,k)) = n - 2k + 2$ by Lovász's proof [6] of the Kneser conjecture.

Stahl made quite a number of observations regarding the multi-chromatic number of Kneser graphs. In particular they showed that the conjectured value for the multi-chromatic number is an upper bound, i.e. $\chi_{qk-r}(K(n,k)) \leq qn - 2r$, and that in order to prove the conjecture it suffices to prove it for $r = k-1$.

In a follow-up paper [10], Stahl also proved a general lower bound for $\chi_{k'}(K(n,k))$:

$$\chi_{qk-r}(K(n,k)) \geq qn - 2r - (k^2 - 3k + 4). \tag{1}$$

Some special values of n, k, k' for which the conjecture is known to be true are as follows.

Theorem 3.
(a) *Conjecture 2 is true for the bipartite Kneser graphs $K(2k,k)$ and for the so-called* odd graphs *$K(2k+1,k)$ (all k') (Stahl [9]).*

(b) *Conjecture 2 is true whenever k' is a multiple of k (all n, k); in other words: $\chi_{qk}(K(n,k)) = qn$ (Stahl [9]).*

(c) *Conjecture 2 is true for all $k' \leq k$ (all n, k); in other words $\chi_{k-r}(K(n,k)) = n - 2r$ (Stahl [9]).*

(d) *Conjecture 2 is true for $k = 2$ and $k = 3$ (all n, k') (Stahl [10]).*

Osztényi [7] proved the conjecture for $2k < n < 3k$, $0 \leq r < \frac{k}{n-2k}$. That proof is long and quite involved. We give a one-paragraph proof in the next section (which also holds for $r = \frac{k}{n-2k}$ and $n \geq 3k$, $r = 0$).

Theorem 4.
If $0 \leq r \leq \dfrac{k}{n-2k}$, then we have $\chi_{k'}(K(n,k)) = qn - 2r$ (where $k' = qk - r$ with $q \geq 1$ and $0 \leq r \leq k-1$).

Another result we want to present in this note is that for a fixed k, only at most $k^3 - k^2$ values $\chi_{k'}(K(n,k))$ need to be determined in order to conclude whether or not Stahl's conjecture is true for that value of k and for *all n and k'*.

Theorem 5.
Fix $k \in \mathbb{N}$. Then there exist $n_0(k)$ and $q_0(n,k)$ such that the following holds.
If $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$ for all $2k \leq n \leq n_0(k)$ and for at least one $q \geq q_0(n,k)$, then we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $n \geq 2k$, $q \geq 1$ and $0 \leq r \leq k-1$.

The functions $n_0(k)$ and $q_0(n,k)$ in Theorem 5 need to satisfy some quite complicated equations we won't give here. From those equations, it is possible to show that we have $n_0(k) < k^3 - k^2 + 2k$ for all k, and $q_0(n,k) < \dfrac{4^k}{\mathrm{e}k}(n-2k)$ for all $k \geq 2$ and $n \geq 2k+1$ (where $\mathrm{e} \approx 2.718$ is Euler's number).

We could replace $q_0(n,k)$ by $q_0'(k) = \max\{q_0(n,k) \mid 2k \leq n \leq n_0(k)\}$ in Theorem 5 to remove the dependency of q_0 on n. We chose to keep $q_0(n,k)$, since for larger values of n we get better bounds for $q_0(n,k)$. For instance, if $n \geq (\log_2 \mathrm{e})\, k^2$, then we can show $q_0(n,k) < n$.

For $k = 4$, our methods show that we only need to find $\chi_{4q-3}(K(n,4))$ for $8 \leq n \leq 10$, $q = 13$, and $11 \leq n \leq 38$, $q = 12$. The cases $n = 8, 9$ follow from Theorem 3 (a). The case $n = 10$ is solved in [5]. So the first open case is to determine whether or not $\chi_{45}(K(11,4)) = 126$. Note that Stahl already showed $\chi_{45}(K(11,4)) \leq 126$, while bounds in the next section give $\chi_{45}(K(11,4)) \geq 124$.

In the next section we also explain that determining $\chi_{45}(K(11,4))$ can be done by finding the chromatic number of the lexicographic product $K(11,4) \bullet K_{45}$. Unfortunately, $K(11,4) \bullet K_{45}$ is a highly symmetric graph with 14,850 vertices and 12,021,075 edges, and none of the publicly available packages for graph colouring we could find seems to be able to deal with this graph within a reasonable amount of time.

2 Main Ideas

In this section we sketch some of the ideas behind our results and methods to approach Stahl's conjecture. In fact, most of these ideas have been observed before, but we haven't seen them used in the way we use them.

For any proper k'-multi-colouring of a graph G with $\chi_{k'}(G)$ colours, it is obvious that each colour class (the set of vertices whose colour set contains some particular colour) is an independent set, hence contains at most $\alpha(G)$ vertices (where $\alpha(G)$ is the *independence number*). Since each vertex appears in k' colour classes, this immediately gives $\chi_{k'}(G) \geq \left\lceil \dfrac{k'|V(G)|}{\alpha(G)} \right\rceil$.

For Kneser graphs we have $|V(K(n,k))| = \binom{n}{k}$ by definition, while the celebrated Erdős-Ko-Rado Theorem [2] gives that $\alpha(K(n,k)) = \binom{n-1}{k-1}$ for $n \geq 2k$. Substituting those values and $k' = qk - r$ in the bound above leads to

$$\chi_{k'}(K(n,k)) \geq \left\lceil \frac{k'\binom{n}{k}}{\binom{n-1}{k-1}} \right\rceil = \left\lceil \frac{k'n}{k} \right\rceil = qn - 2r - \left\lfloor \frac{r(n-2k)}{k} \right\rfloor. \tag{2}$$

This simple inequality is surprisingly powerful. For instance, it gives a better bound than (1) if $n \leq k^2 + 2$. It also more or less directly gives Theorem 4.

We can obtain further results by using more detailed knowledge about independent sets in Kneser graphs. For instance, in [2] it is also proved that if $n \geq 2k+1$, then the only independent sets of order $\binom{n-1}{k-1}$ in the Kneser graph $K(n,k)$ are the so-called *trivial* independent sets: those vertex sets whose vertices correspond to all k-sets in $[n]$ that contain some fixed element $i \in [n]$. Using that information about the structure of independent sets of order $\alpha(K(n,k))$ shows that we can only have equality in (2) in very special cases.

Theorem 6.
We have equality in (2) if and only if $k' = qk$ for some integer q. In those cases we have $\chi_{qk}(K(n,k)) = qn$.

Theorem 6 is not explicitly stated in Stahl [9], although it is implicit in its proof of Theorem 3 (b) (using significantly more involved arguments).

As observed in [3], we have for any graph G that $\chi_{k'}(G) = \chi(G \bullet K_{k'})$, where "$\bullet$" denotes the *lexicographic product* of graphs: $V(G \bullet H) = V(G) \times V(H)$, and $(u_1, v_1)(u_2, v_2) \in E(G \bullet H)$ if and only if either $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$. This allows us to translate the problem of finding multi-chromatic numbers to finding chromatic numbers, and also can give an alternative proof of $\chi_{k'}(G) \geq \left\lceil \frac{k'|V(G)|}{\alpha(G)} \right\rceil$.

One of the essential elements in the proof of Theorem 5 is the result of Hilton and Milner [4] that if an independent set in the Kneser graph $K(n,k)$, $n \geq 2k+1$, is not trivial, then it has order at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. This 'second best' bound is significantly smaller than the Erdős-Ko-Rado bound, which means that for large n and q, many of the colours used in a 'good' $(qk-r)$-multi-colouring of $K(n,k)$ must induce trivial independent sets. This observation allows us to prove relations between the multi-chromatic numbers $\chi_{qn-r}(K(n,k))$ for different values of n and q, and eventually to prove Theorem 5.

Finally, we note that Theorem 5 generalises some known results. Chvátal, Garey and Johnson [1] showed that for fixed k, we only need to find $\chi_{k+1}(K(n,k))$ for finitely many n to decide if Stahl's Conjecture holds for $\chi_{k+1}(K(n,k))$ for all n. And Stahl [9] proved that for fixed n, k and sufficiently large k', the correctness of the conjecture for k' is equivalent to its correctness for $k'-k$. The proof of that result is non-constructive and does not give an explicit bound on the value of k', and hence it can only give a version of Theorem 5 without a bound on the function $q_0(n,k)$.

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