Heroes in Orientations of Complete Multipartite Graphs

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Abstract

We almost characterize digraphs H such that orientations of complete multipartite graphs with no induced copy of H have bounded dichromatic number.

1 Introduction

Long version of this extended abstract can be found in [1].

In this paper, we only consider *directed graphs* (*digraphs* in short) with no digons (a cycle on two vertices), loops nor multi-arcs. If G is a digraph, we denote by $V(G)$ its set of vertices, and by $A(G)$ its set of arcs. For a vertex x of G, we denote by x^+ (resp. x^-, x^o) the set of its out-neighbours (resp. in-neighbours, non-neighbours). For a given set of vertices $X \subseteq V$, we denote by $G[X]$ the subgraph of G induced by X. For two disjoint set of vertices X, Y, we write $X \rightarrow Y$ if there are no arcs from Y to X, and $X \Rightarrow Y$ if for every $x \in X$ and for every $y \in Y$, $xy \in A(G)$. When $X = \{x\}$ we write $x \Rightarrow Y$ and $x \rightarrow Y$. Given two digraphs G_1 and G_2, $G_1 + G_2$ denotes the disjoint union of G_1 and G_2. Given two digraphs G and H, we say that G is *H-free* if it does not contain an induced copy of H. We denote as $Forb_{ind}(H)$ the set of all H-free digraphs.

A *tournament* is an orientation of a complete graph. A *transitive tournament* is an acyclic tournament and we denote by TT_n the unique acyclic tournament on n vertices. Given two tournaments H_1 and H_2, we denote by $\Delta(1, H_1, H_2)$ the tournament obtained from pairwise disjoint copies of H_1 and H_2 plus a vertex x, and all arcs from x to the copy of H_1, all arcs from the copy of H_1 to the copy of H_2, and all arcs from the copy of H_2 to x. When ℓ and k are integers, we write $\Delta(1, k, H)$ for $\Delta(1, TT_k, H)$ and $\Delta(1, \ell, k)$ for $\Delta(1, TT_\ell, TT_k)$. The tournament $\Delta(1, 1, 1)$ is also denoted by C_3. An *OCM graph* is an orientation of a complete multipartite graph, or equivalently a member of $Forb_{ind}(TT_1 + TT_2)$.

A *k-dicolouring* of G is a partition of $V(G)$ into k sets $V_1, \ldots, V_k$ such that $G[V_i]$ is acyclic for $i = 1, \ldots, k$. The *dichromatic number* of G, denoted by $\vec{\chi}(G)$ and introduced by Neuman-Lara [2] is the minimum integer k such that G admits a k-dicolouring. Given an undirected graph H, a *k-colouring* of H is a partition of $V(H)$ into k independent sets. The *chromatic number* of H is the minimum k such that H is k-colourable. For a digraph G we denote by $\chi(G)$ the chromatic number of the underlying graph of G.

Given a class of digraphs $\mathcal{C}$, we say that a digraph H is a *h*ero in $\mathcal{C}$ if H-free digraphs of $\mathcal{C}$ have bounded dichromatic number.

In a seminal paper, Berger, Choromansky, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé gave a recursive characterization of all heroes in tournaments, as follows.

Theorem 1 (Berger et al. [4])**.** *A digraph H is a hero in tournaments if and only if :*

- *$H = TT_1$ (the one-vertex digraph), or*
- *$H = (H_1 \Rightarrow H_2)$, where H_1 and H_2 are heroes in tournaments, or*

- *$H = \Delta(1, k, H')$ or $H = \Delta(1, H', k)$, where $k \geq 1$ and H' is a hero in tournaments.*

Observe that if a class of digraphs $\mathcal{C}$ contains all tournaments, then a hero in $\mathcal{C}$ must be a hero in tournaments. In [3], it is conjectured that heroes in OCM graphs are the same as heroes in tournaments. We disprove this conjecture by showing the following:

Theorem 2. *The digraphs $\Delta(1, 2, C_3)$, $\Delta(1, C_3, 2)$, $\Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in tournaments.*

On the positive side, we prove that:

Theorem 3. *A digraph H is a hero in OCM graphs if :*

- *$H = TT_1$, or*
- *$H = (H_1 \Rightarrow H_2)$, where H_1 and H_2 are heroes in OCM graphs, or*
- *$H = \Delta(1, 1, H')$ where H' is a hero in OCM graphs.*

Observe that the second case ($H = (H_1 \Rightarrow H_2)$) in the theorem above implies that a digraph is a hero in OCM graphs if *and only if* each of its strong connected components are. Indeed, the *only if* part of the assertion holds because a subgraph of a hero in any class is a hero in this class.

Since a hero in OCM graphs must be a hero in tournaments, Theorem 1, Theorem 2 and Theorem 3 imply that, to get a full characterization of heroes in OCM graphs, it suffices to decide whether $\Delta(1, 2, 2)$ is a hero in OCM graphs or not. If it is not, then heroes in OCM graphs are precisely the ones described in Theorem 3. If it is, then a digraph H is a hero in OCM graphs if and only if H can be built as in Theorem 3 but with the base case being $H = TT_1$ *or* $H = \Delta(1, 2, 2)$.

We also managed to prove that if $\Delta(1, 2, 2)$ is a hero, then the following conjecture by Axenovich, Rollin and Ueckerdt in [7] holds :

Conjecture 1. *Let $\mathcal{C}$ be the class of ordered graphs (V, E) such that for every five distinct vertices $a < b < c < d < e$, either $ac \notin E$, $ce \notin E$ or $bd \notin E$. Then $\mathcal{C}$ has bounded chromatic number.*

2 Heroes in tournaments but not in OCM graphs

The goal of this section is to prove that $\Delta(1, 2, C_3)$, $\Delta(1, C_3, 2)$, $\Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in OCM graphs. Since reversing all arcs of a $\Delta(1, 2, C_3)$-free OCM graph results in a $\Delta(1, C_3, 2)$-free OCM graph and does not change the dichromatic number, if $\Delta(1, 2, C_3)$ is not a hero in OCM graphs then neither is $\Delta(1, C_3, 2)$. Similarly, if $\Delta(1, 2, 3)$ is not a hero in OCM graphs then neither is $\Delta(1, 3, 2)$. Hence, we will prove the existence of $\{\Delta(1, 2, C_3), \Delta(1, 2, 3)\}$-free OCM graphs with arbitrarily large dichromatic number. We need the following classical definition : a *feedback arc set* of a given digraph G is a set of arcs F of G such that their deletion from G yields an acylic digraph.

Let $D_n = (V_n, E_n)$ be a digraph where V_n is the set of triples of integers (a, b, c) such that $1 \leq a < b < c \leq n$ and $E_n = F_n \cup B_n$ where :

- $F_n = \{(a, b, c)(b, c, d), a < b < c < d < e\}$
- $B_n = \{(a, b, c)(a', b', c'), b > b'\} \setminus F_n$

If we fix b and denote by V_b the set of triples $(a, b, c) \in V_n$, then D_n is indeed an OCM graph with partition given by the sets V_b. Observe that F_n is a feedback arc set of D_n. The definition of the arcs in F_n imply the following lemma.

Lemma 1. *For every vertex x of D_n, $|\{b \mid (a,b,c)x \in F_n\}| \leq 1$ and $|\{b \mid x(a,b,c) \in F_n\}| \leq 1$.*

Let T be a subdigraph of D_n that induces a tournament. Since the V_b are independent sets, T contains at most one vertex from each V_b. Therefore, Lemma 1 implies that T has a feedback arc set F that induces a digraph where every vertex has out-degree and in-degree at most 1. But a short analysis proves that $\Delta(1, 2, C_3)$ and $\Delta(1, 2, 3)$ do not admit such feedback arc sets, and thus D_n does not contain $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$.

Let us now prove that $(D_n)_{n\in\mathbb{N}}$ has unbounded dichromatic number. For a digraph G, the *line digraph* of G, denoted by $L(G)$, is defined as follows: its vertex set is $A(G)$, and there is an arc from $ab \in A(G)$ to $cd \in A(G)$ if and only if $b = c$. The next lemma links the (usual) chromatic numbers of the underlying graphs of G and $L(G)$.We think it appears for the first time in [5].

Lemma 2. *[5] For every digraph G, we have $\chi(L(G)) \geq \log(\chi(G))$.*

Now one can observe that $D'_n = (V(D_n), F_n)$ is isomorphic to $L(L(TT_n))$ so by the previous lemma we have $\chi(D'_n) \geq \log(\log(n))$. We conclude by proving that $\vec{\chi}(D_n) \geq \chi(D'_n)/2$. For that consider an optimal dicolouring of D_n and let X be a colour class. By definition of F_n, it is easy to see that if $xy \in F_n$ and $yz \in F_n$, then $zx \in B_n$. Therefore $D'_n[X]$ cannot contain any directed path on 3 vertices, as it would imply that $D_n[X]$ contains a directed triangle. Hence the underlying graph of $D'[X_n]$ is bipartite, which concludes the proof.

3 Heroes in OCM graphs

3.1 Strong components

Our main result is the following.

Theorem 4. *If H_1 and H_2 are heroes in $Forb_{ind}(TT_1 + TT_2)$, then so is $H_1 \Rightarrow H_2$.*

We actually prove the following stronger result:

Theorem 5. *Let H_1, H_2 and F be digraphs such that $H_1 \Rightarrow H_2$ is a hero in $Forb_{ind}(F)$ and H_1 and H_2 are heroes in $Forb_{ind}(TT_1 + F)$. Then $H_1 \Rightarrow H_2$ is a hero in $Forb_{ind}(TT_1 + F)$.*

To see that Theorem 5 implies Theorem 4, take $F = TT_2$ and observe that $Forb_{ind}(TT_2)$ is the class of digraphs with no arc and thus every digraph is a hero in $Forb_{ind}(TT_2)$. Note also that by taking $F = TT_1$, we have that $Forb_{ind}(F)$ is empty and that $Forb_{ind}(TT_1 + F)$ is the class of tournaments, so Theorem 5 yields the result of [4] (see **(3.1)**) stating that H is a hero in tournaments if and only if all of its strong components are. Then, by induction, we get the same result for the class of digraphs with bounded independence number, reproving Theorem 1.4 of [6].

For $t \in \mathbb{N}$, we say that a digraph K is a *t-cluster* if $\vec{\chi}(K) \geq t$ and $|V(K)| \leq f(t)$, where $f(t)$ is the function defined recursively by $f(1) = 1$ and $f(t) = 1 + f(t-1)(1 + f(t-1))$.

Our proof is inspired from but simpler than the analogous result for tournaments in [4], even though our result is more general. The idea is the following : we prove first that digraphs in $Forb_{ind}(TT_1 + F, H_1 \Rightarrow H_2)$ that do not contain a t-cluster for some t have bounded dichromatic number. Then, given a t-cluster K and a subset H of K inducing H_2 (resp. H_1) , the set of vertices x such that x^+ (resp. x^-) contains H is H_1-free (resp. H_2-free) and thus has bounded dichromatic number. Since the number of such H is bounded (by $2^{f(t)}$), we win by proving that if t is sufficiently large, every $x \notin K$ will satisfy this property for some such H.

3.2 Growing a hero

We managed to prove the following theorem:

Theorem 6. *If H is a hero in OCM graphs, then so is $\Delta(1, H, 1)$.*

Again, our proof is inspired by the proof of the similar theorem for tournaments from [4]. It begins with the following lemma

Lemma 3. *Let G be a $\Delta(1, 1, H)$-free OCM graph given with a partition $(X_1, \dots, X_n)$ of its vertex set $V(G)$. Suppose that c is an integer such that:*

- *H-free OCM graphs have dichromatic number at most r,*
- $\forall\, 1 \leq i \leq n \;\; \vec{\chi}(X_i) \leq r$,
- $\forall\, 1 \leq i \leq n \; \forall v \in X_i \;\; \vec{\chi}(v^+ \cap (X_1 \cup \dots \cup X_{i-1})) \leq r$,
- $\forall\, 1 \leq i \leq n \; \forall v \in X_i \;\; \vec{\chi}(v^- \cap (X_{i+1} \cup \dots \cup X_n)) \leq r$.

Then $\vec{\chi}(G) \leq 8r + 4$.

We then define a *H-jewel* as a subset of vertices isomorphic to $H \Rightarrow H$, and a *H-jewel-chain* of length n as a sequence $(J_1, \dots, J_n)$ of pairwise disjoint H-jewels such that for $i = 1, \dots, n-1$, $J_i \Rightarrow J_{i+1}$, and for every $1 \leq i < j \leq n$, $J_i \rightarrow J_j$. We consider a H-jewel-chain of maximum length and create a bag X_i for every H-jewel J_i and insert every vertex v in the bag numbered $\min\{i \mid v^+ \cap J_i \neq \emptyset\}$. We then prove that this partition satisfies the conditions of Lemma 3 for some constant r depending only on $\vec{\chi}(Forb_{ind}(TT_1 + TT_2, H))$, thus proving Theorem 6.

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