Note on light 3-stars in embedded graphs

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Abstract

By $g(k,t)$ we denote the smallest integer such that every plane graph with girth $g \geq g(k,t)$, minimum degree at least 2, and no $(k+1)$-paths consisting of vertices of degree 2 has a 3-vertex with at least t neighbors of degree 2. Borodin and Ivanova completed the list of values of $g(k,t)$ for all $k \geq 1$ and $1 \leq t \leq 3$.

By $g^*(k,t)$ we denote the smallest integer for which there exists an (integer) function $f(k)$ such that for every surface $\mathcal{S}$ with non-positive Euler characteristic $\chi(\mathcal{S})$ and every graph G embedded on $\mathcal{S}$ with $|V(G)| > f(k)|\chi(\mathcal{S})|$, $g(G) \geq g^*(k,t)$, $\delta(G) \geq 2$, and no $(k+1)$-path consisting of vertices of degree 2, $k \geq 1$, G has a 3-vertex with at least t neighbors of degree 2. We have showed that $g^*(k,1) = 4k+5$ if $f(k) \geq 10k+5$, $g^*(k,2) = 4k+5$ if $f(k) \geq 16k+11$, and $g^*(k,3) = 4k+7$ if $f(k) \geq 30k+30$. Moreover, we will discuss the quality of our results.

1 Introduction

In this contribution we use a standard graph theory terminology according to the books [1] and [6]. However, we recall some more frequent notions. The degree $\deg(v)$ of a vertex v is the number of edges incident with v, and $\delta(G) = \delta$ is the minimum vertex degree in G. A k-vertex is a vertex v with $\deg(v) = k$. The *girth* $g(G)$ of G is the length of the shortest cycle in G.

An *orientable surface* (or an orientable compact 2-manifold) $\mathbb{S}_k$ of genus k is obtained from the sphere by adding k handles. Analogously, a *non-orientable surface* (or a non-orientable compact 2-manifold) $\mathbb{N}_k$ of genus k is obtained from the sphere by adding k crosscaps. The *Euler characteristic* of surfaces is defined by $\chi(\mathbb{S}_k) = 2 - 2k$ and $\chi(\mathbb{N}_k) = 2 - k$. By a surface $\mathcal{S}$ we mean either an orientable surface $\mathbb{S}_k$ or a non-orientable surface $\mathbb{N}_k$.

Given an embedding of a graph G on a surface $\mathcal{S}$ (a drawing of G in such a way that no edges cross each other), the components of $\mathcal{S} - G$ are called *regions*. If each region is homeomorphic to an open disk, the embedding $G \to \mathcal{S}$ is called a *2-cellular embedding* (or 2-cell embedding) and the regions are also called *faces* of G. For the rest of this paper all embeddings will be 2-cellular embeddings of graphs in surfaces. For a given embedding $G \to \mathcal{S}$ let $V(G)$, $E(G)$, and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. To describe a graph embedding $G \to \mathcal{S}$ it is sufficient to specify the rotation system of G. Therefore we distinguish two (orientations) types of edges: orientation-preserving (type 0) and orientation-reversing edges (type 1). Then rotation system of graph G is an assignment of a rotation to each vertex of G and a designation (labelling) of orientation type to each edge of G. In figures, edges marked with small "x" represent orientation-reversing edges (for more details see [6], pg 113).

For the class of planar graphs there are many results concerning the existence and the structure of small subgraphs containing only vertices of small degrees − see e.g. the survey paper [9].

We are interested in the subgraphs induced by vertices of degree 3 and their neighbors − so called 3-stars (we denote it as S_3). By $g(k,t)$ we denote the smallest integer such that every plane graph with girth $g \geq g(k,t)$, minimum degree at least 2 and no $(k+1)$-paths consisting of vertices of degree 2 (also called $(k+1)$-threads), where $k \geq 1$, has a 3-vertex with at least t neighbors of degree 2, where $1 \leq t \leq 3$. All known results was summarized into the following theorem:

Theorem 1 (Borodin, Ivanova [2]). *For the value of $g(k,t), k \geq 1$, it holds:*

(i) $g(k,1) = 3k+4$,

(ii) $g(k,2) = 3k+5$,

(iii) $g(1,3) = 10$ *and* $g(k,3) = 3k+8$ *for* $k \geq 2$.

Results about the structure of planar graphs can be naturally extended to graphs embeddable on other surfaces – see e.g. papers [7, 8]. Theorem 1 can be extended to embeddings with positive Euler characteristic, since discharging rules and redistribution of initial charges used in [2] can be applied in the same way also for graphs embedded on $\mathbb{N}_1$ (as $\chi(\mathbb{N}_1) = 1$) and each graph showing the optimality of lower bounds can be embedded on $\mathbb{N}_1$ (by adding one crosscap to sphere we obtain non-orientable surface $\mathbb{N}_1$; if we add a crosscap into planar embedding of G in such a way that exactly one edge e incident with two different faces will cross this crosscap, then the resulting embedding is a 2-cellular embedding of G on $\mathbb{N}_1$ and two faces incident with e merge into one face with bigger degree – hence $g(G)$ will not change).

For graphs embedded on surfaces with non-positive Euler characteristic we define $g^*(k,t)$ as the smallest integer for which there exists an (integer) function $f(k)$ such that for every surface $\mathcal{S}$ with non-positive Euler characteristic $\chi(\mathcal{S})$ and every graph G embedded on $\mathcal{S}$ with $|V(G)| > f(k)|\chi(\mathcal{S})|$, $g(G) \geq g^*(k,t)$, $\delta(G) \geq 2$, and no $(k+1)$-path consisting of vertices of degree 2, $k \geq 1$, G has a 3-vertex with at least t neighbors of degree 2. Our aim is to minimize the value of $g^*(k,t)$ in the sense that increasing the lower bound $f(k)$ for the number of vertices will not affect the value of $g^*(k,t)$, and decreasing of $f(k)$ will cause that $g^*(k,t)$ is not defined in general. We have showed the following:

Theorem 2. *Let G be a connected graph on n vertices with $\delta(G) \geq 2$ embedded on a surface $\mathcal{S}$ of Euler characteristic $\chi(\mathcal{S}) \leq 0$. Then for the value of $g^*(k,t)$ it holds:*

(*i*) $g^*(k,1) = 4k+5$, *if* $n > (10k+5)|\chi(\mathcal{S})|$,

(*ii*) $g^*(k,2) = 4k+5$, *if* $k \leq 5$ *and* $n > (16k+11)|\chi(\mathcal{S})|$,
and $g^*(k,2) = 4k+5$, *if* $k \geq 6$ *and* $n > (10k+5)|\chi(\mathcal{S})|$,

(*iii*) $g^*(1,3) = 11$, *if* $n > 44|\chi(\mathcal{S})|$,
and $g^*(k,3) = 4k+7$, *if* $k \geq 2$ *and* $n > (30k+30)|\chi(\mathcal{S})|$.

2 The optimality of $f(k)$

It is easy to see that $g^*(k,1) \leq g^*(k,2) \leq g^*(k,3)$ (for graphs with sufficiently many vertices). To show that $g^*(k,1) \geq 4k+5$ it suffices to put k vertices of degree 2 on every edge of a quadrangular grid of $\mathbb{S}_1$ or $\mathbb{N}_2$ with $g(G) = 4$. The resulting graph G_1 has $g(G_1) = 4k+4$ and does not contain any vertex of degree 3. Similarly, to show that $g^*(k,3) \geq 4k+7$, we put k vertices of degree 2 on every dashed edge of a graph of hexagonal grid with $g(G) = 6$ (see e.g. Figure 1) to obtain a graph G_2 having $g(G_2) = 4k+6$, where each 3-vertex is adjacent to precisely two 2-vertices (note that dashed edges have to be chosen in such a way that every cycle of length 6 contains at least four of them).

In embedding of hexagonal grid in Figure 1, the orientation of rectangle with black arrows corresponds to embeddings on torus and orientation of rectangle with red arrows corresponds to embeddings on $\mathbb{N}_2$. Note that graphs of quadrangular and hexagonal grids can be arbitrarily large as well as graphs G_1 and G_2. Graphs embedded on surfaces with higher genus can be obtained from graphs embedded on torus by adding of edges in appropriate way or by changing the types of some edges (for more details see [3]).

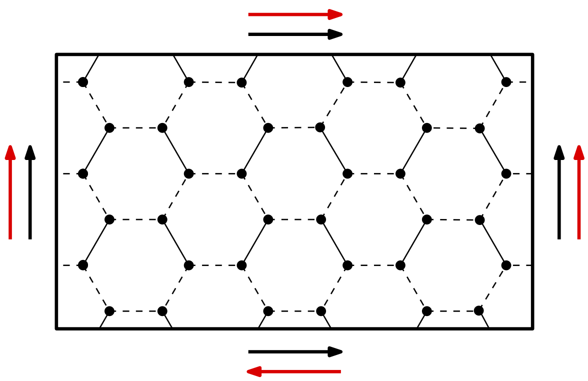

Figure 1: Embedding of hexagonal grid on $\mathbb{S}_1$ and $\mathbb{N}_2$

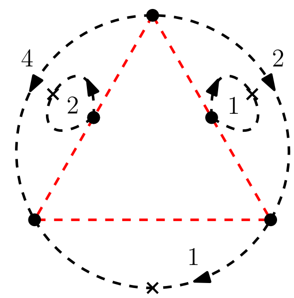

Figure 2: G_3 with voltages from Z_r; H_3^β shows optimality of $f(k)$ in Theorem 2(i), and (ii) for $k \geq 6$

On the other hand, one can ask if the condition on the number of vertices is necessary for surfaces with negative Euler characteristic in general. Clearly, as for every pair of parameters $r \geq 4$ and $g \geq 3$ there exists a finite r-regular graph of girth g (see [4] and [5]), some condition is really needed (although the determining of genus of this graph is hard in general).

We will say that the value of $f(k)$ is *optimal* (for some combinations of k and t) if there is a graph G with $f(k)|\chi(\mathcal{S})|$ vertices, girth $g(G) \geq g^*(k,t)$, having no 3-vertex with t neighbors of degree 2 for infinitely many surfaces.

Graph G_3 (see Figure 2) is a 4-regular voltage graph with voltages $\beta : E(G_3) \to Z_r$, red edges correspond to edges with identity voltage. Moreover, G satisfies Kirchhoff's voltage law (voltage on the boundary walk of every face is identity) and it is embedded on a surface $\mathcal{S} = \mathbb{N}_3$, $\chi(\mathcal{S}) = -1$. Derived graph G_3^β is connected and simple for $r \geq 5$, has $5r$ vertices, $10r$ edges and $4r$ faces of degree 5 and is embedded on a surface $\mathcal{S}^\beta$ with $|\chi(\mathcal{S}^\beta)| = r \cdot |\chi(\mathcal{S})| = r$. We additionally subdivide each dashed edge k times and the resulting graph we denote as H_3^β. Graph H_3^β has $5r$ vertices of degree four, $10rk$ vertices of degree two and every face has degree $5k+5$. Hence, H_3^β has together $(10k+5)r$ vertices, does not contain any 3-vertex, for sufficiently big r ($r \geq 9$) has $g(H_3^\beta) > g^*(k,1) = 4k+5$, and shows the optimality of bound $f(k) = (10k+5)$ in Theorem 2(i), and (ii) for $k \geq 6$.

In Theorem 2(ii) for values $k \in \{2,3,4,5\}$ we have not found the optimal value of $f(k)$, we only know that $f(k) \leq 16k+11$. Note, that we know $f(1) = 27$.

Problem 1. *Find the optimal values of function $f(k)$, $k \in \{2,3,4,5\}$, such that for every surface $\mathcal{S}$ with non-positive Euler characteristic $\chi(\mathcal{S})$ and every connected graph G embedded on $\mathcal{S}$ with $n > f(k)|\chi(\mathcal{S})|$ vertices, $\delta(G) \geq 2$, $g(G) \geq 4k+5$, and no $(k+1)$-path consisting of vertices of degree 2 ($k \geq 1$), G contains a 3-vertex adjacent to at least two 2-vertices.*

As we do not have an example showing the optimality of values of $f(k)$ in Theorem 2(iii), it would be interesting either to find graphs showing the optimality of values 44 or $30k+30$, or prove the better lower bound for $f(k)$. We can formulate this as follows:

Problem 2. *Find the optimal values of function $f(k)$ such that for every surface $\mathcal{S}$ with non-positive Euler characteristic $\chi(\mathcal{S})$ and every connected graph G embedded on $\mathcal{S}$ with $n > f(k)|\chi(\mathcal{S})|$ vertices, $\delta(G) \geq 2$, $g(G) \geq 4k+7$, and no $(k+1)$-path consisting of vertices of degree 2 $(k \geq 1)$, G contains a 3-vertex adjacent to three 2-vertices.*

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