On K_2-hamiltonian graphs

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Abstract

Motivated by a conjecture of Grünbaum and a problem by Katona, Kostochka, Pach and Stechkin, we study K_2-(hypo)hamiltonian graphs. These are (non-hamiltonian) graphs in which the removal of any pair of adjacent vertices yields a hamiltonian graph. Using computational methods we show that Petersen's graph is the smallest K_2-hypohamiltonian graph and classify their existence for all but two orders.

In addition, we fully classify the existence of K_2-hypohamiltonian graphs for all orders in the class of cubic graphs and classify all K_2-hypohamiltonian snarks up to 36 vertices.

Finally, we also determine the smallest planar K_2-hypohamiltonian graph of girth 5 and the smallest cubic planar K_2-hypohamiltonian graph and show that there exist planar K_2-hypohamiltonian graphs for every order from 177 onward.

1 Introduction

Hamiltonian and hypohamiltonian graphs have been intensely studied since the 1960s [2, 4]. In this project we only consider connected simple graphs. A graph is *hamiltonian* if it contains a cycle which visits every vertex. Similarly, a graph is *hypohamiltonian* if it is not hamiltonian, but every vertex-deleted subgraph is. Motivated by a conjecture of Grünbaum and a problem of Katona, Kostochka, Pach and Stechkin, we investigate K_2-hamiltonian graphs based on earlier work by Zamfirescu [8].

Grünbaum [3] defined $\Gamma(j,k)$ with $k \geq j$ to be the family of graphs whose order and *circumference*, i.e. the length of a longest cycle, differ by k and in which any j vertices are missed by some longest cycle. $\Gamma(1,1)$ are then precisely the hypohamiltonian graphs. In 1974, Grünbaum conjectured that $\Gamma(j,j)$ must be empty for $j \geq 2$. So far little is known about the truth of this conjecture. In 1989, Katona, Kostochka, Pach and Stechkin [6] asked whether any graph on n vertices in which any $n-2$ vertices induce a hamiltonian graph must itself be hamiltonian. The difference between this and Grünbaum's conjecture is that the latter does not allow $n-1$ cycles in the graph, while Katona et al. do allow them. K_2-hamiltonian graphs relax this problem further by only looking at subgraphs of order $n-2$ obtained by removing a pair of adjacent vertices.

2 K_2-hamiltonian graphs

A graph is called *K_2-hamiltonian* if the deletion of any two adjacent vertices yields a hamiltonian graph. If this graph is also non-hamiltonian, we call it *K_2-hypohamiltonian.* An example of a K_2-hypohamiltonian graph is Petersen's graph, which is also the smallest such graph. We have designed and implemented an algorithm that checks whether a given input graph is K_2-(hypo)hamiltonian and which turns out to be quite efficient in practice. In particular, we use a backtracking algorithm with certain heuristics to restrict the search space. The program is also able to determine hamiltonicity and hypohamiltonicity of the given input graph.

Using this algorithm we construct and analyse K_2-(hypo)hamiltonian graphs in several classes of graphs.

2.1 General case

We shall focus mainly on K_2-hypohamiltonian graphs. Using existing software we generated all connected simple graphs of a given order (with a given lower bound on the girth) and then filtered these graphs using our program to check whether they are K_2-hypohamiltonian or not. Using this as well as results by Zamfirescu [8], we were able to classify all orders except two for which there exists a K_2-hypohamiltonian graph.

Theorem 1. *For any integer $n \in \{10, 13, 15, 16\}$ or $n \geq 18$, there exists a K_2-hypohamiltonian graph of order n. For $n < 10$ or $n \in \{11, 12\}$, there exist no K_2-hypohamiltonian graphs of order n.*

Hence, the only open orders are 14 and 17. For the former any K_2-hypohamiltonian graph must have girth 3, for the latter any K_2-hypohamiltonian graph must have girth at most 4.

It was shown by Zamfirescu [8] that there exist infinitely many K_2-hypohamiltonian graphs of girth 5 and infinitely many K_2-hypohamiltonian graphs of girth 6. So far none of the K_2-hypohamiltonian graphs we constructed had girth 3 or girth 4 and none are known for girth 7 or higher.

2.2 Cubic case

In a paper by Thomassen [7] a classification of all orders for which a cubic hypohamiltonian graph exists was given. Building on an operation by Zamfirescu [8] we do the same for cubic K_2-hypohamiltonian graphs. We only need to verify K_2-hypohamiltonicity for graphs of girth at least 5 as it is easy to see that there exist no K_2-hypohamiltonian cubic graphs containing triangles or 4-cycles. In Table 1 we give an exhaustive count of all K_2-hypohamiltonian graphs up to 32 vertices as well as the count of all hypohamiltonian K_2-hamiltonian graphs.

n	Girth	Total	Non-hamiltonian	K_2-hypo	hypo- and K_2-
10	≥ 5	1	1	1	1
12 - 16	≥ 5	60	0	0	0
18	≥ 5	455	3	0	0
20	≥ 5	5 783	15	1	1
22	≥ 5	90 938	110	3	3
24	≥ 5	1 620 479	1 130	0	0
26	≥ 5	31 478 584	15 444	6	5
28	≥ 5	656 783 890	239 126	14	12
30	≥ 5	14 621 871 204	4 073 824	15	14
32	≥ 5	345 975 648 562	75 458 941	12	12

Table 1: Counts of all cubic non-hamiltonian and K_2-hypohamiltonian graphs of girth at least 5. In column *K_2-hypo* the counts of cubic K_2-hypohamiltonian graphs can be found. In column *hypo- and K_2-* the counts of cubic hypohamiltonian K_2-hamiltonian graphs can be found.

Using these graphs (as well as some of the graphs in Table 2) and the operation mentioned earlier we obtain the following result.

Theorem 2. *There exists a cubic K_2-hypohamiltonian graph of order $n = 2k$ if and only if $n \in \{10, 20, 22\}$ or $n \geq 26$.*

We also exhaustively classified all *snarks*, i.e. cyclically 4-edge-connected cubic graphs with chromatic index 4, with girth at least 5 up to 36 vertices in Table 2. However, the operation that was used earlier does not preserve the chromatic index of a graph.

n	Girth	Total	K_2-hypohamiltonian
10	≥ 5	1	1
12 - 18	≥ 5	2	0
20	≥ 5	6	1
22	≥ 5	20	2
24	≥ 5	38	0
26	≥ 5	280	6
28	≥ 5	2 900	14
30	≥ 5	28 399	9
32	≥ 5	293 059	11
34	≥ 5	3 833 587	1 036
36	≥ 5	60 167 732	3 849
38	≥ 6	39	20

Table 2: Counts of K_2-hypohamiltonian snarks.

The snarks in Table 2 were downloaded from the *House of Graphs* [1] at `https://hog.grinvin.org/Snarks`.

2.3 Planar case

In [4] Holton and Sheehan ask what the smallest order of a planar hypohamiltonian graph is. This problem is still open, but in a paper [5] by Jooyandeh et al. it was shown that the smallest planar hypohamiltonian graph of girth 5 has order 45. We determined the counts for planar K_2-hypohamiltonian graphs of girth 5 and list them in Table 3. Up to 48 vertices these counts are exhaustive. The counts mentioned for 50-vertices or higher will likely not be complete.

Order	≤ 47	48	49	50	51	52	53
K_2-hypohamiltonian	0	1	≥ 9	≥ 11	≥ 0	≥ 6	≥ 9

Table 3: Counts of K_2-hypohamiltonian graphs among the planar graphs of girth 5.

It is easy to see that a K_2-hypohamiltonian graph is always 3-connected. Hence, it follows by Euler's formula that planar K_2-hypohamiltonian graphs have girth at most 5.

From Table 3 we obtain the following.

Theorem 3. *The smallest planar K_2-hypohamiltonian graph of girth 5 has order 48. There is precisely one such graph on 48 vertices, namely the graph on the left-hand side of Figure 1.*

Using operations from [8] we show the following.

Theorem 4. *For every integer $n \geq 177$ there exists a planar K_2-hypohamiltonian graph of order n.*

We also determined the counts of all cubic planar K_2-hypohamiltonian graphs up to 78 vertices. They can be found in Table 4.

Order	≤ 66	68	70	72	74	76	78
K_2-hypohamiltonian	0	1	1	0	1	3	3

Table 4: Counts of all K_2-hypohamiltonian graphs among the cubic planar graphs up to 78 vertices.

Using this we were able to determine the smallest cubic planar K_2-hypohamiltonian graph.

Theorem 5. *The smallest cubic planar K_2-hypohamiltonian graph has order* 68*. There is only one such graph on* 68 *vertices, namely the graph on the right-hand side of Figure 1.*

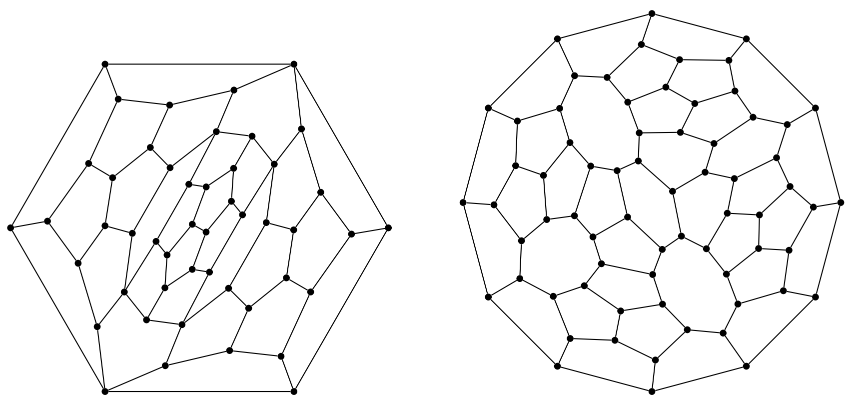

Figure 1: Left: The smallest planar K_2-hypohamiltonian graph of girth 5. Right: The smallest cubic planar K_2-hypohamiltonian graph.

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