On edge colorings of graphs with no color-rich cycles

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Abstract

For a graph G and an integer i, let $\mathcal{K}_i^\circ(G)$ be the maximum number of colors in an edge-coloring of G such that each its cycle contains at most i colors. We explore basic properties and estimates of this coloring invariant. We give a greedy-based degree-related general lower bounds for $\mathcal{K}_i^\circ(G)$ when $i \geq 3$ and discuss their sharpness. In particular, we provide the exact value for $\mathcal{K}_3^\circ(G)$ and $\mathcal{K}_2^\circ(G)$ for G being 5-connected or 3-connected, respectively. Furthermore, we show that, for a graph G of vertex connectivity 2, the exact value of $\mathcal{K}_2^\circ(G)$ can be found by maximizing the number of components after deleting a 2-cut, and, if G is highly connected, then the exact value is 2. We also provide upper bounds in terms of anti-Ramsey numbers for the cycles of length $i+1$ and discuss the effectiveness of these bounds.

In the research of edge colorings, a great attention is paid on the study of colorings defined by constraints on the number of colors appearing on particular subgraphs within a graph. For example, the classical proper edge coloring satisfies the condition that, for each maximal star S_x centered at a vertex x, the number of colors on S_x is $\deg(x)$; the related coloring invariant – the chromatic index – is the minimum number of colors of such coloring. If one asks to see at least (at most) i colors on each S_x for a fixed positive integer i, the corresponding colorings are m_i- and M_i-colorings defined first in [3]; the latter one was further extensively investigated in [4], [2], [7], [8], [14]. Other kinds of edge colorings operate with cycles within graphs – the requirement of at least three colors on every cycle (while keeping the coloring to be proper) yields the notion of acyclic edge coloring, first considered in [10] (and independently in [1]); a generalization where each cycle C sees at least $\min\{|C|, r\}$ colors for a fixed $r \geq 4$ was introduced in [12]. A variant of acyclic edge coloring where, instead of cycles, the facial cycles (or, in general, face-induced subgraphs) of plane graphs were considered, has been studied in [5]. Very recently, a facial variation where each face of a plane graph sees at most $i \geq 3$ colors (and the total number of used colors is to be maximized) was introduced in [6].

Motivated by the above coloring concepts, we consider the following problem:

Problem 1. *Given a graph G and a positive integer i, determine the maximum number $\mathcal{K}_i^\circ(G)$ of colors in an edge-coloring of G such that every cycle of G sees at most i colors.*

To deal with Problem 1, it is enough to consider 2-connected graphs, as the colors used in particular block of a graph have no influence on color sets of cycles in other blocks. Thus $\mathcal{K}_i^\circ(G) = \sum_{B \in B(G)} \mathcal{K}_i^\circ(B)$, where $B(G)$ is the set of blocks of a graph G; and therefore, $\mathcal{K}_i^\circ(G)$ is an additive coloring invariant. On the other hand, it is not hereditary, because, for each fixed i, taking an n-cycle C_n with $n > i+1$, we have $\mathcal{K}_i^\circ(C_n) = i$, but, for its subgraph P_n, the n-vertex path, $\mathcal{K}_i^\circ(P_n) = n-1$.

We give a greedy-based degree-related general lower bound for $\mathcal{K}_i^\circ(G)$ when $i \geq 3$ is odd:

Theorem 1. *Let $i \geq 3$ be odd, G be a connected graph of order at least 2 and let $\mathcal{S} = \{S \subseteq V(G) : |S| = \frac{i-1}{2}\}$. Then* $\mathcal{K}_i^\circ(G) \geq 1 + \max_{S \in \mathcal{S}} \left\{ \sum_{x \in S} \deg(x) - |E(G[S])| \right\}$.

To prove Theorem 1, it is sufficient to take any subset $S \subseteq V(G)$ of $\frac{i-1}{2}$ vertices and color all edges incident with vertices of S with distinct colors; the remaining edges of G color with one extra color. In so obtained coloring, the number of colors equals the sum of degrees of vertices from S minus the number of edges with both endvertices in S. Now, each cycle of G visits each vertex of S at most once, and so it sees at most $2 \cdot |S| + 1 = i$ colors. By maximizing the number of colors through all possible S, the result follows.

A similar approach can be used also for i being even: an admissible coloring which uses $2 + \max_{S' \in \mathcal{S}'} \left\{ \sum_{x \in S'} \deg(x) - |E(G[S'])| \right\}$ colors can be constructed.

There are graphs for which the lower bound provided by Theorem 1 is sharp – consider, for example, the complete bipartite graph $K_{2,r}$ and $i = 3$. We obtain that $\mathcal{K}_i^\circ(K_{2,r}) \geq r + 1$; this, in particular, implies that there exists a 2-valent vertex y_1 such that its incident edges are colored differently. Then, to avoid a rainbow 4-cycle, we have that, for each 2-valent vertex $y_k \neq y_1$, its incident edges have the same color, and so the total number of colors is at most $2 + (r - 1) = r + 1$.

For an upper bound on $\mathcal{K}_i^\circ(G)$, one can use the value of anti-Ramsey number $\mathrm{ar}(G, C_{i+1})$, which is equal to the maximum number of colors in an edge-coloring of G without any rainbow copy of C_{i+1} (for the survey of results, see [11]). It can be easily seen that $\mathcal{K}_i^\circ(G) \leq \mathrm{ar}(G, C_{i+1}) + 1$. For complete n-vertex graph, we thus obtain

$$\frac{i-1}{2}n - \frac{i^2 - 9}{8} \leq \mathcal{K}_i^\circ(K_n) \leq \left(\frac{i-1}{2} + \frac{1}{i}\right) n + O(1),$$

where the lower bound is derived from Theorem 1 and the upper bound is from the result of [16].

Let us note here that this anti-Ramsey based upper bound may be distant from the exact value of $\mathcal{K}_i^\circ(G)$. As an example, consider the (classical) Petersen graph $P(5,2)$: by [15], $\mathrm{ar}(P(5,2), C_5) = 10$. On the other hand, it is possible to show that any edge 6-coloring of $P(5,2)$ induces a 5-colored cycle, which yields that $\mathcal{K}_4^\circ(P(5,2)) = 5$.

The exact value of $\mathcal{K}_i^\circ(G)$ can be determined (for small i) for graphs with sufficiently high vertex connectivity. We provide several results for such graphs; following results on cycles of k-connected graphs are used in their respective proofs.

Theorem 2 (Häggkvist, Thomassen [13]). *Let G be a k-connected graph ($k \geq 2$).*

1. *For any set S of independent edges of size $k - 1$, there is a cycle in G containing all edges of S;*

2. *For any set S of independent paths (except single vertices) of total length $k - 1$ there is a cycle in G containing all elements of S.*

Theorem 3 (Denley, Wu [9]). *Let G be a k-connected graph where $k \geq 2$. Let S be a set of independent paths with a total of s edges and T be a set of t vertices, where $s + t = k$ and $t \geq 1$. Then there is a cycle of G containing each path of S and each vertex of T.*

The following theorem shows that the bound in Theorem 1 is sharp in the case of $i = 3$ and G being a 5-connected graph.

Theorem 4. *If G is a 5-connected graph then $\mathcal{K}_3^\circ(G) = \Delta(G) + 1$.*

It can be shown, using Theorem 1, that $\mathcal{K}_3^\circ(G) \geq \Delta(G)+1$ for every connected graph G of order at least 2. To prove Theorem 4, it therefore suffice to show that in each $(\Delta(G)+2)$-coloring of a 5-connected graph G there is a subgraph consisting of four edges of mutually distinct colors whose maximum degree is at most 2. Then Theorem 2 yields an existence of a four-colored cycle.

In the case of colorings with only monochromatic and bichromatic cycles allowed, we provide following two results. The first one deals with the case when G is 3-connected and the second one deals with the case when the vertex connectivity of G is 2. Since $\mathcal{K}_i^\circ(G) = \sum_{B \in B(G)} \mathcal{K}_i^\circ(B)$, where $B(G)$ is the set of blocks of a graph G, Problem 1 is fully solved in the case when $i=2$.

Theorem 5. *If G is a 3-connected graph then $\mathcal{K}_2^\circ(G) = 2$.*

Theorem 6. *Let G be a graph with $\kappa(G)=2$. Then $\mathcal{K}_2^\circ(G) = \max_X (c(G,X)+\varepsilon(G,X))$, where the maximum is taken over all 2-cuts of G, $c(G,X)$ denotes the number of components of $G-X$, and $\varepsilon(G,X)=1$ if the vertices of X are connected by an edge and $\varepsilon(G,X)=0$ otherwise.*

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