Fractional forcing number of graphs

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Abstract

In this work, we introduce the notion of the forcing function of fractional perfect matchings, which is continuous analogous to forcing sets defined over the perfect matching polytope of graphs. We study analytic properties of this function. Finally, we deduce bounds and results for the integral forcing number from these analytic properties.

1 Introduction

The notion of defining set is an important concept in studying combinatorial structures. Roughly speaking, when we talk about a defining set for a particular object, we mean a part of it which uniquely extends to the entire object. As an example, a defining set for a perfect matching M of a graph (also known as a *forcing set*) is a subset of M such that M is the unique perfect matching containing it. The size of the smallest forcing sets of a perfect matching is called the *forcing number* of it. The smallest and the largest forcing numbers over all possible perfect matchings of a graph are, respectively, called the *forcing number* and the *maximum forcing number* of the graph and have been studied intensively for various families of graphs. This parameter is particularly important in the theory of resonance in Chemistry. (See [2] for more details and historical notes).

In this work, we first introduce the concept of *fractional forcing function* for g-factors. When $g = 1$, g-factors are also called *fractional perfect matchings*. For the special case of $g = 1$, we call *fractional forcing functions*, fractional forcing number of g. We prove that fractional forcing number is indeed a continuous extension of the forcing number of perfect matchings.

Notice that the set of fractional perfect matchings is a polytope. Thus, fractional forcing number is a function defined over this polytope. We study analytic properties of this function and prove that it is continuous and concave. Such analytic properties imply several combinatorial implications which we overview in this paper.

Particularly, since the fractional perfect matching polytope contains all the perfect matchings, the maximum fractional forcing number is an upper bound to the maximum forcing number. When G is edge and vertex transitive, the fractional perfect matching polytope is symmetric and due to the symmetry and concavity of the fractional forcing, the maximum is attained at the center of symmetry. This will provide a systematic way to find an upper bound for the integral maximum forcing number of such graphs which is generally hard problem even for special family of graphs such as Cartesian product of even cycles or K_2's. (See [3]). We also prove that when G is bipartite, the minimum fractional forcing number is equal to the minimum forcing number of G. All the proofs can be found in [2], which is the long version of this paper.

2 Preliminaries

We assume the knowledge bases of graph theory and follow the notation of [1].

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Let $g : V(G) \to \mathbb{R}^{\geq 0}$ be a function. The function $\gamma : E(G) \to \mathbb{R}^{\geq 0}$ is called a *partial g-factor* if for every vertex $v \in V(G)$, $\sum\limits_{e:e\ni v} \gamma(e) \leq g(v)$. γ is called a *g-factor* if for every vertex $v \in V(G)$, $\sum\limits_{e:e\ni v} \gamma(e) = g(v)$. Any convex combination of two g-factors is again a g-factor.

Denote by $\mathbb{1}_G$ the constant function 1 over V. Any partial $\mathbb{1}_G$-factor is called a *fractional matching* and any $\mathbb{1}_G$-factor is called *fractional perfect matching*. Every fractional perfect matching γ, with $\gamma(e) \in \mathbb{Z}$, for all $e \in E(G)$, is called a *perfect matching*.

Let M be a perfect matching of a graph G. A subset $S \subseteq M$ is called a *forcing set* for M if M is the unique perfect matching of G containing S.

Definition 1. *Let G be a graph and M be any perfect matching of G. We define the quantities forcing number of M, $f(G, M)$ the forcing number of G, $f(G)$ the maximum forcing number of G, $F(G)$ and the spectrum of the forcing numbers of G,* $\mathrm{Spec}(G)$ *as follows*

$$\begin{aligned} f(G, M) &:= \min\{|S| : S \textit{ is a forcing set for } M\} \\ f(G) &:= \min\{f(G, M) : M \textit{ is a perfect matching of } G\} \\ F(G) &:= \max\{f(G, M) : M \textit{ is a perfect matching of } G\} \\ \mathrm{Spec}(G) &:= \{f(G, M) : M \textit{ is a perfect matching of } G\} \end{aligned}$$

Observe that $f(G) = \min\limits_{x\in \mathrm{Spec}(G)} x$, and $F(G) = \max\limits_{x\in \mathrm{Spec}(G)} x$.

A subset C of $\mathbb{R}^n$ is said to be *convex* if $\lambda x+(1-\lambda)y$ belongs to C for all $x, y \in C$ and $0 \leq \lambda \leq 1$. The *convex hull* of a set $X \subseteq \mathbb{R}^n$, denoted by $\mathrm{Conv}(X)$, is the smallest convex set containing X . A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many point in $\mathbb{R}^n$.

For every subset $M \subseteq E$, denote by $\mathcal{X}^M$ the characteristic function of M. The *perfect matching polytope* of G is defined as $\mathcal{P}(G) := \mathrm{Conv}\{\mathcal{X}^M : M$ is a perfect matching of $G\}$.

It is straightforward to observe that fractional perfect matchings with integer coordinates are precisely the characteristic vectors of perfect matchings. The set of all fractional perfect matchings forms a polytope called *fractional perfect matching polytope* of G and is denoted by $\mathcal{P}_f(G)$. If G is bipartite then, $\mathcal{P}_f(G) = \mathcal{P}(G)$.

Let $g : V(G) \to \mathbb{R}^{\geq 0}$ be a function. The set of all g-factors is called the *g-polytope*.

3 Main Results

In this section, we outline the results of this paper. To better explain the results and their connections, we partition them into three main categories. The first one is about structures of the fractional forcing function. Next we talk about connection between fractional and integral parameters. Finally, we establish analytic results about fractional forcing function. In order to explain the main results of this paper, we need the following definitions.

Definition 2. Let G be a graph and $\alpha, \alpha' : E(G) \to \mathbb{R}^{\geq 0}$ are two functions. Define the relation "$\preceq$" as follows

$$\alpha \preceq \alpha' \to \forall e \in E(G) : \alpha(e) \leq \alpha'(e)$$

One can easily observe that indeed, $\preceq$ is a partial order on the set $(\mathbb{R}^{\geq 0})^E$.

If $\alpha : E(G) \to \mathbb{R}$, define $|\alpha| := \sum\limits_{e\in E(G)} \alpha(e)$.

Let $g : V(G) \to \mathbb{R}^{\geq 0}$ be a function, α be a partial g-factor and γ be a g-factor in a graph G. We say α is *g-extendable* (or simply extendable if g is clear from the context) to γ if $\alpha \preceq \gamma$. In this case, we say γ is a *g-extension* (or extension, when g is clear from the context) of α.

α is a *forcing function* for γ if α is uniquely g-extendable to γ (i.e. $\alpha \preceq \gamma$ and γ is the unique extention of α). In this situation, we write $\alpha \uparrow \gamma$. α is a *minimal forcing function* for γ if $\alpha \uparrow \gamma$ and, if $\alpha' \preceq \alpha$ and $\alpha' \uparrow \gamma$ then, $\alpha = \alpha'$. In this case, we write $\alpha \Uparrow \gamma$.

Definition 3. Let G be a graph and γ be any fractional perfect matching of G. We define the quantities *fractional forcing number* of γ, $f_f(G, \gamma)$ *fractional forcing number* of G, $f_f(G)$ *maximum fractional forcing number* of G, $F_f(G)$ and the *spectrum* of forcing numbers of G, $\mathrm{Spec}_f(G)$ as follows

$$f_f(G,\gamma) := \min_{\alpha:\alpha\uparrow\gamma} \sum_{e\in E} \alpha(e),\ \ f_f(G) := \min_{\gamma\in\mathcal{P}_f(G)} f_f(G,\gamma)$$
$$F_f(G) := \max_{\gamma\in\mathcal{P}_f(G)} f_f(G,\gamma),\ \ \mathrm{Spec}_f(G) := \{f_f(G,\gamma) : \gamma \in \mathcal{P}_f(G)\}$$

We first explain results concerning the structure of fractional forcing functions. It turns out that a useful tool is the notion of saturation of an edge.

Definition 4. In an extention γ of the partial g-factor α, an edge e is called *saturated* if $\alpha(e) = \gamma(e)$.

The next lemma is the key result in this part.

Lemma 1 (**Saturated Edges**). *Let G be a graph, γ be a g-factor and $\alpha \Uparrow \gamma$. Then, for every edge $e \in E(G)$, $\alpha(e) \in \{0, \gamma(e)\}$.*

The above lemma implies that the only way one can extend a minimal uniquely extendable partial g-factor α to a g-factor γ is by increasing the value of α on the edges with $\alpha(e) = 0$.

Theorem 5. *Suppose that $\gamma, \gamma' \in g$-polytope,* $\mathrm{Supp}(\gamma) = \mathrm{Supp}(\gamma')$ *and $\alpha \Uparrow \gamma$. Then, $\alpha' \Uparrow \gamma'$, where*

$$\alpha'(e) = \begin{cases} \gamma'(e) & \alpha(e) = \gamma(e) \\ 0 & \textit{otherwise} \end{cases}$$

In words, Theorem 5 says that if α is a minimal forcing set for some g-factor γ then, if we alter γ on some of the edges to get a new g-factor γ', while preserving the support, then the same alteration will turn α to a minimal forcing function for the resulting g-factor γ'.

Theorem 5 combined with Lemma 1 imply that the minimal forcing functions for every $\gamma \in \mathcal{P}_f(G)$ can be obtained in a two-stage process. In the first stage, we only need to know the support of G. Having access only to $\mathrm{Supp}(G)$, the support of every minimal forcing function for G is specified in the sence that the set $\mathcal{D}_\gamma := \mathrm{Supp}(\alpha) : \alpha \Uparrow \gamma\}$, only depends on $\mathrm{Supp}(\gamma)$ i.e. if $\mathrm{Supp}(\gamma) = \mathrm{Supp}(\gamma')$ then $\mathcal{D}_\gamma = \mathcal{D}'_\gamma$. In the second stage, once we have access to $\mathrm{Supp}(\alpha)$ and the values of $\gamma(e)$, we know by Lemma 1, that α is uniquely identified.

This observation raises the following question. If G is a graph, $\gamma \in \mathcal{P}_f(G)$ and $S \subseteq E(G)$, is there a fractional matching α such that $\alpha \uparrow \gamma$ and $\mathrm{Supp}(\alpha) = S$? The next theorem answers this question for bipartite graphs.

Theorem 6. *Let G be a bipartite garph, $\gamma \in \mathcal{P}_f(G)$, $S \subseteq E(G)$ and $T = E(G) \setminus \operatorname{Supp}(\gamma)$. There exsist a fractional matching α such that $\alpha \uparrow \gamma$ and $\operatorname{Supp}(\alpha) = S$ if and only if the following conditions are satisfied. 1)$S \subseteq \operatorname{Supp}(\gamma)$ 2) For every cycle C of G with a proper 2-coloring of the edges of C, each color class intersects $T \cup S$.*

The next theorem relates the fractional parameters to the integral ones.

Theorem 7. *For every graph G we have $F_f(G) \geq F(G)$. If, in addition, G is bipartite, then, $f_f(G) = f(G)$.*

A proof of this theorem utilises the following lemmas.

Lemma 2. *If $M \in \mathcal{P}(G)$ and $\alpha \uparrow M$ then $\operatorname{Supp}(\alpha)$ is a forcing set for M.*

Lemma 3. *For every perfect matching M of G, $f_f(G, M) \geq f(G, M)$. Furthermore, if G is bipartite then $f_f(G, M) = f(G, M)$.*

Finally, we state the main result of the paper in the following theorem.

Theorem 8. *The function $f_f : \mathcal{P}_f(G) \to \mathbb{R}^{\geq 0}$ is continuous and concave with respect to the Euclidian metric.*

A quick corollary of the continuity of f_f is that the range of f_f is an interval on the real line. Such property for the case of integral forcing number of perfect matching is not always true. For some families of graphs, however, it has been shown or conjectured that the forcing numbers of perfect matchings are integers within certain real interval.

4 Application

In this section we show that the fractional forcing function can be used to obtain results about the integral case. The next theorem in particular provides a systematic way to obtain upper bound on the maximum fractional forcing number for symmetric graphs.

Theorem 9. *Let G be a vertex and edge-transitive graph, and $v \in V(G)$. Then, the fractional perfect matching that assign the value $\frac{1}{\deg(v)}$ to all edges, has the maximum fractional forcing number.*

As a corollary of this theorem, we are able to prove the first non-trivial upper bound on the maximum forcing number of hypercube graph as follows.

Theorem 10. *For any integer n, $F(Q_n) \leq \lfloor \frac{n2^{n-1} - 5\times 2^{n-3}}{n} \rfloor$ in which Q_n is the n-dimensional hypercube.*

For further application of this theory, as well as the details of the proofs, see [2].

References

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