Certain extremal uniformly connected graphs

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Abstract

A graph is uniformly k-connected if each pair of its vertices is connected by exactly k independent paths. Building on a recent constructive characterization of uniformly 3-connected graphs, we reinvestigate a bound on the minimum number of vertices of minimum degree. We study the structure of those graphs that attain this bound and provide results about their crossing and chromatic numbers.

Among the many connectivity concepts in graph theory, requiring the same connectivity between each pair of a graph's vertices may seem to be quite restrictive. Yet this may be a valuable feature of a network and uniform connectivity nicely complements the notions of ordinary, minimal, or average connectivity. When studying the latter, Beineke, Oellermann, and Pippert [1] introduced uniformly connected graphs as they became interested for which graphs the connectivity equals the average connectivity. To proceed, let us recall the following definition, whereas we refer to the monograph of Diestel [2] for basic graph theoretical terminology.

Definition. *A Graph is called uniformly k-connected if $k \in \mathbb{N}$ is the maximum number of independent paths between any two vertices.*

It is not hard to see that uniformly 1-connected graphs are exactly all trees and uniformly 2-connected graphs are exactly all cycles. Further examples are wheel graphs or k-regular, k-connected graphs. Such relations are discussed in more detail by Göring, Hofmann, and Streicher [4]. In this article is proven too that uniformly 3-connected graphs are exactly the graphs from the following recursively defined class $\mathcal{C}$.

(i) If a graph G is 3-regular and 3-connected, then G shall be contained in $\mathcal{C}$.

(ii) For graphs $G_1, G_2 \in \mathcal{C}$ with vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ whose neighborhoods are $N(v_1) = \{x_1, x_2, x_3\}$ and $N(v_2) = \{y_1, y_2, y_3\}$, we include in $\mathcal{C}$ the graph

$$(G_1 - v_1) \cup (G_2 - v_2) + x_1y_1 + x_2y_2 + x_3y_3.$$

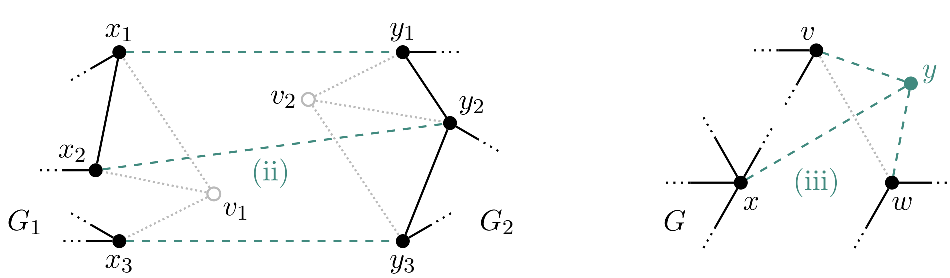

Figure 1: Constructing uniformly 3-connected graphs

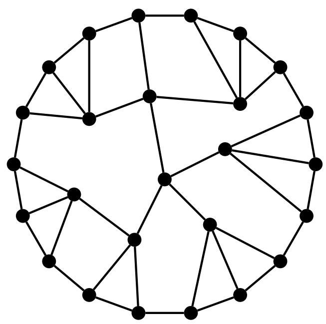

Figure 2: An extremal uniformly 3-connected graphs

(iii) For a graph $G \in \mathcal{C}$ with distinct vertices $v, w, x \in V(G)$, containing $vw \in E(G)$, and satisfying $\deg(y) = 3$ for all $y \in V(G) \setminus \{x\}$, we include in $\mathcal{C}$ the graph

$$G + u - vw + uw + uv + ux$$

where $u \notin V(G)$ is a new vertex to be added to G.

The operations (ii) and (iii) are illustrated in Figure 1. They complement the classical constructions by Tutte [6, 7] for 3-connected, 3-regular, and 3-connected graphs. A question from extremal graph theory that aroused a lot of interest is what graphs from a certain class have minimum number of vertices of minimum degree. So for a graph G one asks for the parameter

$$\nu(G) := \left|\left\{v \in V(G) : \deg(v) = \min_{v \in V(G)} \deg(v)\right\}\right|.$$

A corner stone on which many related investigations build on is the result by Halin [3], who proved that a minimally k-connected graph contains a vertex of degree k. A series of results on that topic is concluded by Mader [5], who gave the tight bound $\nu(G) \geq ((k-1)n + 2k)/(2k-1)$ for minimally k-connected graphs G on n vertices. This result does also hold for uniformly 3-connected graphs, since those can be seen to be minimally k-connected. But for them, we find an even stronger bound.

Theorem. *A uniformly* 3*-connected graph* G *on* n *vertices satisfies*

$$\nu(G) \geq \frac{2n+2}{3}.$$

The purpose of our talk is to reinvestigate this theorem from [4]. We provide a more detailed result that relates $\nu(G)$ to the operations that can be used to construct a uniformly 3-connected graph G. Building on this, we study the structure of graphs that attain this bound. One example is illustrated in Figure 2, which is obtained by embedding a tree in the plane whose inner vertices are of degree four by joining its leafs to a cycle. Besides giving constructive insights, we shall focus on the crossing and chromatic numbers of those extremal graphs.

References

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