Rainbow spanning trees for small color classes

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Abstract

Let G be a graph and $\phi : E(G) \to S$ an edge-coloring of G for some set S. A spanning tree T of G is called rainbow if $|\phi^{-1}(s) \cap E(T)| \leq 1$ for every $s \in S$. We consider several packing and covering problems on rainbow spanning trees and similar objects for colorings in which each color class has a small constant size.

1 Introduction

We present some results for packing and covering problems on rainbow spanning trees with respect to edge colorings with small color classes.

Given a graph G and a coloring $\phi : E(G) \to S$ for some set S, we say that a subgraph H of G is rainbow with respect to ϕ if $|\phi^{-1}(s) \cap E(H)| \leq 1$ for every $s \in S$. We call the pair (G, ϕ) a *colored graph* and say that H is a *rainbow* subgraph of (G, ϕ). In this article, we are interested in finding rainbow spanning trees for colored graphs. A characterization of colored graphs containing a rainbow spanning tree has been proven by Broersma and Li [2] using matroid intersection while an elementary proof has been given by Suzuki in [9].

The most famous open problem on rainbow spanning trees is the following conjecture due to Brualdi and Hollingsworth.

Conjecture 1. *Let n be an even positive integer and $\phi : E(K_n) \to \{1, \dots, n-1\}$ a coloring of $E(K_n)$ such that $\phi^{-1}(i)$ is a perfect matching of K_n for all $i = 1, \dots, n-1$. Then (K_n, ϕ) contains a set of $\frac{n}{2}$ edge-disjoint rainbow spanning trees.*

A significant amount of work has been put into this conjecture. Among other works, some progress has been made by Horn [6] and Pokrovskiy and Sudakov [8]. Finally, Conjecture 1 has been proven for sufficiently large n by Glock, Kühn, Montgomery and Osthus [5].

Despite the attention received by Conjecture 1, rather little effort has been made to study rainbow spanning trees in the general setting of arbitrary graphs rather than in complete graphs. One such packing problem has recently been dealt with by Horn and Nelsen [7]. The objective of our work is to consider a wider range of problems in this field. To start with, observe that Conjecture 1 can be viewed both as a packing and a covering problem. It can easily be seen that every set of $\frac{n}{2}$ edge-disjoint spanning trees of K_n is also a set of $\frac{n}{2}$ spanning trees covering the edge set of K_n and vice-versa. Graphs with this property play a significant role for the problems we consider here. Formally, we say that a graph G is a k-multiple tree for some positive integer k if there is a set $\{T_1, \dots, T_k\}$ of edge-disjoint spanning trees of G such that $\bigcup_{i=1}^{k} E(T_i) = E(G)$. We call such a set of spanning trees a *spanning tree factorization*. Whenever we speak of packing and covering in this work, we refer to the edge sets of the corresponding graphs.

Clearly, packing and covering problems related to rainbow spanning trees are only interesting if we impose certain restrictions on the edge colorings. The restriction we consider in this work is the total size of the color classes. Formally, given a set E, we say that a coloring $\phi : E \to S$ for some set S is *ℓ-bounded* for some positive integer ℓ if $|\phi^{-1}(s)| \leq \ell$ for every $s \in S$. All these problems are also interesting in a more general matroid setting. A matroid M is called a *k-base matroid* for some positive integer k if there is a partition $\{B_1, \ldots, B_k\}$ of $E(M)$ such that B_i is a basis of M for $i = 1, \ldots, k$. A *colored matroid* is a matroid M together with a coloring $\phi : E(M) \to S$ for some set S.

In Section 2, we give results concerning the covering of certain colored graphs by rainbow forests. In Section 3, we deal with the more restrictive problem of covering colored graphs by rainbow spanning trees. In Section 4, we give a negative result concerning the algorithmic tractability of packing and covering problems for rainbow spanning trees. Finally, in Section 5, we show how this result can be applied to obtain a negative result for a related problem in directed graphs answering a question of Frank [4].

2 Covering by rainbow forests

In this section, we consider the problem of covering a given colored graph by a set of rainbow forests which are not necessarily spanning trees.We are interested in questions of the following form: Given two positive integers k, ℓ, what is the minimum number $\mu(k, \ell)$ for which every colored graph (G, ϕ) such that G can be covered by k forests and ϕ is ℓ-bounded can be covered by $\mu(k, \ell)$ rainbow forests? It can easily be observed that $\mu(k, \ell) \geq \max\{k, \ell\}$. On the other hand, given a colored graph (G, ϕ) such that G can be covered by k forests $F_1, \ldots, F_k$ and ϕ is ℓ-bounded, we can partition F_i into ℓ rainbow forests for every $i = 1, \ldots, k$. This yields $\mu(k, \ell) \leq k\ell$. We here focus on the case $\ell = 2$ and improve these trivial upper bounds. Observe that, in order to make such improvements, we may restrict to k-multiple trees as additional edges can only make the task more difficult. We prove the following two results yielding $\mu(2, 2) \leq 3$ and $\mu(3, 2) \leq 4$:

Theorem 1. *Let (G, ϕ) be a colored 2-multiple tree such that ϕ is 2-bounded. Then (G, ϕ) can be covered by 3 rainbow forests.*

Theorem 2. *Let (G, ϕ) be a colored 3-multiple tree such that ϕ is 2-bounded. Then (G, ϕ) can be covered by 4 rainbow forests.*

The following is an immediate consequence of Theorems 1 and 2.

Corollary 1. *Let (G, ϕ) be a colored t-multiple tree for some positive integer t such that ϕ is 2-bounded. Further suppose that $t = 2\alpha + 3\beta$ for some positve integers α and β. Then (G, ϕ) can be covered by $3\alpha + 4\beta$ rainbow forests.*

We next wish to mention that Theorem 1 yields $\mu(2, 2) = 3$ because the example of K_4 together with the edge coloring in which every color class is a matching of size two shows that $\mu(2, 2) \geq 3$. It is an interesting open problem to determine further values of $\mu(k, \ell)$. In particular, we are interested in determining $\mu(3, 2)$, a problem which can be reduced to the following question:

Problem 1. *Given a colored 3-multiple tree (G, ϕ) such that ϕ is 2-bounded, is there always a factorization of G into 3 rainbow spanning trees?*

Observe that Problem 1 can also be viewed as a packing problem.

In the matroidal setting, similar questions can be asked. While a natural generalization of Theorem 1 holds for matroids, the proof of Theorem 2 cannot easily be adapated to work for matroids. The following natural matroidal analogue of Problem 1 is also open.

Problem 2. *Given a colored 3-base matroid (M, ϕ) such that ϕ is 2-bounded, is there always a factorization of M into 3 rainbow bases?*

3 Covering by rainbow spanning trees

In this section, we are interested in the problem if a given colored graph (G, ϕ) can be covered by a set of rainbow spanning trees and if it can, how many rainbow spanning trees we need at least to do so. Again, k-multiple trees turn out to be a particularly interesting class of graphs to consider. As before, we focus on 2-bounded colorings. For $k \geq 4$, the following lemma is very helpful.

Lemma 1. *Let (G, ϕ) be a colored graph such that G contains 4 edge-disjoint spanning trees and ϕ is 2-bounded. Further, let F be a rainbow forest in (G, ϕ). Then there are two rainbow spanning trees U_1, U_2 of (G, ϕ) such that $E(F) \subseteq E(U_1) \cup E(U_2)$.*

The following result can be obtained by combining Corollary 3 and Lemma 1.

Corollary 2. *Let (G, ϕ) be a colored t-multiple tree for some positive integer $t \geq 4$ such that ϕ is 2-bounded. Further suppose that $t = 2\alpha + 3\beta$ for some positve integers α and β. Then (G, ϕ) can be covered by $6\alpha + 8\beta$ rainbow spanning trees.*

Using an adapted version of Lemma 1, we can also prove the following statement on covering colored 3-multiple trees by rainbow spanning trees.

Corollary 3. *Let (G, ϕ) be a colored 3-multiple tree such that ϕ is 2-bounded. Then (G, ϕ) can be covered by 12 rainbow spanning trees.*

None of these methods work for $k = 2$. We hence leave the following open question:

Problem 3. *Is there a fixed integer γ such that every colored 2-multiple tree (G, ϕ) such that ϕ is 2-bounded can be covered by γ rainbow spanning trees?*

Using some more involved techniques, we manage to make some progress toward Problem 3. More concretely, we prove the following result.

Theorem 3. *Let (G, ϕ) be a colored 2-multiple tree such that ϕ is 2-bounded. Then (G, ϕ) can be covered by $O(\log |V(G)|)$ rainbow spanning trees.*

All the methods mentioned in this section also work for the matroidal version. The matroidal version of Problem 3 is also open.

4 Complexity considerations

In this section, we deal with the algorithmic tractability of packing and covering problems on rainbow spanning trees. Given a fixed intger k, we consider the following decision problem which can be viewed as the intersection of several covering and packing problems:

k-Rainbow spanning tree factorization (kRSTF):

Input: A colored k-multiple tree (G, ϕ)

Question: Can (G, ϕ) be factorized in k rainbow spanning trees?

We prove the following result using a reduction from Not All Equal 3SAT for $k = 2$ and a reduction from 3-Colorability for $k \geq 3$. It generalizes a matroidal result of Bérczi and Schwarz [3] and answers a question raised in the same article.

Theorem 4. *kRSTF is NP-hard for any $k \geq 2$.*

As kRSTF is a special case of several packing and covering problems, we have the following corollary:

Corollary 4. *All of the following problems are NP-hard:*

- *Deciding if a colored graph (G, ϕ) has a packing of k edge-disjoint rainbow spanning trees,*
- *deciding if the edge set of a colored graph (G, ϕ) can be covered by k rainbow forests,*
- *deciding if the edge set of a colored graph (G, ϕ) can be covered by k rainbow spanning trees.*

5 Application to g-bounded spanning trees

We here deal with the following algorithmic problem in digraphs where for a given digraph D, we let $UG(D)$ denote the underlying graph of D.

k g-bounded spanning trees (kgBST):

Input: A digraph D, a function $g : V(D) \to \mathbb{Z}_{\geq 0}$.

Question: Is there a set of arc-disjoint subdigraphs $\{T_1, \dots, T_k\}$ of D such that for all $i \in \{1, \dots, k\}$, $UG(T_i)$ is a spanning tree of $UG(D)$ and $d^-_{A(T_i)}(v) \leq g(v)$ for all $v \in V(G)$?

The question about the computational complexity of kgBST was raised by Frank [4]. Using Theorem 4, we are able to give the following negative answer to it:

Theorem 5. *kgBST is NP-hard for every positive integer $k \geq 2$.*

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