Transversals of Maximum Independent Sets

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Abstract

For a given family of vertex sets $\mathcal{C}$, a d-transversal of $\mathcal{C}$ is a set of vertices intersecting every set in $\mathcal{C}$ in at least d elements. Well known transversal problems are for example Vertex Cover, Feedback Vertex Set and Odd Cycle Transversal. We consider d-transversals of maximum independent sets, where we want the set S to be of minimum size and to intersect every maximum independent set in d elements. We show how to represent independent sets in interval graphs as paths in a directed acyclic graph. By joining multiple copies of this directed acyclic graph we construct a graph on which a minimum vertex cut corresponds to a d-transversal of the original graph. Further, we give some complexity results for other subclasses of perfect graphs.

1 Introduction

Vertex Cover, Feedback Vertex Set and Odd Cycle Transversal are well investigated graph problems. In all three problems, we are looking for a vertex set S of minimum size such that S contains at least one vertex of each edge, cycle or odd cycle. Instead of edges, cycles and odd cyles one may consider maximal cliques, maximum matchings or maximum independent sets: we want to find a set S containing at least one vertex of every maximal clique, every maximum matching or every maximum independent set. One may extend the problem such that we want the set S to contain at least d vertices of every maximal clique, maximum matching or maximum independent set.

> d-Transversal(α)
>
> **Instance:** A graph $G = (V, E)$ and an integer $k \geq 0$.
>
> **Question:** Is there a set $S \subseteq V$ of cardinality $|S| \leq k$ such that for every maximum independent set I we have $|I \cap S| \geq d$?

While transversals of maximal cliques and maximum matchings are well investigated, not much research has been done on transversals of maximum independent sets. In [1] it was shown that d-Transversal(α) is polynomial time solvable in trees and later in [2] for bipartite graphs. We extend the investigation to different well known subclasses of perfect graphs. For interval graphs we show how to transform d-Transversal(α) to Vertex Cut. We then present results on other subclasses of perfect graphs. Those results are either easy to see or follow from results on related problems.

2 Transversals of Independent Sets in Interval Graphs

Let $G = (V, E)$ be an interval graph. We will construct a directed graph $G' = (V', E')$ such that independent sets in G correspond to paths in G' and an instance (G, k) is a Yes-instance of d-Transversal(α) if and only if (G', k) is a Yes-instance of Vertex Cut. To get this result, we first need to examine the structure of maximum independent sets in interval graphs.

Lemma 1. *Let I_1, I_2 be two maximum independent sets in an interval graph G. Let $\alpha = \alpha(G)$ be the size of a maximum independent set of G and let $I_1 = \{u_1, \dots, u_\alpha\}$ and $I_2 = \{v_1, \dots, v_\alpha\}$, where the vertices are sorted by increasing right endpoints of their corresponding intervals. Assume $\exists\ i, j \in \{1, \dots, \alpha\}$, such that $u_i = v_j$. Then $i = j$.*

Proof. Assume that there are $i, j \in \{1, \dots, \alpha\}$, $i < j$, such that $u_i = v_j$. It follows that $\{v_1, \dots, v_{j-1}, u_i, \dots, u_\alpha\}$ is an independent set of size at least $\alpha + 1$, a contradiction. $\square$

Lemma 1 allows us to define the position of a vertex, which does not depend on the chosen maximum independent set.

Definition 1. *Let $G = (V, E)$ be an interval graph and $v \in V$ be a vertex in a maximum independent set I of G. The vertex v is said to be at position i, if there are $i-1$ vertices in I whose corresponding intervals are to the left of the interval representing v. We denote the position of v by* $\text{pos}(v)$.

Furthermore, this gives us the possibility to partition the vertices that are part of a maximum independent set into sets $L_1, \dots, L_\alpha \subseteq V$, where $\alpha = \alpha(G)$, such that for $v \in V$ we have $v \in L_i$ if and only if there exists an independent set $\{u_1, \dots, u_{i-1}, v, u_{i+1}, \dots, u_\alpha\}$, where the vertices are sorted by increasing right endpoints of their corresponding intervals. In other words, $L_i = \{v \in V(G) | \text{pos}(v) = i\}$, for $i \in \{1, \dots, \alpha\}$. Since two non-adjacent vertices in the same set L_i would result in an independent set of size strictly larger than α, all L_i's are clearly cliques. Figure 1 gives an example of an interval graph together with its interval representation and the partition of the vertices into the sets $L_1, \dots, L_\alpha$.

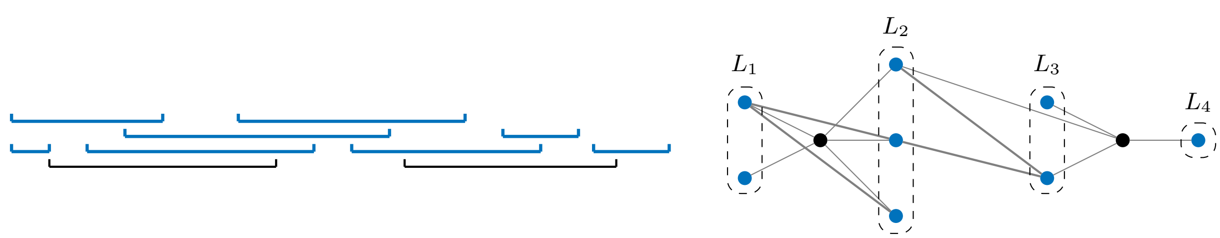

Figure 1: The figure shows an interval graph and its interval representation. The blue vertices and intervals are those which belong to some maximum independent set. Each set L_i is a complete graph. The edges inside the L_i are not represented.

Using this partition we can easily construct a directed graph G^* with $V(G) = V(G^*) \cup \{s, t\}$ where every maximum independent set of G corresponds to an s-t-path in G^*. This can be done by introducing an arc (u, v) if there is $i \in \{1, \dots, \alpha - 1\}$ such that $u \in L_i$ and $v \in L_{i+1}$ and u and v are non-adjacent. Further, we add arcs (s, u) for $u \in L_1$ and arcs (u, t) for $u \in L_\alpha$. Then, every s-t-path clearly corresponds to an independent set, which is maximum since it has to pass every set L_i for $i \in \{1, \dots, \alpha\}$. Thus, a 1-transversal of G corresponds to a vertex set C in G^* such that for every s-t-path P in G^* we have $C \cup P \neq \varnothing$. Hence, a minimum 1-transversal of G can be obtained by computing a minimum vertex cut in G^*.

In Figure 2, we can see the graph G^* corresponding to the graph from Figure 1. Definition 2 and Observation 1 will help us to extend this idea to d-transversals.

Definition 2. *Let G be a graph. We call an independent set I an extendable independent set if there is a maximum independent set I^e of G with $I \subseteq I^e$.*

Observation 1. *A vertex set S is a d-transversal of G if and only if it intersects every extendable independent set of size $\alpha - d + 1$ in at least one vertex.*

Using Observation 1, we can see that instead of considering d-transversals of maximum independent sets in G we can consider 1-transversals of extendable independent sets of size $\alpha - d + 1$ in G. To be able to transform d-Transversal(α) to Vertex Cut again, we want to construct a directed graph G' with a source s and a sink t, such that every s-t-path in G' corresponds to an extendable independent set in G of size $\alpha - d + 1$ and vice versa. For the construction, we use the graph G^*, which we defined above, and we denote with G_α the graph $G^* - \{s, t\}$.

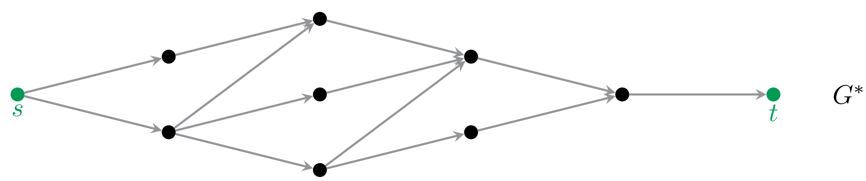

Figure 2: The graph G^* corresponding to the graph G from Figure 1.

We take d copies $G_\alpha^1, \dots, G_\alpha^d$ of G_α and we define the graph G'_α as the disjoint union $G'_\alpha = \bigoplus_{1 \le h \le d} G_\alpha^h$. We denote by $L_{i,h}$ the set L_i in G_α^h, for $i \in \{1, \dots, \alpha\}$ and $h \in \{1, \dots, d\}$, and we say that h is the *level* of the vertices in $L_{i,h}$. We want to connect the vertices such that every s-t-path corresponds to an extendable independent set of size $\alpha - d + 1$. To do this, we add an arc (u, v) if $\text{level}(v) - \text{level}(u) = \text{pos}(v) - \text{pos}(u)$ and $\{u, v\}$ is an extendable independent set. This means that in an extendable independent set I for any two vertices u and v with $\text{pos}(u) < \text{pos}(v)$ the number of positions between u and v that do not contain a vertex in I equals the increase of levels from u to v on the path corresponding to I. Since the maximum level is d we omit exactly $d - 1$ positions. More formally, to connect the vertices of the graphs $G_\alpha^1, \dots, G_\alpha^d$, we add an arc from $u \in L_{i,h}$ to $v \in L_{i',h'}$ if $i' = i + g + 1, h' = h + g$, with $g > 0$, and we can extend $\{u, v\}$ to a maximum independent set of G. Figure 3 shows the graph G'_α for the example from Figure 1 with $d = 2$, which consists of all vertices in $G_\alpha^1 \cup G_\alpha^2$ and contains all arcs shown in grey and green.

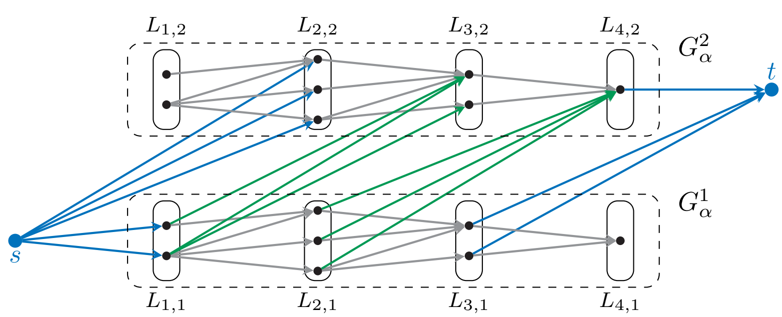

Figure 3: Structure of the graph G' for $d = 2, \alpha = 4$. All arcs starting in s or ending in t are shown in blue. The arcs in G_α^1 and G_α^2 are grey and the arcs between G_α^1 and G_α^2 are green.

For the vertex set of the graph G', we introduce the source s and the sink t and we define $V(G') = V(G'_\alpha) \cup \{s, t\}$.

Finally, to connect s and t to G'_α, we add arcs (s,v) for every $v \in L_{i,i}$ and arcs (v,t) for every $v \in L_{\alpha-d+i,i}$ for $i \in \{1,\ldots,d\}$. In the example in Figure 3, these arcs starting in s or ending in t are shown in blue. We say that s is on level 1 and t on level d. Notice that a vertex $u \in L_{i,h}$, $i \in \{1,\ldots,\alpha\}, h \in \{1,\ldots,d\}$ corresponds to a unique vertex v in L_i in G.

Lemma 2. *Let $G = (V,E)$ be an interval graph, $\alpha = \alpha(G)$, $d > 0$ an integer and G' constructed as above. Every extendable independent set of size $\alpha - d + 1$ in G corresponds to an s-t-path in G' and vice versa.*

Lemma 3. *Let G be an interval graph, G' as constructed above, $d,k > 0$ integers. (G,k) is a* Yes*-instance of d-*Transversal*(α) if and only if (G',k) is a* Yes*-instance of* Vertex Cut*.*

It is well known that a minimum vertex cut can be found in polynomial time. The size of a minimum vertex cut in G' then clearly tells us if (G',k) is a Yes-instance of Vertex Cut. Hence, together with Lemma 3, we get that d-Transversal(α) is solvable in polynomial time on interval graphs.

Theorem 1. *d-*Transversal*(α) is polynomial-time solvable for interval graphs.*

3 Transversals of Independent Sets in Other Subclasses of Perfect Graphs

Given a graph $G = (V,E)$, the independence blocker problem asks for a set $S \subseteq V$ of size $\leq k$ such that $\alpha(G-S) \leq \alpha(G) - d$. In [4] it was shown that independence blocker is NP-complete for chordal and hence for perfect graphs. It is easy to see that independence blocker with $d = 1$ is the same problem as 1-Transversal(α) and hence we get that 1-Transversal(α) is NP-hard for chordal and perfect graphs. It is also easy to see that d-Transversal(α) on cobipartite graphs is the same problem as vertex cover on bipartite graphs, for which polynomial time algorithms are known. Hence, d-Transversal(α) is polynomial time solvable on cobipartite graphs.

Let $G = (V,E)$ be split graph with $V = I \cup C$, where I is a maximum independent set and C a clique. For split graphs we get that any $d+1$ vertices of I are a d-transversal of G. If there are d vertices in I which are part of every maximum independent set of G, those vertices clearly are a d-transversal. Since a d-transversal consists of at least d vertices, this solution would be optimal. Checking if a vertex is in every maximum independent set can be done in polynomial time in split graphs and hence d-Transversal(α) is solvable in polynomial time in split graphs.

References

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