Burling Graphs

Pegah Pournajafi — **École Normale Supérieure de Lyon, France**
Nicolas Trotignon — **École Normale Supérieure de Lyon, France**

Abstract

The class of Burling graphs is a triangle-free class of graphs with arbitrarily large chromatic number. Since 1965, when Burling graphs were first introduced by Burling [1], they have been useful in solving many problems in graph theory, specifically in the two fields of χ-boundedness and geometric graph theory. Meanwhile, they have been studied in many different ways: as intersection graphs of several different geometric objects and as subgraphs of an inductively defined sequence of graphs.

In this talk, after reviewing some applications of Burling graphs and recalling their classical definition, we see how to unify the previously existing views of Burling graphs by providing some new equivalent definitions. Moreover, we discuss how each of these definitions can be useful to obtain new results in graph theory, based on their respective features.

This talk is based on [6, 7, 8].

Prerequisites

A *clique* in a graph is a set of pairwise adjacent vertices. The *size* of a clique is the number of its vertices. The *clique number* of a graph G, denoted by $\omega(G)$, is the size of the biggest clique in G. A *triangle* in a graph is a clique of size 3, and a graph is *triangle-free* if it contains no triangle as an induced subgraph. A class of graphs is *triangle-free* if every graph in the class is triangle-free.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k such that we can partition the vertex set of G into k stable sets. Notice that for every graph G, we have $\chi(G) \geq \omega(G)$, but the converse is not true in general. In fact, there exist many examples of sequences of graphs which are triangle-free but have increasing chromatic number.

A class $\mathcal{C}$ of graphs is said to be *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$, we have $\chi(G) \leq f(\omega(G))$. Notice that if $\mathcal{C}$ is triangle-free, then $\mathcal{C}$ being χ-bounded is equivalent with the existence of a number c such that every $G \in \mathcal{C}$ has chromatic number at most c.

A *subdivision* of a graph H is a graph obtained from H by replacing some of its edges by paths of length at least 1. For a graph H, we denote by $Forb^*(H)$ the class of graphs which do not contain any subdivision of H as an induced subgraph. The graph H is said to be *weakly pervasive* if $Forb^*(H)$ is χ-bounded, and it is said to be *non-weakly pervasive* otherwise.

The *intersection graph* G of a set $V = \{X_1, X_2, \dots X_n\}$ of subsets of $\mathbb{R}^d$ is the graph whose vertex set is V and two vertices X_i and X_j are adjacent in it if and only if $i \neq j$ and $X_i \cap X_j \neq \varnothing$.

History of Burling graphs

In his Ph.D. thesis, Burling [1] studied the χ-boundedness of the intersection graphs of some specific polytopes in $\mathbb{R}^n$. In particular, he proved that in many cases the class is not χ-bounded, and in order to do so, he first reduced the problem to the case of intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ (*box graphs* for short) which are triangle free, and second built a sequence $\{G_k\}_{k\geq 1}$ of such graphs with $\chi(G_k) \geq k$ for each $k \geq 1$. This sequence is known as the *Burling sequence.* The graph of Burling sequence and all their induced subgraphs form the *class of Burling graphs.*

In 2012, Pawlik, Kozik, Krawczyk, Lason, Micek, Trotter and Walczak [4] used Burling graphs to answer a question of Erdős. Indeed, they rediscovered Burling sequence as intersection graphs of line segments in $\mathbb{R}^2$ (*line segment graphs* for short), and thus showed that the triangle-free line segment graphs have unbounded chromatic number. Also, in a later paper [5], they showed that the Burling sequence can be regarded as the intersection graph of other geometric object: any arc-connected compact set in $\mathbb{R}^2$ which is not an axis-aligned rectangle.

Following the work of Burling and Pawlik et. al., one can also give a more combinatorial definition of the Burling sequence in an inductive way. See for example [9] for definition. Using this definition, Davies [3] proved that Burling graphs are wheel-free.

Another importance of the work of Pawlik et. al. was that they provided the first counterexamples to Scott's conjecture which was open for about 20 years at the time. Scott's conjecture states that every graph is weakly pervasive. Let H be a subdivision of a non-planar graph in which every edge is subdivided at least once. One can show that no subdivision of H is a line segment graph. So, $Forb^*(H)$ contains all the graphs of the Burling sequence, thus H is non-weakly pervasive, disproving Scott's conjecture. After that, however, the question of deciding which graphs are weakly pervasive and which are not remained of interest, a question which is still wildly open.

What was essential in finding the first examples of non-weakly pervasive graphs was the fact that there exist graphs H such that none of the subdivisions of H are Burling graphs, and mathematicians found this thanks to regarding Burling graphs in an appropriate way, i.e. as line-segment graphs. However, triangle-free line-segment graphs are a proper super-class of Burling graphs. So, it is important to understand Burling graphs better in order to find more non-weakly pervasive graphs.

In 2016, Chalopin, Esperet, Li and Ossona de Mendez [2], studied Burling graphs as triangle-free intersection graphs of *frames* (borders of axis-aligned rectangles) in $\mathbb{R}^2$. Frames are compact and path-connected, so by [5], they contain all Burling graphs, but they contain other graphs too. By setting several constraints on how the frames can intersect, Chalopin et. al. defined the class of *restricted frame graphs*, a strict subclass of triangle-free intersection graphs of frames, which still contains Burling graphs. This new class, being smaller than the line-segment graphs, enabled them to provide much more examples of non-weakly pervasive graphs. However, this class is still a strict super-class of Burling graphs.

Our contribution

So far, we have seen that the class of Burling graphs is defined graph-theoretically in an inductive way, and also is seen as a subclass of intersection graphs of geometric objects, including boxes, line segments, and frames. In [6], we give five new equivalent definitions of Burling graphs: three of them are geometrical, one is of a more graph-theoretical flavour, and one is more axiomatic. See Figure 1.

First of all, setting some restrictions on how the objects can intersect, we restricted the classes of box graphs, line-segment graphs, and frame graphs to *strict box graphs*, *strict line-segment graphs*, and *strict frame graphs* and proved that each class is equal to the class of Burling graphs.

Second, we define the class of *derived graphs*, a class of graphs with a purely combinatorial definition: we *derive* a graph from some tree structure using several simple rules. We prove that the class of derived graphs is equal to the class of Burling graphs.

Finally, we define the *abstract Burling graphs*, a class of graphs defined using sets with two

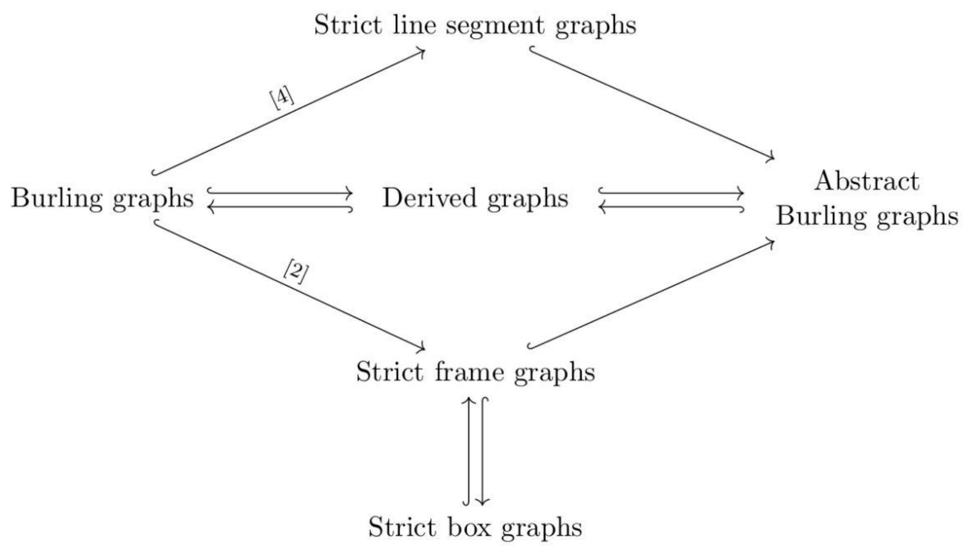

Figure 1: The definitions of Burling graphs

binary relations on them which satisfy a set of four axioms.

Derived graphs seem to be specific and well structured, while abstract Burling graphs seem to be general (though they are equivalent). Being more "general", abstract Burling graphs enable us to easily check that geometrical objects satisfy the axioms in their definition, which is how we prove that strict geometric objects classes are a subset of Burling graphs. On the other hand, derived graphs, being more structured, turn out to be useful to study the structure of Burling graphs. This is mostly done in [7], where we prove several structural theorems about derived graphs. We find several results about the structure of holes in Burling graphs and their interactions, as well as the role of star cutsets in the class. Moreover, we give a short proof of the fact that Burling graphs are wheel-free.

Finally, in [8], we use the results from [7] to find new non-weakly pervasive graphs, among which are K_5, some necklace graphs, and "dumbbell graphs", which contain vertex cuts.

Open problems

There are still many open problems about Burling graphs, which can lead to a better understanding of the class. We conclude this note by one of them:

Question. Does every hereditary proper subclass of Burling graphs have bounded chromatic number? Equivalently, let H be a Burling graph. Does the class of H-free Burling graphs have bounded chromatic number?

References

[1] James Perkins Burling, *On coloring problems of families of prototypes (PhD thesis)*, University of Colorado, Boulder (1965).

[2] Jérémie Chalopin, Louis Esperet, Zhentao Li, and Patrice Ossona de Mendez, *Restricted frame graphs and a conjecture of scott*, Electron. J. Comb. **23** (2016), no. 1, P1.30.

[3] James Davies, *Triangle-free graphs with large chromatic number and no induced wheel*, arXiv:2104.05907, 2021.

[4] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T. Trotter, and Bartosz Walczak, *Triangle-free intersection graphs of line segments with large chromatic number*, J. Comb. Theory, Ser. B **105** (2014), 6–10.

[5] Arkadiusz Pawlik, Jakub Kozik, Tomasz Krawczyk, Michał Lasoń, Piotr Micek, William T. Trotter, and Bartosz Walczak, *Triangle-free geometric intersection graphs with large chromatic number*, J. Comb. Theory, Ser. B **105** (2014).

[6] Pegah Pournajafi and Nicolas Trotignon. *Burling graphs revisited, part I: New characterizations.* arXiv:2104.07001, 2021.

[7] Pegah Pournajafi and Nicolas Trotignon. *Burling graphs revisited, part II: Structure.* arXiv:2106.16089, 2021.

[8] Pegah Pournajafi and Nicolas Trotignon. *Burling graphs revisited, part III: Applications to χ-boundedness.* arXiv:2112.11970, 2021.

[9] Alex Scott and Paul D. Seymour, *A survey of χ-boundedness*, J. Graph Theory **95** (2020), no. 3, 473–504.