Feedback vertex sets in (directed) graphs of bounded degeneracy or treewidth

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Abstract

We study the minimum size f of a feedback vertex set in directed and undirected n-vertex graphs of given degeneracy or treewidth. In the undirected setting the bound $\frac{k-1}{k+1}n$ is known to be tight for graphs with bounded treewidth k or bounded odd degeneracy k. We show that neither of the easy upper and lower bounds $\frac{k-1}{k+1}n$ and $\frac{k}{k+2}n$ can be exact for the case of even degeneracy. More precisely, for even degeneracy k we prove that $\frac{3k-2}{3k+4}n \leq f < \frac{k}{k+2}n$.

For directed graphs of bounded degeneracy k, we prove that $f \leq \frac{k-1}{k+1}n$ and that this inequality is strict when k is odd. For directed graphs of bounded treewidth $k \geq 2$, we show that $\frac{k-2\lfloor \log_2(k) \rfloor}{k+1}n \leq f \leq \frac{k}{k+3}n$. Further, we provide several constructions of low degeneracy or treewidth and large f.

We consider only simple graphs and oriented directed graphs, i.e. our graphs do not have loops or multiple edges or arcs. A set $F \subseteq V(G)$ of vertices of a (directed) graph G, is a *feedback vertex set* if deleting F results in a (directed) graph without (directed) cycles. The complement of a feedback vertex set is called *acyclic set*, and some results in the literature are formulated in terms of acyclic sets. Deciding whether a graph has a feedback vertex set of a given size is among the 21 original NP-complete problems of Karp [1]. Thus finding the minimum size of a feedback vertex set or equivalently, the largest acyclic set, is a challenging algorithmic problem and was extensively studied in the literature.

Because of its hardness, a natural class to study the minimum size of a feedback vertex set are sparse (directed) graphs. A particular example are planar graphs. The size of a minimum feedback vertex set in a planar graph is conjectured to be at most half the vertices by Albertson and Berman [2]. Up to date the best-known upper bound is $\frac{3}{5}n$ achieved through acyclic colorings with Borodin's result [3]. Further, it is known that if true this bound is best-possible [5]. Moreover, it is noteworthy that the best known upper bound coincides with the above mentioned $\frac{3}{5}n$ from the undirected setting [3]. This conjecture remains open even in the directed setting. Note that, in this setting, it is a weakening of the Neumann-Lara conjecture.

Conjecture 1 (Neumann-Lara [4])**.** *Every planar oriented graph can be vertex-partitioned into two acyclic sets.*

Another class that has received attention in the directed setting is the class of tournaments. Already Stearns [6] and Erdős and Moser [7] have shown that any tournament on n vertices admits a feedback vertex set of size $n - \lfloor \log_2(n) \rfloor - 1$, while there are tournaments where no feedback vertex set on less than $n - 2\lfloor \log_2(n) \rfloor - 1$ vertices exists. More precise bounds for small values of n have been obtained by Sanchez-Flores [8, 9] and recently more work has been done into that direction by Neiman, Mackey and Heule [10] and by Lidický and Pfender [11]. Improving the asymptotic upper and lower bounds remains an open problem.

We focus on the class of (directed) graphs of bounded treewidth or degeneracy. Here, the treewidth or degeneracy of a directed graph is simply the treewidth or degeneracy of its underlying undirected graph. Recall that every graph of treewidth k also has degeneracy k. In the undirected setting, the minimum feedback vertex set of graphs of bounded treewidth has been determined by Fertin, Godard and Raspaud [12]: for a graph of order n, treewidth k, the size of a minimum feedback vertex set is at most $\frac{k-1}{k+1}n$ and this bound is best-possible. Moreover, for odd degeneracy k it is easy to achieve the same upper bound. However, for even degeneracy the same argument only yields an upper bound of $\frac{k}{k+2}n$, and a lower bound of $\frac{k-1}{k+1}n$. Indeed, in [13] Borowiecki, Drgas-Burchardt, and Sidorowicz show that the true value for $k = 2$ is $\frac{2}{5}n$ which lies strictly between the above bounds.

Our main contribution here is to construct for any even k a family of graphs of degeneracy k, whose members of large order n have minimum feedback vertex sets whose size comes arbitrarily close to $\frac{3k-2}{3k+4}n$. Let $n(G)$ be the number of vertices of G and $f(G)$ be the size of a minimum feedback vertex set of G.

Theorem 1. *For every even k there exists a family of k-degenerate graphs $(G_i)_{i\in\mathbb{N}}$ such that $n(G_i) = \frac{3k+6}{2} + i\frac{3k+4}{2}$ and $f(G_i) = \frac{3k-2}{2} + i\frac{3k-2}{2}$.*

On the other hand we know that there exists no graph of order n and even degeneracy k whose minimum feedback vertex set is of size $\frac{k}{k+2}n$.

Proposition 1. *For every even $k \geq 2$ there is a graph G with degeneracy k, $n(G) = \frac{(k+2)k}{2} + 1$ and $f(G) = \frac{k^2}{2}$.*

In the directed setting, to our knowledge, apart from the above mentioned results in planar digraphs and tournaments no classes of given degeneracy or treewidth have been studied previously. We give an upper bound for the smallest feedback vertex sets of n-vertex graphs of degeneracy k.

Theorem 2. *If D is a k-degenerate directed graph, then $f(D) \leq \frac{k-1}{k+1}n(D)$.*

For $k = 2$ and $k = 3$, this yields tight bounds $\frac{1}{3}n$ and $\frac{1}{2}n$, respectively. For $k = 2$, the directed triangle is a simple example reaching the upper bound and for $k = 3$, the construction from [5] yields $\frac{1}{2}n$ for degeneracy 3. Unlike the undirected setting, we know that there exists no graph of order n and odd degeneracy k whose minimum feedback vertex set is of size $\frac{k-1}{k+1}n$.

Proposition 2. *If D is a directed graph of odd degeneracy $k \geq 3$, then $f(D) < \frac{k-1}{k+1}n(D)$.*

We also present constructions for digraphs with large minimum feedback vertex set and given small degeneracy or treewidth that improve on the bounds obtained from using just tournaments from [8, 9, 10].

For general treewidth, taking disjoint unions of the tournaments of [7], one can find n-vertex digraphs of treewidth k and $f \geq \frac{k-2\lfloor\log_2(k+1)\rfloor}{k+1}n$. However, we show that on general graphs of treewidth k one can force slightly larger minimum feedback vertex sets.

Theorem 3. *For every k, there exists a family of directed graphs $(D_i)_{i\in\mathbb{N}}$ of treewidth k, such that $n(D_i) = k + 2 + i(k+1)$ and $f(D_i) \geq (i+1)(k - 2\lfloor\log(k)\rfloor)$.*

On the other hand, we show that every n-vertex digraph of treewidth k has a feedback vertex set of size at most $\frac{k}{k+3}n$.

Theorem 4. *If G has treewidth k, then $f(G) \leq \frac{k}{k+3}n(G)$.*

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