# On the minimum number of arcs in 3-dicritical oriented graphs

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**Abstract**

We prove that every 3-dicritical oriented graph on $n$ vertices has at least $\frac{7n+2}{3}$ arcs. We also give a construction of 3-dicritical oriented graphs on $n$ vertices with $\frac{5n}{2}$ arcs for all even $n \geq 12$.

Let $G$ be a graph. We denote by $V(G)$ its vertex set and by $E(G)$ its edge set, and we set $n(G) = |V(G)|$ and $m(G) = |E(G)|$. A **subgraph** of $G$ is a graph $G'$ such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. A **proper subgraph** of $G$ is a subgraph $G'$ of $G$ such that $V(G') \neq V(G)$ or $E(G') \neq E(G)$.

A **proper $k$-colouring** of a graph $G$ is a partition of the vertex set of $G$ into $k$ disjoint **stable sets** (i.e. sets of pairwise non-adjacent vertices). A graph is **$k$-colourable** if it has a $k$-colouring. The **chromatic number** of a graph $G$, denoted by $\chi(G)$, is the least integer $k$ such that $G$ is $k$-colourable. The chromatic number is monotone in the sense that if $G'$ is a subgraph of $G$, then $\chi(G') \leq \chi(G)$. A graph $G$ is said to be **critical** and **$k$-critical** if every proper subgraph $G'$ of $G$ satisfies $\chi(G') < \chi(G) = k$. Clearly, every graph contains a critical subgraph with the same chromatic number. Hence many problems concerning the chromatic number can be reduced to critical graphs. The study of critical graphs was initiated by G. A. Dirac in the 1950s and has attracted a lot of attention since then. Dirac [3, 4, 5] established the basic properties of critical graphs and started to investigate the minimum number of edges possible in a $k$-critical graph of order $n$, denoted by $g_k(n)$. It is easy to show that the minimum degree of a $k$-critical graph is at least $k-1$. Consequently, $g_k(n) \geq \frac{1}{2}(k-1)n$ for all $n \geq k$. In 2014, using the potential method, Kostochka and Yancey [7] proved that $g_4(n) = \lceil \frac{5n-2}{3} \rceil$ for all $n \geq 4$, $n \neq 5$. Furthermore, they [8] determined the best linear approximation for the function $g_k(n)$. In particular, they established a lower bound for $g_k(n)$ that is sharp when $k \geq 4$ and $n \equiv 1 \mod (k-1)$. In particular, it proved that $g_5(n) \geq \frac{9}{4}n - \frac{5}{4}$ with equality when $n \equiv 1 \mod 4$.

The **girth** of a graph $G$ is the minimum length of a cycle in $G$ or $+\infty$ if $G$ is acyclic. Also using the potential method, Liu and Postle [10] showed that the minimum number of edges of a 4-critical graph is larger than $g_4(n)$ if we impose the graph to have girth 5: If $G$ is a 4-critical graph of girth at least 5, then $m(G) \geq \frac{5n(G)+2}{3}$. Likewise, Postle [13] showed that every 5-critical graph with girth at least 4 satisfies $m(G) \geq (\frac{9}{4} + \epsilon)n(G) - \frac{5}{4}$ for $\epsilon = \frac{1}{84}$.

Let $D$ be a digraph. We denote by $V(D)$ its vertex set and by $A(D)$ its arc set, and we set $n(D) = |V(D)|$ and $m(D) = |A(D)|$. A **subdigraph** of $D$ is a digraph $D'$ such that $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. A **proper subdigraph** of $D$ is a subdigraph $D'$ of $D$ such that $V(D') \neq V(D)$ or $A(D') \neq A(D)$.

A **$k$-dicolouring** of a digraph is a partition of its vertex set into $k$ subsets inducing acyclic subdigraphs. A digraph is **$k$-dicolourable** if it has a $k$-dicolouring. The **dichromatic number** of a digraph $D$, denoted by $\vec{\chi}(D)$, is the least integer $k$ such that $D$ is $k$-dicolourable. This notion was introduced and investigated by Neumann-Lara [12]. It can be seen as a generalization of the chromatic number. Indeed, for a graph $G$, the **bidirected graph** $\overleftrightarrow{G}$ is the digraph obtained from $G$ by replacing each edge by a **digon**, that is a pair of oppositely directed arcs between the same end-vertices. Observe that $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$ since any two adjacent vertices in $\overleftrightarrow{G}$ induce a directed cycle of length 2.

Similarly to the chromatic number, the dichromatic number is monotone. A digraph $D$ is said to be **dicritical** and **$k$-dicritical** if every proper subdigraph $D'$ of $D$ satisfies $\vec{\chi}(D') < \vec{\chi}(D) = k$. Clearly, every digraph contains a critical subdigraph with the same dichromatic number. Dicritical digraphs were introduced in Neumann-Lara's seminal paper [12]. Observe that $\vec{\chi}(D) = 1$ if and only if $D$ is acyclic. As a consequence, a digraph $D$ is 2-dicritical if and only if $D$ is a directed cycle. Bokal, Fijavž, Juvan, Kayll and Mohar [1] proved that deciding whether a given digraph is $k$-dicolourable is NP-complete for all $k \geq 2$. Hence a characterization of the class of $k$-dicritical digraphs with fixed $k \geq 3$ is unlikely. However, it might be possible to derive bounds on the number of arcs in a $k$-dicritical digraph. Kostochka and Stiebitz [9] deduced the following from a Brooks-type result for digraphs due to Mohar [11]: if $D$ is a 3-dicritical digraph of order $n \geq 3$, then $m(D) \geq 2n$ and equality holds if and only if n is odd and D is a bidirected odd cycle.

For integers $k$ and $n$, let $d_k(n)$ denote the minimum number of arcs in a $k$-dicritical digraph of order $n$. By the above observations, $d_2(n) = n$ for all $n \geq 2$, and $d_3(n) \geq 2n$ for all possible $n$, and equality holds if and only if $n$ is odd and $n \geq 3$.

If $G$ is a $k$-critical graph, then $\overleftrightarrow{G}$ is $k$-dicritical, so $d_k(n) \leq 2g_k(n)$ provided that there is a $k$-critical graph of order $n$. (It is known that, for $k \geq 4$, there is a $k$-critical graph of order $n$ if and only if $n \geq k$ and $n \neq k+1$).

Kostochka and Stiebitz proved that if $D$ is a 4-dicritical digraph then $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3} = 2g_4(n)$. They also proposed the following conjecture.

**Conjecture 1** (Kostochka and Stiebitz [9])**.** If $D$ is a $k$-dicritical digraph of order $n$ with $k \geq 4$ and $n \geq k$, then $m(D) \geq 2g_k(n)$ and equality implies that $D$ is a bidirected $k$-critical graph. As a consequence, $d_k(n) = 2g_k(n)$ when $n \geq k$ and $n \neq k+1$.

Similarly to the undirected case, it is expected that the minimum number of arcs in a $k$-dicritical digraph of order $n$ is larger than $d_k(n)$ if we impose this digraph to have no short directed cycles, and in particular if the digraph is an **oriented graph**, that is a digraph with no digon. Let $o_k(n)$ denote the minimum number of arcs possible in a $k$-dicritical oriented graph of order $n$. Clearly $o_k(n) \geq d_k(n)$.

**Conjecture 2** (Kostochka and Stiebitz [9])**.** There is a constant $c > 1$ such that $o_k(n) > c \cdot d_k(n)$ for $k \geq 3$ and $n$ sufficiently large.

As observed by Hoshino and Kawarabayashi [6], using iteratively an analogue of Hajós construction for oriented graphs, for each $k \geq 3$, one can construct an infinite family of sparse $k$-dicritical oriented graphs such that $m(D) < \frac{1}{2}(k^2 - k + 1)n(D)$. Consequently, $o_k(n) < \frac{1}{2}(k^2 - k + 1)n$ for infinitely many values of $n$. When $k = 3$, a better result can be obtained using the unique 3-dicritical oriented graph with 20 arcs. It yields 3-dicritical oriented graphs with $n$ vertices and $\frac{19n}{6} + 1$ arcs for all $n \equiv 1 \mod 6$. Consequently, $o_3(n) \leq \frac{19n}{6} + 1$ for all $n \equiv 1 \mod 6$.

In this paper, we give better lower and upper bounds on $o_3(n)$.

In [2], the authors conjecture that there is no 3-dicritical digraph on $n$ vertices with less than $\frac{5}{2}n(G)$ arcs. We give a construction that matches this bound.

The **knob of height 1** is the tournament $\vec{K}_1$ defined by $V(\vec{K}_1) = \{x_1, x_2, y_1, y_2, y_3\}$, and $A(\vec{K}_1) = \{x_1x_2, y_1y_2, y_2y_3, y_3y_1, y_1x_1, y_2x_1, y_3x_1, x_2y_1, x_2y_2, x_2y_3\}$. The **base** of the knob is the arc $x_1x_2$. For all integer $i \geq 2$, The **knob of height** $i$, denoted by $\vec{K}_i$, is the oriented graph obtained from $\vec{K}_{i-1}$ by adding two new vertices $z_1z_2$ and the arcs of the two directed 3-cycle $(z_1, z_2, x, z_1)$ for all end-vertex $x$ of the base of $\vec{K}_{i-1}$. The **base** of $\vec{K}_i$ is the arc $z_1z_2$. A **knob** is a knob of height $i$ for some positive integer $i$.

Let $\mathcal{O}_3$ be the family of the oriented graphs that are obtained from an odd directed cycle by adding a copy of a knob with base $a$ for every arc $a$ of this cycle. See Figure 1. Since there are knobs of any odd order at least 5, there are elements of $\mathcal{O}_3$ of every even order at least 12.

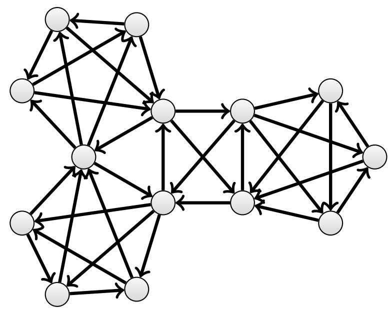

Figure 1: A digraph of $\mathcal{O}_3$ of order 14.

**Proposition 3.** *If $D \in \mathcal{O}_3$, then $m(D) = \frac{5}{2}n(D)$ and $D$ is 3-dicritical.*

We then prove the main result of the paper stating $o_3(n) \geq \frac{7n+2}{3}$.

**Theorem 4.** *If $D$ is a 3-dicritical oriented graph, then $m(D) \geq \frac{7n(D)+2}{3}$.*

To prove this theorem, we use the so-called potential method introduced by Kostochka and Yancey [7, 9], as well as some ideas introduced by Liu and Postle [13, 10]. The **digon graph** of $D$, denoted by $B(D)$, is the graph with vertex set $V(D)$ in which $uv$ is an edge if and only if there is a digon between $u$ and $v$ in $D$. We denote by $\pi(D)$ the maximum size of a matching in $B(D)$. The **potential** of a set $R$ of vertices in a digraph $D$ is $\rho_D(R) = 7|R| - 3m(D[R]) - 2\pi(D[R])$ and we write $\rho(D) = \rho_D(V(D))$. We actually prove the following stronger result than Theorem 4, where an odd 3-wheel is a digraph obtained by connecting a vertex $c$ to a directed 3-cycle by three odd bidirected paths that are disjoint except in $c$.

**Theorem 5.** *If $D$ is a 3-dicritical digraph, then $\rho(D) = 1$ if $D$ is a bidirected odd cycle, $\rho(D) = -1$ if $D$ is in an odd 3-wheel, $\rho(D) \leq -2$ otherwise.*

This result implies Theorem 4 because $\pi(D) = 0$ for every oriented graph $D$.

The proof of Theorem 5 is by considering a minimal counterexample $D$. Using a usual trick in the potential method, we prove that every set $R$ of vertices such that $3 \leq |R| \leq n(D) - 1$ has potential at least 4. We then deduce that many digraphs cannot be subdigraphs of $D$. Using those forbidden configurations and the discharging method we then derive a contradiction. An important role is played by the **out-chelou** arcs, which are arcs $xy$ such that $d^+(x) = d^-(y) = 2$ and $y$ is incident to no digon (and their directional dual the **in-chelou** arcs). We show that if $xy$ is an out-chelou arc and $d(x) \geq 5$ or is incident to at least two chelou arcs, then the structure around $y$ is very constrained. This structure is intensively used to show forbidden configurations and then design the dedicated discharging rules.

# References


[1] D. Bokal, G. Fijavz, M. Juvan, P. M. Kayll, and B. Mohar. The circular chromatic number of a digraph. *Journal of Graph Theory*, 46(3):227–240, 2004.

[2] J. Bang-Jensen, T. Bellitto, M. Stiebitz, and T. Schweser. Hajós and ore constructions for digraphs. *The Electronic Journal of Combinatorics*, 27(1):1–63, 2020.

[3] G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *Journal of the London Mathematical Society*, s1-27(1):85–92, 1952.

[4] G. A. Dirac. A theorem of R. L. Brooks and a conjecture of H. Hadwiger. *Proceedings of the London Mathematical Society*, s3-7(1):161–195, 1957.

[5] G. A. Dirac. On the structure of k-chromatic graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 63(3):683–691, 1967.

[6] R. Hoshino and K.-i. Kawarabayashi. The edge density of critical digraphs. *Combinatorica*, 35(5):619–631, 2015.

[7] A. Kostochka and M. Yancey. Ore's conjecture for $k = 4$ and Grötzsch's theorem. *Combinatorica*, 34(3):323–329, 2014.

[8] A. Kostochka and M. Yancey. Ore's conjecture on color-critical graphs is almost true. *Journal of Combinatorial Theory, Series B*, 109:73–101, 2014.

[9] A. V. Kostochka and M. Stiebitz. The minimum number of edges in 4-critical digraphs of given order. *Graphs and Combinatorics*, pages 1–16, 2020.

[10] C.-H. Liu and L. Postle. On the minimum edge-density of 4-critical graphs of girth five. *Journal of Graph Theory*, 86(4):387–405, 2017.

[11] B. Mohar. Eigenvalues and colorings of digraphs. *Linear Algebra and its Applications*, 432(9):2273–2277, 2010. Special Issue devoted to Selected Papers presented at the Workshop on Spectral Graph Theory with Applications on Computer Science, Combinatorial Optimization and Chemistry (Rio de Janeiro, 2008).

[12] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265–270, 1982.

[13] L. Postle. On the minimum number of edges in triangle-free 5-critical graphs. *European Journal of Combinatorics*, 66:264–280, 2017.