Maximum size of a triangle-free graph with bounded maximum degree and matching number

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Abstract

Determining the maximum number of edges under degree and matching number constraints have been solved for general graphs ([4, 1]). Since extremal graphs contain claws, C_4's and triangles, it is interesting to ask if the maximum number of edges decreases when these structures are forbidden separately. The first two cases being already settled ([5, 3]), in this paper we focus on triangle-free graphs. We show that unlike most cases for claw-free graphs and C_4-free graphs, forbidding triangles from extremal graphs causes a strict decrease in the number of edges and adds to the hardness of the problem. We provide a formula giving the maximum number of edges in a triangle-free graph with degree at most d and matching number at most m for all cases where $d \geq m$, and for the cases where $d < m$ with either $d \leq 6$ or $Z(d) \leq m < 2d$ where $Z(d)$ is roughly $5d/4$. We also provide an integer programming formulation for the remaining cases and as a result of further discussion on this formulation, we conjecture that our formula giving the size of triangle-free extremal graphs is also valid for these open cases.

1 Introduction

Determining the maximum number of edges of a graph when its maximum degree is at most d and its matching number is at most m for two given integers d and m, is a special case of a more general problem posed by Erdős and Rado [6]. This problem is also equivalent to determining Ramsey numbers for line graphs [2]. This question has been first solved in 1974 by Chvátal and Hanson [4] using some optimization techniques. A constructive proof has only came out much later in 2009 by Balachandran and Khare [1].

Let $\Delta(G)$ denote the maximum degree of G and $\nu(G)$ the size of a maximum matching of G. Using Vizing's Theorem, we obtain $|E(G)| \leq (\Delta(G) + 1)\nu(G)$ for any graph. In [1], it has been shown that this upper bound is actually met when some divisibility conditions hold, and we are "pretty close" to it otherwise. More precisely, the authors show that an extremal graph has $dm + \lfloor \frac{d}{2} \rfloor \left\lfloor \frac{m}{\lceil \frac{d}{2} \rceil} \right\rfloor$ edges; moreover, they exhibit an extremal graph (with maximum number of edges under given degree and matching number constraints) whose connected components consist of stars, complete graphs and in some cases "almost complete graphs" that contain C_4's (cycles of length 4), but do not inform us on the unicity of these extremal graphs. This gives rise to a natural question: Can the same maximum number of edges be still achieved if we restrict the structure of extremal graphs? This question is especially interesting for three classes of graphs obtained by restricting the above-mentioned types of components: claw-free graphs obtained by forbidding the smallest star, triangle-free graphs obtained by forbidding the smallest complete graph, and C_4-free graphs (since C_4's occur in "almost complete graphs"). Among these directions, the situation of claw-free graphs has been settled by Dibek et al. in [5] and the situation of chordal graphs (which are much more restricted than C_4-free graphs) by Blair et al. in [3]. In both graph classes, the size of edge-extremal graphs are the same as the general upper bound in most of the cases.

In this paper, we investigate the direction that remained open and consider triangle-free graphs from the same perspective. Unlike previous results, the size of edge-extremal triangle-free graphs that we find in this paper does not achieve the general upper bound apart from two simple cases, namely $d = 1$ or $m < \lfloor d/2 \rfloor$. This adds to the difficulty of proving the optimality in our results. We provide a single formula giving the maximum number of edges of a triangle-free graph when its maximum degree is at most d and its matching number is at most m for two given integers d and m such that $d \geq m$ or $m > d$ with either $d \leq 6$ or $Z(d) \leq m < 2d$ where $Z(d)$ is roughly $5d/4$. Then, we investigate the remaining cases, namely for natural numbers m and d such that $7 \leq d < m$ with either $m < Z(d)$ or $m \geq 2d$. For these open cases, we suggest an integer programming formulation based on our earlier observations. With further discussion on this formulation, we conjecture that the formula we provide is valid in general, with no condition on d and m.

2 Triangle-free extremal graphs

For a given graph class $\mathbf{C}$ and two given positive integers d and m, we define $\mathbb{M}_{\mathbf{C}}(d, m)$ to be the set of all graphs G in $\mathbf{C}$ satisfying $\Delta(G) \leq d$ and $\nu(G) \leq m$. A graph in $\mathbb{M}_{\mathbf{C}}(d, m)$ with the maximum number of edges is called *(edge)-extremal*, and the number of edges of an edge-extremal graph in $\mathbb{M}_{\mathbf{C}}(d, m)$ is denoted by $f_{\mathbf{C}}(d, m)$. A graph G is said to be *factor-critical* if $G \setminus v$ has a perfect matching for all $v \in V(G)$. $\mathcal{G}_{\mathbf{C}}(d, m)$ is the subclass of the set of edge-extremal graphs in $\mathbb{M}_{\mathbf{C}}(d, m)$ which consists of the graphs having maximum number of connected components isomorphic to a d-star. The following is a key lemma that describes the structure of edge-extremal graphs.

Lemma 1. *[1, 3] Let d, m be natural numbers, and let $\mathbf{C}$ be a graph class that is closed under vertex deletion and closed under taking disjoint union with stars. Take a graph $G \in \mathcal{G}_{\mathbf{C}}(d, m)$. Then, every connected component of G that is not a d-star is factor-critical.*

Let $\triangle$ be the class of triangle-free graphs. We first strengthen Lemma 1 for triangle-free graphs to derive the exact value of $f_{\triangle}(d, m)$ for various cases.

Proposition 1. *Let d and m be natural numbers. For any edge-extremal graph in $\mathcal{G}_{\triangle}(d, m)$ (whose number of d-star components is maximum), every component H of it which is not a d-star is a factor-critical and edge-extremal graph in $\mathbb{M}_{\triangle}(d, \nu(H))$ with matching number $\nu(H) \geq d$.*

Using Proposition 1 and constructing extremal graphs, we settle the problem for $d \geq m$.

Theorem 1. *With the preceding notation, $f_{\triangle}(d, m) = dm$ for $d > m \geq 1$, $f_{\triangle}(1, 1) = 1$ and $f_{\triangle}(d, d) = d^2 + 1$ for $d \geq 2$.*

Next, we consider $m > d$. After settling the trivial case $d = 1$, we assume $d \geq 2$ in the rest.

Theorem 2. *With the preceding notation, we have $f_{\triangle}(1, m) = m$ for all $m \geq 1$.*

For any $d \geq 2$, let $Z(d)$ be the smallest natural number n such that there exists a d-regular (if d is even) or almost d-regular (if d is odd) triangle-free and factor-critical graph G with $\nu(G) = n$.

Proposition 2. *For every $d \geq 2$, the value $Z(d)$ and a triangle-free factor-critical (almost) d-regular graph C_d with matching number $Z(d)$ exist.*

Our results are based on the following key property which states that if H is a connected component of a graph $G \in \mathcal{G}_{\triangle}(d, m)$ and if it is not a d-star, then in addition to the assumption $\nu(H) \geq d$ given in Proposition 1, we can also bound $\nu(H)$ from above by $Z(d)$.

Lemma 2. *Let d and m be natural numbers with $d \geq 2$, and let $G \in \mathcal{G}_\triangle(d,m)$. Then, for every connected component H of G, we have $\nu(H) \leq Z(d)$.*

We can find the exact value of $Z(d)$ for small values of d and for even d, and identify a very restricted interval for $Z(d)$ if d is odd.

Lemma 3. *We have $Z(d) = d$ for $d \in \{2,3\}$, and $Z(d) = d+1$ for $d \in \{4,5\}$. Moreover, $Z(d) \geq d+1$ holds for all $d \geq 4$.*

Lemma 4. *For $d \geq 2$, if d is even then we have $Z(d) = \lfloor 5d/4 \rfloor$; if d is odd then we have $\lfloor 5(d-1)/4 \rfloor \leq Z(d) \leq \lfloor 5(d+1)/4 \rfloor$.*

For $d \leq 6$, the value of $Z(d)$ given in Lemmas 3 and 4 allows us to compute $f_\triangle(d,m)$ for $m > d$. Besides, the case $d \leq Z(d) < 2d$ for any $d \geq 2$ can be derived using Proposition 1. It turns out that all our findings can be summarized in a single formula that we state as our main result. Recall that C_d is a (almost) d-regular triangle-free factor-critical graph with matching number $Z(d)$ whose existence is guaranteed by Proposition 2.

Theorem 3. *[Main Theorem] Let d and m be natural numbers with $d \geq 2$, and let k and r be non-negative integers such that $m = kZ(d) + r$ with $0 \leq r < Z(d)$. Then, for all the cases with $d \geq m$, and for the cases $d < m$ with either $d \leq 6$ or $Z(d) \leq m < 2d$, we have*

$$f_\triangle(d,m) = \begin{cases} dm + k\lfloor d/2 \rfloor & \text{if } r < d, \\ dm + k\lfloor d/2 \rfloor + r - d + 1 & \text{if } r \geq d, \end{cases}$$

where a graph in $\mathcal{G}_\triangle(d,m)$ can be constructed as the disjoint union of k copies of C_d and

(i) A_d if $r \geq d$, where A_d is a graph on $2d+1$ vertices obtained by replacing each one of two adjacent vertices of a C_5 with $d-1$ copies so that the copies of two vertices adjacent if and only if the originals are (a blow-up of C_5).

(ii) r copies of d-stars if $r < d$.

We observe that apart from two simple cases (where $m < \lfloor d/2 \rfloor$ or $1 = d < m$), we loose edges (with respect to the general case) by restricting the extremal graphs to be triangle-free.

3 An integer programming formulation and further discussions

In light of Theorem 3, the remaining open cases are for $7 \leq d < m$, and either $m < Z(d)$ or $m \geq 2d$. To solve these open cases, we develop an integer programming formulation based on our earlier observations. By Proposition 1 and Lemma 2, there is an edge-extremal graph $G \in \mathcal{G}_\triangle(d,m)$ whose components are either d-stars, or edge-extremal factor-critical triangle-free graphs H where $d \leq \nu(H) \leq Z(d)$. In other words, by letting x_i to be the number of connected components of G whose matching number is i, we have :

$$f_\triangle(d,m) = d\Big(m - \sum_{i=d}^{Z(d)} ix_i\Big) + \sum_{i=d}^{Z(d)} f_\triangle(d,i)x_i = dm + \sum_{i=d}^{Z(d)} (f_\triangle(d,i) - di)x_i.$$

It follows that, for a fixed d, the value of $f_\triangle(d,m)$ can be determined for all natural numbers m by finding the values of $f_\triangle(d,i)$ and corresponding x_i values only for $d \leq i \leq Z(d)$. Accordingly, $f_\triangle(d,m)$ can be computed as the optimal value of the following integer programming:

$$\textbf{Model 1:} \qquad \max \; dm + \sum_{i=d}^{Z(d)} (f_\triangle(d,i) - di)x_i$$

$$\text{subject to } \sum_{i=d}^{Z(d)} ix_i \leq m$$

$$x_i \geq 0, x_i \in \mathbb{Z}$$

This formulation can be seen as a bounded knapsack problem where there is a bounded number of items of each type. The utilities of the items are $(f_\triangle(d,i) - di)$ for $d \leq i \leq Z(d)$ and the volumes of the items range from d to $Z(d)$ which is yet unknown if d is odd (see Lemma 4). Since $f_\triangle(d,d) = d^2 + 1$ for $d \geq 2$, and $f_\triangle(d, Z(d)) = dZ(d) + \lfloor d/2 \rfloor$ for $d \geq 2$ by Theorem 3, it remains to compute $f_\triangle(d,i)$ for $d < i < Z(d)$. We conjecture that $f_\triangle(d,i)$ follows the same trend as what we identified in Theorem 3, which we could verify the correctness for small values by computer search.

Conjecture 1. *Theorem 3 holds also for $7 \leq d < m < Z(d)$. In other words, for $7 \leq d < i < Z(d)$, we have $f_\triangle(d,i) = di + i - d + 1$.*

We show that if Conjecture 1 holds, then Model 1 admits an optimal solution with nice properties. In particular, one can reach $f_\triangle(d,m)$ edges by taking the graph C_d as much as possible and adding either one more graph that is extremal for $f_\triangle(d,r)$ or r many stars, depending on $r \geq d$ where r is the remainder of m when divided by $Z(d)$. Notice that this is exactly how we construct an extremal graph in Theorem 3. Therefore, the formula in Theorem 3 would be valid for all integers d and m if Conjecture 1 is true:

Conjecture 2. *Let $m = kZ(d) + r$ for some $0 \leq r < Z(d)$. Then, we have*

$$f_\triangle(d,m) = \begin{cases} dm + k\lfloor d/2 \rfloor & \text{if } r < d \\ dm + k\lfloor d/2 \rfloor + r - d + 1 & \text{if } r \geq d. \end{cases}$$

We also conjecture about the value of $Z(d)$ which plays a crucial role in the computation of $f_\triangle(d,m)$ and the construction of extremal graphs.

Conjecture 3. *For $d \geq 21$ and odd, we have $Z(d) = \lfloor 5(d+1)/4 \rfloor$.*

Lastly, we note that computing $f_\triangle(d,i)$ for $d \leq i \leq Z(d)$ can be seen as a version of the Erdős-Stone's Theorem where we consider the maximum number of edges in a graph on $2i+1$ vertices not containing a subgraph isomorphic to any graph in the family $\mathcal{F} = \{K_3, K_{1,d}\}$.

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