On the Vertices Forced to Be in Every Metric Basis*

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Abstract

We study vertices that belong to all (strong) metric bases of a graph, i.e., to all (strong) resolving sets of minimum cardinality; such vertices are called *(strong) basis forced*. Although we also consider them in general graphs, we mainly focus on unicyclic graphs. In particular, we give a characterization of branch-active unicyclic graphs containing basis forced vertices.

1 Introduction

Let $G = (V(G), E(G)) = (V, E)$ be a connected, simple and undirected graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The *distance* $d_G(u, v) = d(u, v)$ between the vertices $u, v \in V$ is defined as the number of edges in any shortest path from u to v. A set $R \subseteq V$ is called a *resolving set* of G if for all distinct vertices $u, v \in V$ we have $d(r, u) \neq d(r, v)$ for some $r \in R$. A smallest resolving set of G is a *metric basis* of G. The cardinality of any metric basis of G is called the *metric dimension* and is denoted by $\dim(G)$. Originally, these concepts have been independently introduced in [4] and [9]. Resolving sets are connected to various applications such as network discovery and verification [1], robot navigation [5], chemistry [2] as well as embedding biological sequence data such as DNA, RNA and amino acid sequence data [10].

Observe that no vertex of G is required to be in all resolving sets of G since the set $V \setminus \{u\}$ is resolving for any $u \in V$. However, the situation is different if we consider metric bases instead of resolving sets. Indeed, for some graphs G, there exist vertices which belong to all metric bases of G as previously discussed in [3] (see Figure 1). Furthermore, a vertex $u \in V$ is called a *basis forced vertex* if u is contained in every metric basis of G.

In this paper, we further study the basis forced vertices building on the results of [3]. In Section 2, we first present a couple of results of basis forced vertices in general graphs. In Section 3, we proceed by presenting various results on unicyclic graphs; in particular, we give a characterization of basis forced vertices in branch-active unicyclic graphs. Throughout the paper, the proofs are omitted due to space limitations. However, they are given in the full version of this paper.

2 General graphs

In this section, we present two results concerning basis forced vertices in general graphs. In the first theorem, it is shown that if G is a graph with basis forced vertices, then we can construct a new graph based on G with the same basis forced vertices but with more vertices in total than G.

Theorem 1. *Let G be a graph of order n and $B \neq \emptyset$ be the set of basis forced vertices of G. Choose $b \in B$ and let $v \in V(G) \setminus B$ be such that $d(b, v) = \max\{d(b, u) \mid u \in V(G) \setminus B\}$. If H is a graph with*

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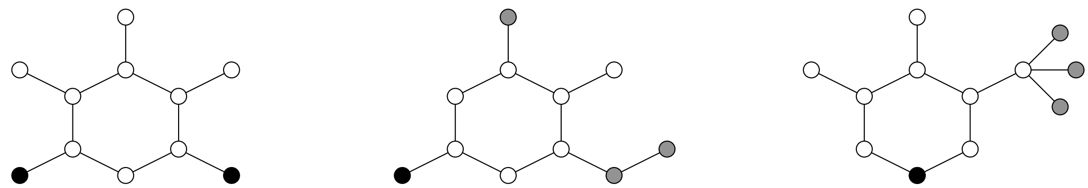

(a) (b) (c)

Figure 1: Three examples of unicyclic graphs that contain basis forced vertices. The black vertices are basis forced vertices and the gray vertices are in some metric bases but not all.

$V(H) = V(G) \cup V(P_m)$ and $E(G) = E(G) \cup E(P_m) \cup \{\{v, v_1\}\}$, where P_m is a path $v_1v_2 \dots v_m$, then B is also the set of basis forced vertices of H, $\dim(H) = \dim(G)$ *and* $|V(H)| = |V(G)| + m$.

A vertex u of G is called a *pendant* if the degree $\deg(u) = 1$. In the following theorem, we consider when a basis forced vertex can be moved to its pendant just added to the graph.

Theorem 2. *Let G be a graph with a basis forced vertex v and H be the graph obtained from G by attaching a pendant u to v. The vertex u is a basis forced vertex of H if and only if for every metric basis R of G there exists a vertex $w \in N_G(v)$ such that $d_H(r, w) = d_H(r, u)$ for all $r \in R$.*

3 Unicyclic graphs

In this section, the basis forced vertices are considered in unicyclic graphs. Hence, for the rest of the section, assume that G is a unicyclic graph, i.e., a graph with exactly one cycle C consisting of the vertices $v_0, v_1, \dots, v_{g-1}$, where g denotes the girth of G. In what follows, we first present some notation and terminology concerning unicyclic graphs (following the ones used in [7, 8]). The connected component of $G - E(C)$ containing v_i is denoted by T_{v_i}. A *thread* in G is a path $u_1u_2 \cdots u_k$ ($k \geq 1$), where $\deg(u_k) = 1$, the degrees of (possible) other vertices are equal to 2 and u_1 is attached to a vertex $v \in V(G)$ such that $\deg(v) \geq 3$. We say that $v \in V(G)$ is a *branching vertex* if $v \in V(C)$ and $\deg(v) \geq 4$, or $v \notin V(C)$ and $\deg(v) \geq 3$. A vertex $v_i \in V(C)$ is called *branch-active* if T_{v_i} contains a branching vertex. The number of branch-active vertices on the cycle of G is denoted by $b(G)$. Further denote the number of threads attached to v by $\ell(v)$. The set S is *branch-resolving* if for every $v \in V(G)$ of degree at least 3 the set S contains a vertex from at least $\ell(v) - 1$ threads attached to v. We denote

$$L(G) = \sum_{v \in V(G), \ell(v) > 1} (\ell(v) - 1).$$

For a subset $S \subseteq V(G)$, we say a vertex $v_i \in V(C)$ is *S-active* if $S \cap T_{v_i} \neq \emptyset$. The number of S-active elements on the cycle is denoted by $a(S)$. A thread attached to $v \in V(G)$ is S-free if it does not contain any element of S (observe that v is not included in the thread). If $v_i, v_j, v_k \in V(C)$ and $d(v_i, v_j) + d(v_j, v_k) + d(v_k, v_i) = |V(C)|$, then the vertices v_i, v_j and v_k form a *geodesic triple* on C. In [7], it is shown that each resolving set S of G is also branch-resolving and $a(S) \geq 2$, and that each branch-resolving set S of G with three S-active vertices on C forming a geodesic triple is also a resolving set of G. For the further study of a branch-resolving set S of G with $a(S) \geq 2$, we say that the cycle $C = v_0v_1 \cdots v_{g-1}v_0$ is *canonically* labelled with respect to S if v_0 is S-active and $k = \max\{i \mid v_i \text{ is } S\text{-active}\}$ is as small as possible. Furthermore, based on the canonical labelling, a characterization for resolving sets of unicyclic graphs have been given in [8]; many of the proofs of this paper rely on this characterization. In [7], it is also shown that for a unicyclic graph G the metric dimension $\dim(G)$ is equal to

$$L(G) + \max\{2 - b(G), 0\} \text{ or } L(G) + \max\{2 - b(G), 0\} + 1. \tag{1}$$

The first of following theorems is a reformulation of the results in [3] and the second one is presented in [3] (as a direct corollary of the results in [7]).

Theorem 3 ([3]). *Let $v \in V(G)$ be a basis forced vertex of a unicyclic graph G. Then either (i) $v = v_i$ for some $v_i \in V(C)$ and $V(T_{v_i}) = \{v_i\}$ or (ii) v is a pendant attached to some $v_i \in V(C)$ and $V(T_{v_i}) = \{v_i, v\}$.*

Theorem 4 ([3]). *If G is a unicyclic graph, then G contains at most two basis forced vertices.*

Combining Equation (1) and Theorem 3, we obtain that if the unicyclic graph G contains f basis forced vertices and the set $R \subseteq V(G)$ is a minimum branch-resolving set of G, then R contains no basis forced vertices and $\dim(G) \geq L(G) + f$. Furthermore, it can be shown that a unicyclic graph with basis forced vertices has *even girth*. Moreover, the unicyclic graphs containing basis forced vertices can shown to be of one of the following types: (1) $b(G) = 0$, $\dim(G) = 2$ and G has a unique metric basis (Figure 1(a)), (2) $b(G) = 0$ and G contains exactly one basis forced vertex (Figure 1(b)) and (3) $b(G) = 1$ and G contains exactly one basis forced vertex (Figure 1(c)). The following theorems can be obtained by a careful analysis using the previous observations.

Theorem 5. *If G is a unicyclic graph with the girth $g \geq 4$ and at least one basis forced vertex, then $\dim(G) = L(G) + \max\{2 - b(G), 0\}$.*

Theorem 6. *If S is a metric basis of a unicyclic graph G with basis forced vertices, then $a(S) = 2$.*

By the previous theorem, if S is a metric basis of G with basis forced vertices, then using the canonical labelling for C we know that v_0 and v_k are the only S-active vertices. In the following lemma, we give restrictions (which cannot be generally improved) for the value k.

Lemma 7. *If G is a unicyclic graph containing at least one basis forced vertex, then $2 \leq k < g/2$.*

The following two lemmas further discuss the structure of G containing basis forced vertices.

Lemma 8. *If G is a unicyclic graph with at least one basis forced vertex and $i \in [1, k-1]$, then following properties hold: (i) Either $\deg(v_i) = 2$ or there is exactly one thread at v_i. (ii) A thread at v_i is of length at most $g/2 - k - 1$. (iii) There exists a thread of length $g/2 - k - 1$ at some v_i or $k = g/2 - 2$ and there is no basis forced vertex on the cycle. (iv) For each $i \in [k+1, g/2-1] \cup [g/2+k+1, g-1]$, we have $\deg(v_i) = 2$.*

Lemma 9. *Let G be a unicyclic graph with at least one basis forced vertex v, S a metric basis of G and C (canonically) labelled so that T_{v_k} contains the basis forced vertex v. Denoting $m = \min\{j \geq 1 \mid \deg(v_j) \geq 3$ or $\deg(v_{g/2+j}) \geq 3\}$, we have $m < k$ and there exists a thread of length $\geq m$ at some v_i where $i \in [g/2 + m + 1, g/2 + k]$.*

Recall by Theorem 3 that a basis forced vertex v on G is either on the cycle C with $\deg(v) = 2$ or v is a pendant attached to a vertex on the cycle. In what follows, we focus on the case, where $b(G) = 1$ and the (single) basis forced vertex is a pendant. Before our main theorem, we still require one additional lemma.

Lemma 10. *If G is a unicyclic graph containing basis forced vertices with $b(G) = 1$ and v is a branch-active vertex, then there is no thread attached to v.*

In the following theorem, we finally show that the properties obtained for unicyclic graphs containing basis forced vertices in Lemmas 7–10 (together with the fact that G has even girth) are actually sufficient when $b(G) = 1$ and pendants are considered; hence, we are able to characterize the basis forced vertices in the case of pendants.

Theorem 11. *Let G be a unicyclic graph with $b(G) = 1$ and C be its cycle labelled in such a way that v_0 is branch-active. Assume further that v is a pendant attached to $v_j \in C$ and $V(T_{v_j}) = \{v_j, v\}$. Now v is a basis forced vertex of G if and only if (1) the girth g of G is even, (2) no thread is attached to v_0, (3) $j \in [2, g/2-1]$, (4) $\deg(v_i) = 2$ for all $i \in [j+1, g/2-1] \cup [g/2+j+1, g-1]$, (5) every thread attached to some v_i where $i \in [1, j-1]$ is at most of length $g/2-j-1$, (6) for $m = \min\{l \geq 1 \mid \deg(v_l) \geq 3 \text{ or } \deg(v_{g/2+l}) \geq 3\}$ we have $m < j$ and there exists a thread at least of length m at some v_i where $i \in [g/2+m+1, g/2+j]$, and (7) $j = g/2-2$ or there exists a thread of length $g/2-j-1$ at some v_i where $i \in [1, j-1]$.*

In addition to the results above, we also investigate the vertices forced to belong to every strong metric basis of unicyclic graphs (see [6] for the formal definitions of strong metric basis and strong metric dimension of graphs). Among other results, and in contrast to the classical version, we prove that unicyclic graphs can have as many strong basis forced vertices as we would require. We also give some structural properties of unicyclic graphs having strong basis forced vertices.

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