Open Problems

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Open Problem #1 (*posed by Ignasi Sau*):

We consider the Directed Disjoint Paths problem, where we are given as input a directed graph G and $2k$ vertices $s_1, \ldots, s_k, t_1, \ldots, t_k \in V(G)$ and we ask whether G contains pairwise vertex-disjoint paths $P_1, \ldots, P_k$ such that, for every $i \in \{1, \ldots, k\}$, P_i is an (s_i, t_i)-path.

The problem is known to be NP-hard for $k = 2$. If n is the number of vertices of the input graph G, and w is the *directed treewidth* $\mathbf{dtw}(G)$ of G, then the Directed Disjoint Paths problem can be solved in time $n^{O(w+k)}$. It is known that, unless $\mathsf{FPT} = \mathsf{W}[1]$, there is no algorithm running in time $f(k) \cdot n^{g(w)}$, for any two functions $f, g : \mathbb{N} \to \mathbb{N}$.

Question: Is there an algorithm running in time $f(w) \cdot n^{g(k)}$, for some functions $f, g : \mathbb{N} \to \mathbb{N}$?

Open Problem #2 (*posed by Pierre Aboulker*):

Consider a set of n points $a_1, \ldots, a_n$ in the plane, that are not all contained in a single straight line. Each pair of points defines a unique line. Suppose the set of points defines m distinct lines, which we denote by $\ell_1, \ldots, \ell_m$.

As first observed by Erdős, the Sylvester-Gallai Theorem implies that $m \geq n$.

For each $i \neq j \in \{1, \ldots, m\}$, we denote by $\ell_i \cap \ell_j$ the number of points among $a_1, \ldots, a_n$ that are included in both lines ℓ_i and ℓ_j (which is 0 or 1). We set $i_{av} = \frac{\sum_{i \neq j} \ell_i \cap \ell_j}{\binom{m}{2}} \leq 1$. This corresponds to the average intersection of the lines.

Question: Is it true that $i_{av} \cdot m \geq n$?

Denote by $d(v)$ the number of lines going through a point v. Observing that in the sum $\sum_{i \neq j} \ell_i \cap \ell_j$ each point v is counted exactly $\binom{d(v)}{2}$ times, one can easily see that the above inequality is equivalent to the following one:

$$\sum_{i=1}^{n} d(a_i)(d(a_i) - 1) \geq n(m - 1),$$

Open Problem #3 (*posed by Colin Geniet*):

We know that a graph G has twinwidth at most $k \iff$ there is an ordering of $V(G)$ such that the adjacency matrix of G with respect to this ordering cannot be separated to k^2 blocks, each of rank more than k.

We know that bounded stack/queue number implies bounded twinwidth. We also know that cubic graphs can have unbounded twinwidth. The proof of the latter uses some counting arguments.

Question: Can we have an explicit construction of a cubic graph of large twinwidth?

Open Problem #4 (*posed by David Wood*):

The following Ramsey-theoretic conjecture about points in the plane is known as *Big Line Big Clique Conjecture*, given by Jan Kara, Attilla Por, and David Wood.

Conjecture: For $k, \ell \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that every finite set of at least n points in $\mathbb{R}^2$ has ℓ collinear points or k pairwise visible points.

Standard Ramsey-theoretic arguments fail to solve the above problem, since visibility graphs are not closed under induced subgraphs.

J. Kára, A. Pór, D. R. Wood. On the chromatic number of the visibility graph of a set of points in the plane, *Discrete & Computational Geometry*, 34.3:497–506, 2005.

J. Matoušek. Blocking visibility for points in general position. *Discrete & Computational Geometry* 42:219–223, 2009.

A. Pór, D. R. Wood. On visibility and blockers, *J. Computational Geometry* 1:29–40, 2010.

A. Pór, D. R. Wood, The big-line-big-clique conjecture is false for infinite point sets. arXiv:1008.2988

B. Hujter, S. Kisfaludi-Bak. 5-Colorable visibility graphs have bounded size or 4 collinear points. arXiv:1410.7273.

Open Problem #5 (*posed by Jørgen Bang-Jensen*):

We define the following problem:

Given a directed graph that is not strongly connected, our task is to reverse the minimum number of arcs so as to make it strongly connected.

This problem can be solved in polynomial time using an algorithm for finding a minimum cost submodular flow.

A directed graph is *k-strong* if it has at least $k+1$ vertices and for any set of at most $k-1$ vertices, the digraph obtained after their removal is strongly connected.

We define a variant of the problem above, when we ask for the minimum number of arcs needed to be reversed to make it k-strong.

Clearly D has a set of arcs whose reversal makes the new digraph k-strong if and only if its underlying undirected graph $UG(D)$ has a k-strong orientation.

Durand de Gevigney proved that for every $k \geq 3$ it is NP-complete to decide whether a given input graph has a k-strong orientation. Hence for $k \geq 3$ it is NP-complete even to decide whether there is any set of arcs whose reversal makes a given digraph k-strong.

Thomassen proved that a graph $G = (V, E)$ has a 2-strong orientation if and only if it is 4-edge-connected and $G - v$ is 2-edge-connected for every $v \in V$. Together with the remark above this implies that we can check in polynomial time whether a given digraph D has a set of arcs whose reversal makes the resulting digraph 2-strong.

Question: Is the problem (i.e., the minimization version) solvable in polynomial time for $k = 2$?

O. Durand de Gevigney: On Frank's conjecture on k-connected orientations. J. Comb. Theory, Ser. B 141: 105-114 (2020)

J. Edmonds and R. Giles. A min-max relation for submodular functions on graphs. In Studies in integer programming (Proc. Workshop, Bonn, 1975), pages 185–204. Ann. of Discrete Math., Vol. 1. North-Holland, Amsterdam, 1977.

C. Thomassen, Strongly 2-connected orientations of graphs. J. Comb. Theory, Ser. B 110: 67-78 (2015)

Open Problem #6 (*posed by Fedor V. Fomin*):

We define the Detour problem, where we are given a directed graph G and $s, t \in V(G)$ and we ask if there is an (s, t)-path in G of length strictly greater than $\mathbf{dist}_G(s, t)$?

Question: Is Detour NP-complete? Is it in P? If we ask for an (s, t)-path of length at least $\mathbf{dist}_G(s, t) + k$, is the problem FPT parameterized by k?

The variant of the problem where we ask for an (s,t)-path of length *exactly* $\mathbf{dist}_G(s,t) + k$ is FPT parameterized by k. Also, for the undirected variant, the problem parameterized by k is FPT.

Open Problem #7 (*posed by Nacim Oijid*):

We define the following game: given a hypergraph $H = (V, E)$, Alice and Bob play by selecting, one after the other, a vertex of the hypergraph. Alice wins if she occupies all vertices in a hyperedge, otherwise Bob wins (i.e., vertices selected by Bob are a hitting set of all hyperedges). Rahman and Watson [STACS 2021] proved that determining the winner is PSPACE-complete on 6-uniform hypergraphs. Galliot, Gravier, and Sivignon proved that it is polynomial on 3-uniform hypergraphs.

Question: What is the complexity of the problem for 4-uniform and 5-uniform hypergraphs?

Open Problem #8 (*posed by Christophe Paul & Frédéric Havet*):

The following problem originates in the work of Pouzet on Tournaments. It is motivated by the problem of Sorting by Reversal, where we are given a permutation of a set of integers, and we ask to sort them by reversing intervals of the permutation.

We can also consider the k-Tournament Inversion problem: Given a tournament T, one aims to turn T to a transitive tournament. The operation allowed is to select a set of vertices S and reverse the arcs of $T[S]$ (i.e., reverse all arcs of T with both endpoints in S).

This problem is FPT parameterized by the number of reversals (Belkhechine, Bouaziz, Boudabbous, and Pouzet, 2010).

Question: Is Tournament Inversion NP-complete or not, when k is part of the input?

We also consider the k-Inversion problem, where we are given a directed graph D and we ask whether the inversion number of D is at most k. It is known that 1-Inversion and 2-Inversion are NP-complete (Bang-Jensen, Ferreira da Silva, and Havet, arXiv 2021). The following was conjectured by Bang-Jensen, Ferreira da Silva, and Havet [arXiv 2021].

Conjecture: k-Inversion is NP-complete for every $k \in \mathbb{N}$.

Open Problem #9 (*posed by Marco Caoduro*):

Axis-parallel rectangles on the plane have the *Helly property*, hence any set of pairwise intersecting rectangles can be hit by one point. Allowing the rectangles to rotate this property is lost and it is possible to construct families of pairwise intersecting rectangles where an arbitrarily large number of hit points is needed.

The situation changes by restricting the attention to squares on the plane (either all units or with different sizes). We ask the following question:

Question: Determine the minimum number of points needed to hit any finite set of pairwise intersecting (unit or not) squares.

In Caoduro and Sebő [ArXiv 2022], it is observed that the optimal values lie in the set $\{3, 4\}$ for unit squares, and $\{4, \dots, 10\}$ for squares of different sizes. The analogous problem for disks is settled: three points (resp. four points) are always sufficient and sometimes necessary to hit any finite set of pairwise intersecting unit (not unit) disks on the plane [Hadwiger, Debrunner, and Klee, 1964; Danzer, 1956].