Product structure of planar graphs

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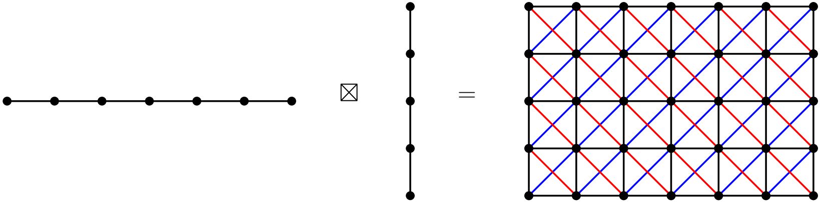
Product structure of planar graphs

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19

Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth $\leqslant 8$ and some path P



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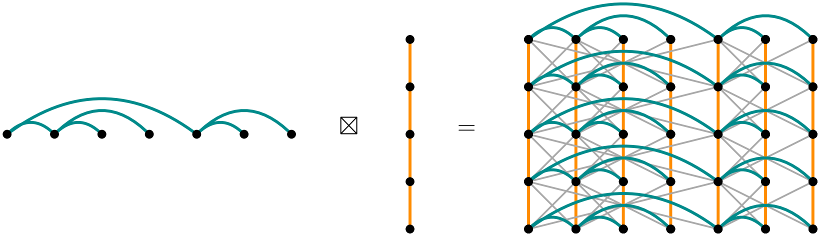
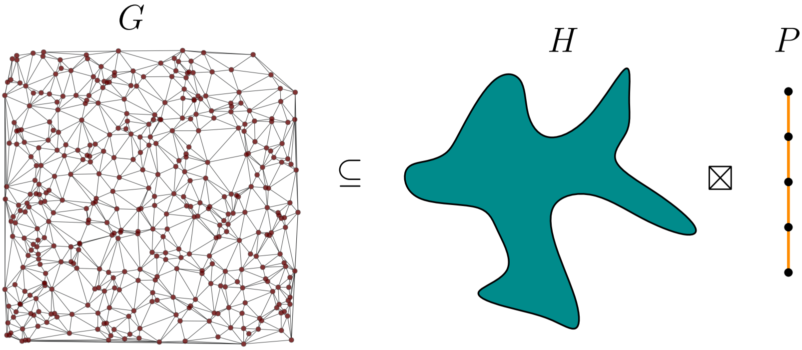


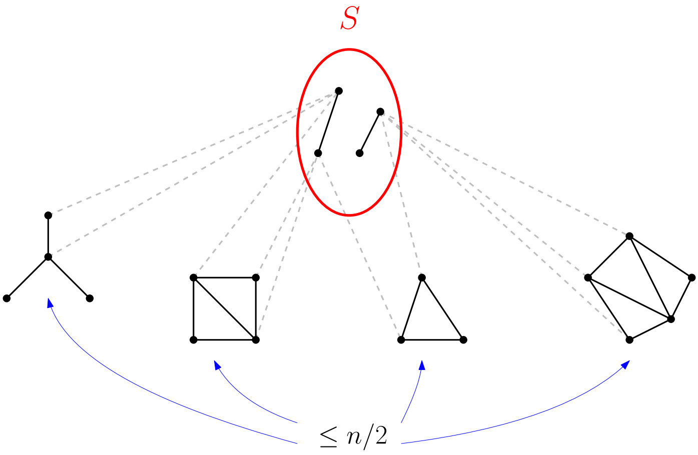

G
⊆
H
⊠
P

Outline

- Background
- Proof
- Applications
- Research directions

How to decompose planar graphs?

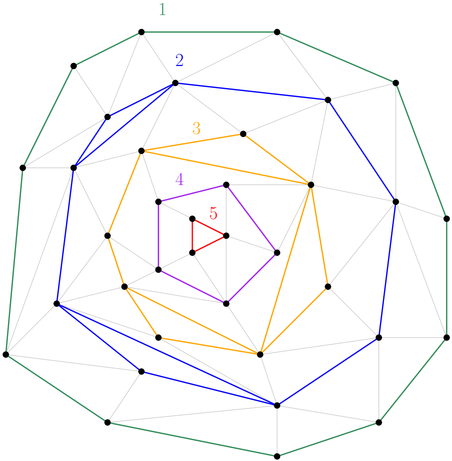
Separator:



Lipton & Tarjan '79

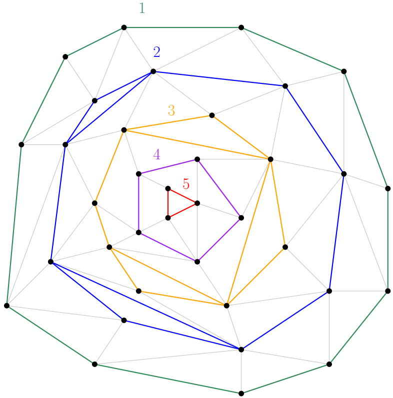
Every n-vertex planar graph has a $O(\sqrt{n})$-size separator

Baker's technique ('94)



Fact: Union of ℓ consecutive layers has treewidth $O(\ell)$

Baker's technique ('94)

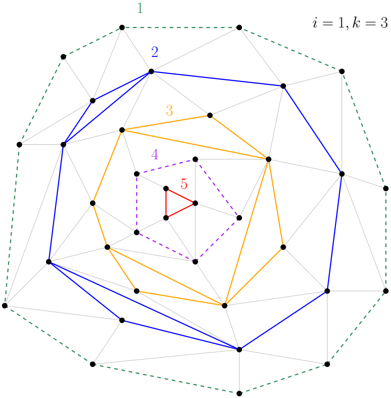


Remove all layers numbered i mod k

Choose i so that $\leqslant n/k$ vertices are removed

Solve problem on remaining graph, which has treewidth $O(k)$

Baker's technique ('94)



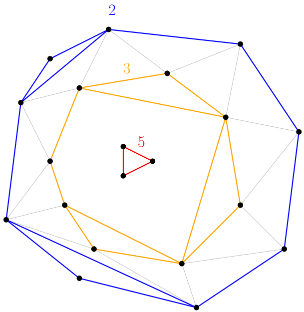
Remove all layers numbered i **mod** k

Choose i so that $\leqslant n/k$ vertices are removed

Solve problem on remaining graph, which has treewidth $O(k)$

Baker's technique ('94)

$i = 1, k = 3$



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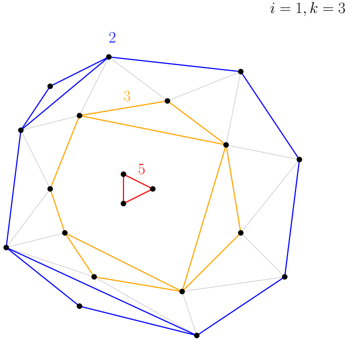


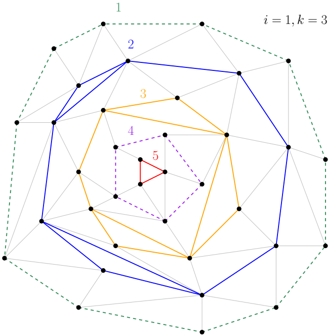
Remove all layers numbered i mod k

Choose i so that $\leqslant n/k$ vertices are removed

Solve problem on remaining graph, which has treewidth $O(k)$

E.g. with $k = \log n$, this gives a $\left(1 - \frac{c}{\log n}\right)$-approximation algorithm for Max Independent Set

Baker $\Rightarrow$ Lipton-Tarjan



Take $k = \sqrt{n}$, choose i so that $\leqslant n/k = \sqrt{n}$ vertices are removed

Remaining graph has treewidth $O(k) = O(\sqrt{n})$

Take a separator S' of size $O(\sqrt{n})$ in remaining graph

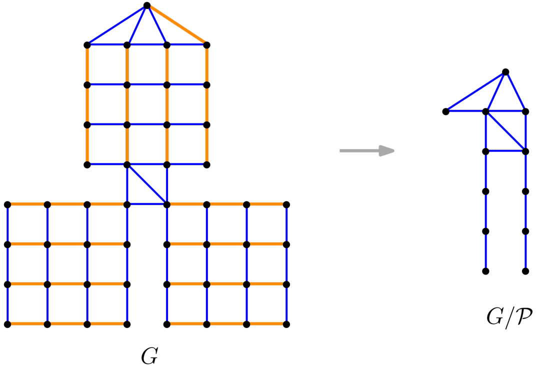
Union of vertices removed and S' is a separator of size $O(\sqrt{n})$

A new way of decomposing planar graphs

Mi. Pilipczuk & Siebertz '18
Every planar graph G has a vertex partition $\mathcal{P}$ into geodesics such that $G/\mathcal{P}$ has treewidth $\leqslant 8$

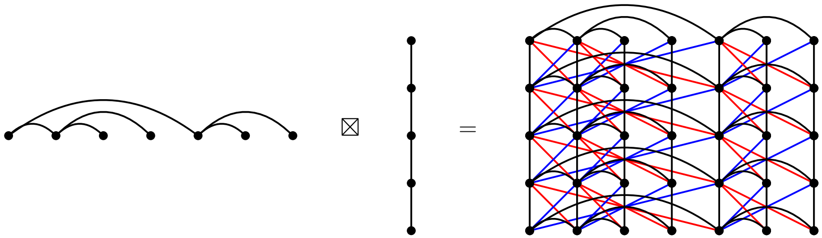
geodesic = shortest path (between its endpoints)

$G/\mathcal{P}$ = graph obtained by contracting each path in $\mathcal{P}$ into a vertex



Product structure of planar graphs

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19
Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth $\leqslant 8$ and some path P



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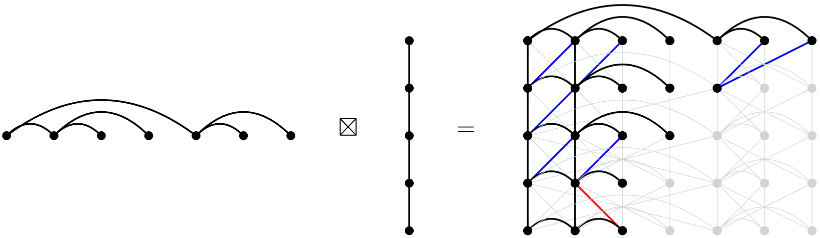
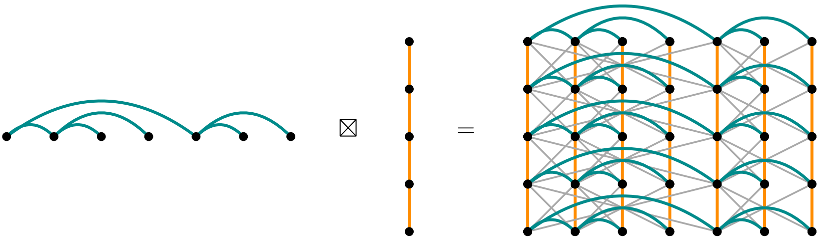


Product structure $\Rightarrow$ Baker



Union of ℓ consecutive layers has treewidth $\leqslant 9\ell - 1$

Mi. Pilipczuk & Siebertz '18

Every planar graph G has a vertex partition $\mathcal{P}$ into geodesics s.t. $\mathrm{tw}(G/\mathcal{P}) \leqslant 8$

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19

Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant 8$ and some path P

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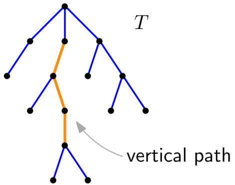
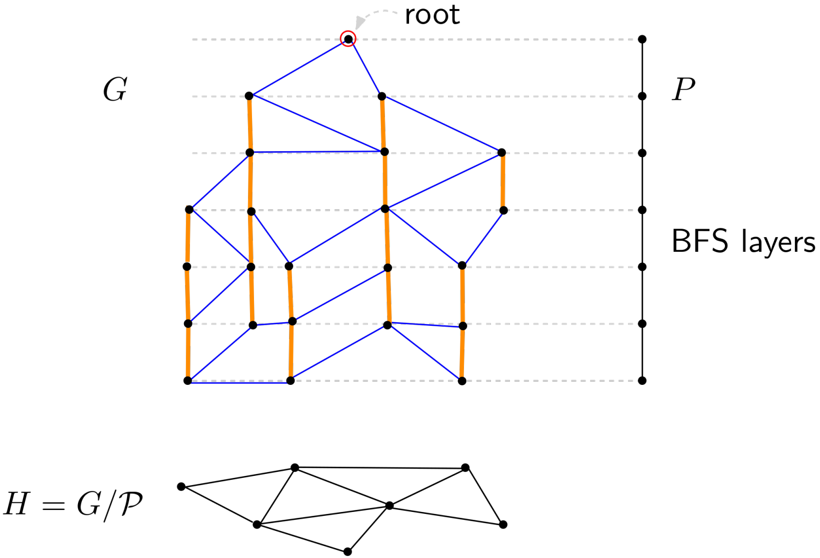
Dujmović, J., Micek, Morin, Ueckerdt, Wood '19

G connected planar graph, T rooted spanning tree $\Rightarrow \exists$ vertex partition $\mathcal{P}$ of G into **vertical paths** of T s.t. $\mathrm{tw}(G/\mathcal{P}) \leqslant 8$

Theorems above follow by taking $T =$ BFS tree





root
G
P
BFS layers
$H = G/\mathcal{P}$

Proof setup

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19

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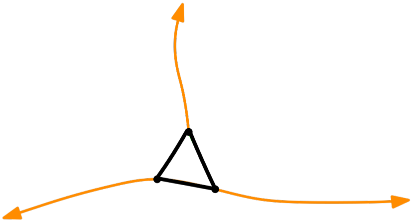
Proof setup

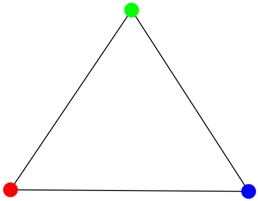
Dujmović, J., Micek, Morin, Ueckerdt, Wood '19
G plane triangulation, T rooted spanning tree $\Rightarrow \exists$ vertex partition $\mathcal{P}$ of G into tripods of T s.t. $\mathrm{tw}(G/\mathcal{P}) \leqslant 3$

Tripod: facial triangle sending out 3 disjoint vertical paths



Degenerated cases: an edge (a vertex) sending 2 (1) disjoint vertical paths

Setting: Region bounded by 3 tripods, root on boundary or outside



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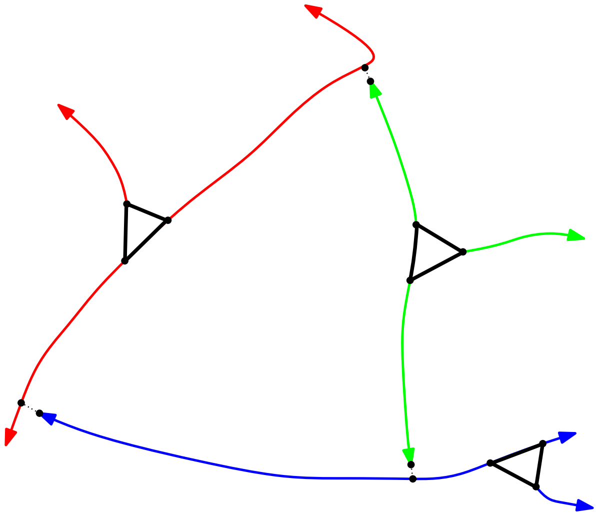

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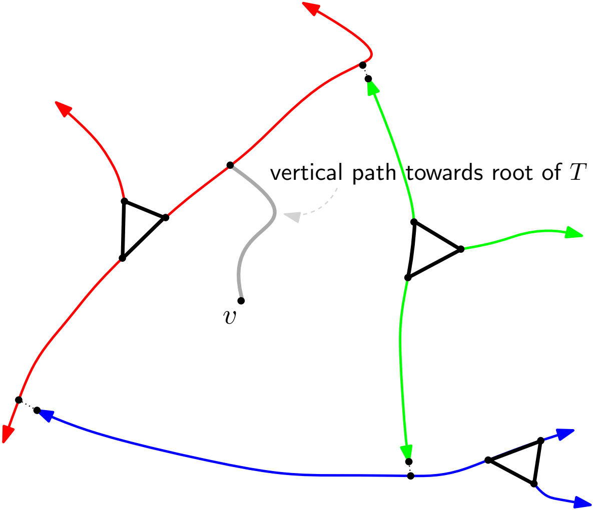

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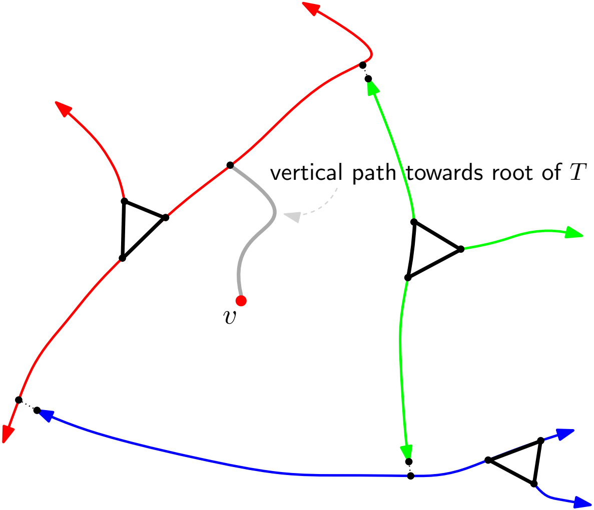

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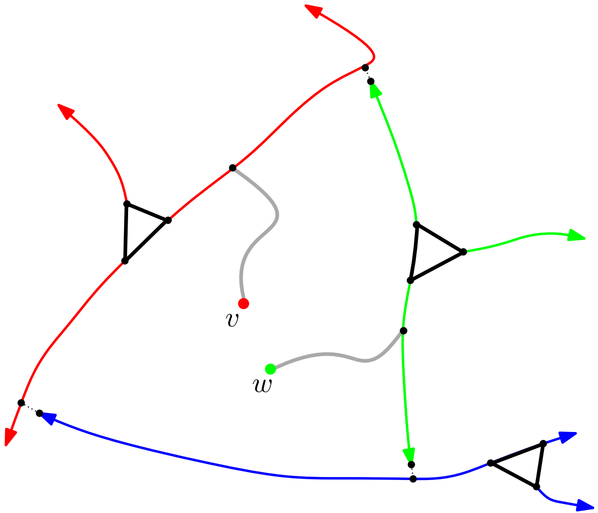

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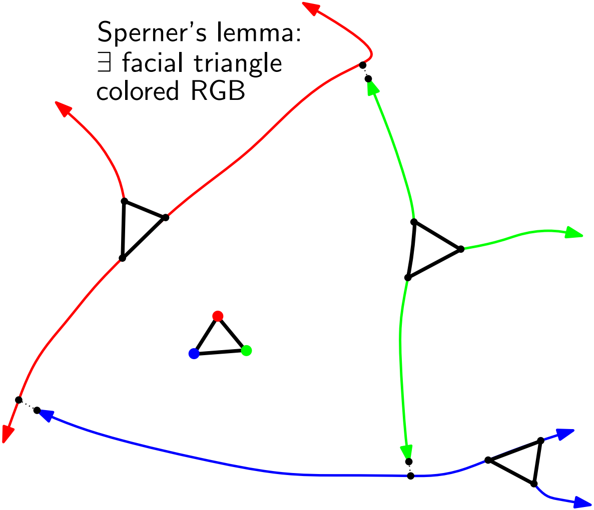

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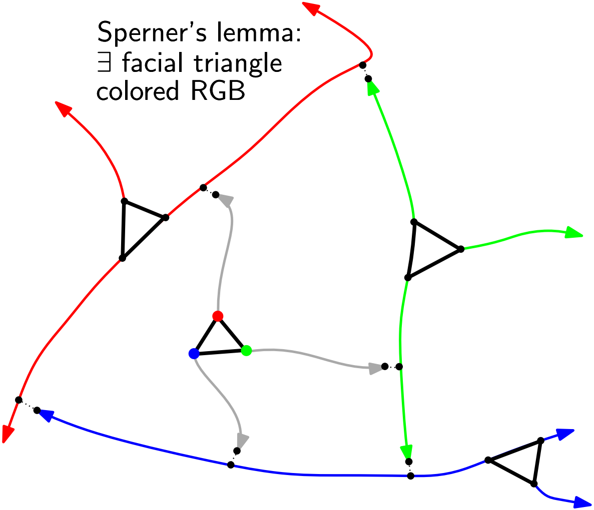

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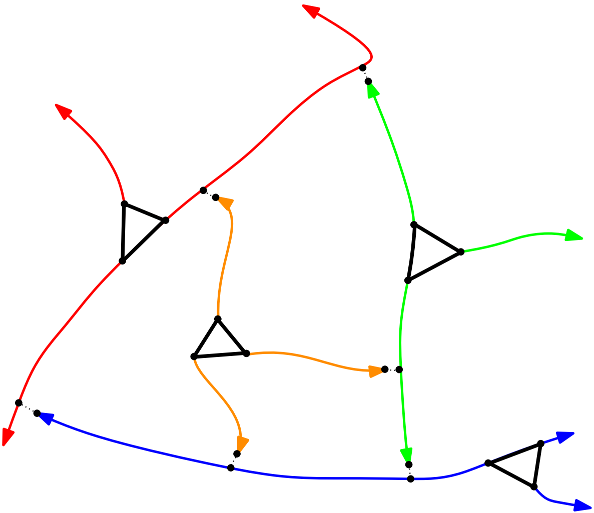
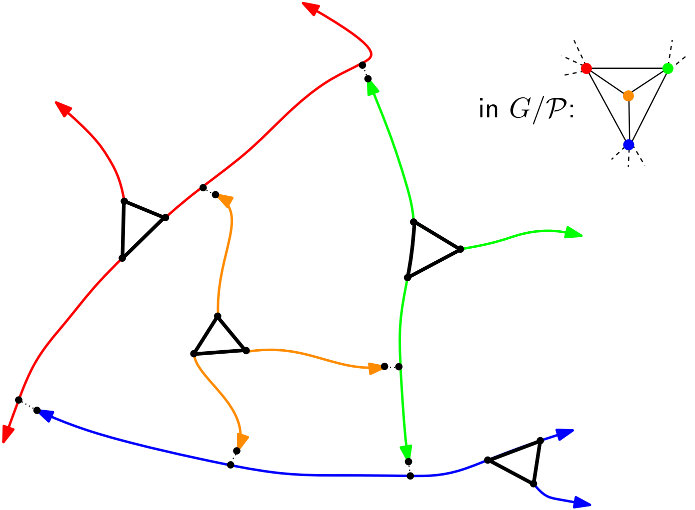


Setting: Region bounded by 3 tripods, root on boundary or outside



in $G/\mathcal{P}$:

Product structure: Summary

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19

Every planar graph is

- a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant 8$ and some path P

Product structure: Summary

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Every planar graph is

- a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant 8$ and some path P
- a subgraph of $H \boxtimes P \boxtimes K_3$ for some graph H with $\mathrm{tw}(H) \leqslant 3$ and some path P

Graphs on surfaces

(Figure: Felix Reidl)

Euler genus of sphere $+\ g$ handles $= 2g$
Euler genus of sphere $+\ g$ crosscaps $= g$

DJMMUW '19

Every graph of Euler genus $g \geqslant 1$ is

- a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant 2g + 8$ and some path P
- a subgraph of $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with $\mathrm{tw}(H) \leqslant 4$ and some path P

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(Figure: Felix Reidl) 

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Distel, Hickingbotham, Huynh, Wood '21

Every graph of Euler genus $g \geqslant 1$ is a subgraph of $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with $\mathrm{tw}(H) \leqslant 3$ and some path P

Apex-minor free graphs

G is apex if $G - v$ is planar for some $v \in V(G)$

DJMMUW '19

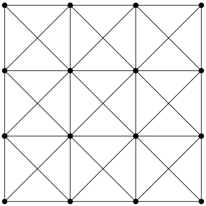
For every apex graph X there exists $c \geqslant 1$ s.t. every X-minor free graph is a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant c$ and some path P

No such theorem holds if X is not apex

Dujmović, Esperet, Morin, Walczak, Wood '20

For every graph X there exists $c \geqslant 1$ s.t. for every $\Delta \geqslant 1$, every X-minor free graph with maximum degree $\leqslant \Delta$ is a subgraph of $H \boxtimes P$ for some graph H with $\mathrm{tw}(H) \leqslant c\Delta$ and some path P

k-Planar graphs



Dujmović, Morin, Wood '19

Every k-planar graph is

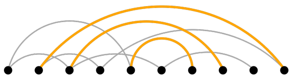
- a subgraph of $H \boxtimes P$ for some graph H with treewidth $O(k^5)$ and some path P
- a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some graph H with treewidth $O(k^3)$, for some path P, and some $\ell \in O(k^2)$

Dujmović, Morin, Wood '19

Every 1-planar graph is a subgraph of $H \boxtimes P \boxtimes K_7$ for some graph H with treewidth 3 and some path P

Application: Queue-numbers

Rainbow in a vertex ordering $v_1, \ldots, v_n$:



Size of rainbow = number of edges

Queue-number $\mathbf{qn}(G)$ = smallest k s.t. there is a vertex ordering where every rainbow has size $\leqslant k$

Conjecture (Heath, Leighton, Rosenberg '92)
Planar graphs have bounded queue-number

DJMMUW '19
Planar graphs have queue-number $\leqslant 49$

Bounded queue-number: Sketch of proof

Dujmović, Morin, Wood '05

If H has treewidth k then $\mathsf{qn}(H) \leqslant f(k)$

Wiechert '17

If H has treewidth k then $\mathsf{qn}(H) \leqslant 2^k - 1$

Lemma

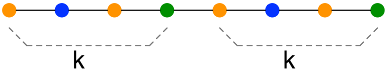
$\mathsf{qn}(H \boxtimes P) \leqslant 3\,\mathsf{qn}(H) + 1$ for every path P

Corollary

If G planar then $G \subseteq H \boxtimes P$ for some H with $\mathsf{tw}(H) \leqslant 8$ and some path P, thus

$$\mathsf{qn}(G) \leqslant \mathsf{qn}(H \boxtimes P) \leqslant 3\,\mathsf{qn}(H) + 1 \leqslant 3 \cdot (2^8 - 1) + 1 = 766$$

Application: Nonrepetitive colorings



repetitively colored path

Vertex coloring nonrepetitive if $\nexists$ repetitively colored paths

Conjecture (Alon, Grytczuk, Hałuszczak, Riordan '02)
Planar graphs have bounded nonrepetitive chromatic number

Dujmović, Esperet, J., Walczak, Wood '19
Planar graphs have nonrepetitive chromatic number $\leqslant 768$

Application: p-Centered colorings

p-Centered coloring of G: Vertex coloring s.t. in every connected subgraph X of G, either some color appears exactly once on X, or more than p distinct colors appear on X

$\chi_p(G) :=$ min. number of colors in a p-centered coloring of G

Mi. Pilipczuk & Siebertz '18
If G planar then $\chi_p(G) = O(p^{19})$

Debski, Felsner, Micek, Schröder '19
If G planar then $\chi_p(G) = O(p^3 \log p)$

Best known lower bound: $\Omega(p^2 \log p)$

Application: Fractional treedepth-fragility

Given G and $a \geqslant 1$, let $r(G, a)$ be smallest positive integer s.t. $\exists$ probability distribution on $2^{V(G)}$ satisfying:

1. Each $v \in V(G)$ has probability $\leqslant \frac{1}{a}$ of belonging to a random subset
2. $G - X$ has treedepth $\leqslant r(G, a)$ for each X in support of the distribution

Class of graphs $\mathcal{G}$ is fractionally treedepth-fragile at rate f if $r(G, a) \leqslant f(a)$ for all $G \in \mathcal{G}$ and $a \geqslant 1$

Dvořák & Sereni '20
Planar graphs are fractionally treedepth-fragile at rate $f(a) = O(a^3 \log a)$

Best known lower bound: $\Omega(a^2 \log a)$

Application: Vertex rankings

Vertex ℓ-ranking of G: Coloring $\varphi : V(G) \to \mathbb{N}$ s.t. for every path $v_1, v_2, \ldots, v_k$ of length between 1 and ℓ,

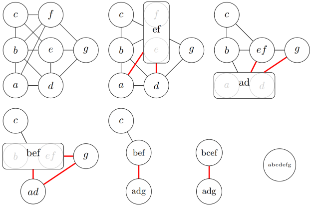
$$\varphi(v_1) \neq \varphi(v_k) \qquad \text{or} \qquad \varphi(v_1) < \max\{\varphi(v_2), \ldots, \varphi(v_{k-1})\}$$

Bose, Dujmović, Javarsineh, Morin '20
For fixed $\ell \geqslant 2$, every n-vertex planar graph has an ℓ-ranking with $O(\log n / \log\log\log n)$ colors, and this is best possible

Improves earlier $O(\log n)$ bound (Karpas-Neiman-Smorodinsky '15)

Application: Twin-width

Reduction sequence of G: Pairs of vertices are successively identified until only one vertex left



(image credit: Twin-width I)

When identifying u and v, each edge incident to exactly one of u and v is colored red

Twin-width of G: Min. k s.t. $\exists$ reduction sequence where every vertex has red degree $\leqslant k$ at all times

Bonnet, Kim, Thomassé, Watrigant '20
Planar graphs have twin-width $O(1)$

Using tripod decomposition:

Bonnet, Kwon, Wood '22
Planar graphs have twin-width $\leqslant 583$

Jacob and Ma. Pilipczuk '22
Planar graphs have twin-width $\leqslant 183$

Bekos, Da Lozzo, Hliněný, Kaufmann '22
Planar graphs have twin-width $\leqslant 37$

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Using a variant, fine-tuned for twin-width:

Hliněný (June '22)
Planar graphs have twin-width $\leqslant 9$

Application: Induced-universal graphs

G is induced-universal for a set $\mathcal{F}$ of graphs if G contains every member of $\mathcal{F}$ as an induced subgraph

What is the minimum number of vertices in an induced-universal graph for n-vertex planar graphs?

A.k.a. adjacency labeling schemes for planar graphs

- $O(n^6)$ (Muller 1988)
- $O(n^{4+o(1)})$ (Kannan, Naor, Rudich 1988)
- $O(n^{2+o(1)})$ (Gavoille & Labourel 2007)
- $O(n^{4/3+o(1)})$ using product structure (Bonamy, Gavoille, Mi. Pilipczuk 2019)

Dujmović, Esperet, J., Gavoille, Micek, Morin '20
Induced-universal graphs with $O(n^{1+o(1)})$ vertices for n-vertex planar graphs

Application: Universal graphs

G is universal for a set $\mathcal{F}$ of graphs if G contains every member of $\mathcal{F}$ as a subgraph (not necessarily induced)

What is the minimum number of edges in a universal graph for n-vertex planar graphs?

- $O(n^{3/2})$ (Babai, Chung, Erdős, Graham, Spencer 1982)

Esperet, J., Morin '20
Universal graphs with $O(n^{1+o(1)})$ edges for n-vertex planar graphs

Research direction 1: Improve bounds

Dujmović, J., Micek, Morin, Ueckerdt, Wood '19
Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth $\leqslant 8$ and some path P

Ueckerdt, Wood, Yi '21
Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth $\leqslant 6$ and some path P

What is the smallest possible bound t on the treewidth of H?

$3 \leqslant t \leqslant 6$

Research direction 2: Product structure for other classes

Are there other graph classes of interest that satisfy some form of a product structure? E.g.

$$H \boxtimes \underbrace{P \boxtimes P \boxtimes \cdots \boxtimes P}_{d}$$

with H of bounded treewidth, P path, and d bounded

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with H of bounded treewidth, P path, and d bounded

Krauthgamer-Lee '07

Let $c \geqslant 1$. If $\forall r$ all radius-r balls in a graph G have size $O(r^c)$ then $G \subseteq \underbrace{P \boxtimes P \boxtimes \cdots \boxtimes P}_{d}$ with $d = O(c \log c)$

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Conjecture (Wood)

If G k-nearest neighbor graph in $\mathbb{R}^d$ then $G \subseteq H \boxtimes \underbrace{P \boxtimes P \boxtimes \cdots \boxtimes P}_{d-1}$ for some graph H of treewidth $f(k, d)$

True for $d = 2$ (Dujmović-Morin-Wood '19), open for $d \geqslant 3$

Research direction 3: New applications

Are there other open problems about planar graphs that can be solved using the product structure?

What about algorithmic applications?

For approximation algorithms, could the product structure lead to better approximation ratios than, say, Baker's technique?

Thank you!