GRAPH STRUCTURE AND ALGORITHMS

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$\mathcal{H}$ HEREDITARY CLOSED UNDER TAKING INDUCED SUBGRAPHS

G CONTAINS F IF F IS ISOMORPHIC TO AN INDUCED SUBGRAPH OF G

G IS F-FREE IF IT DOES NOT CONTAIN F

G IS $\mathcal{F}$-FREE IF IT IS F-FREE $\forall F \in \mathcal{F}$

A HOLE IS A CHORDLESS CYCLE OF LENGTH ≥ 4

EVEN

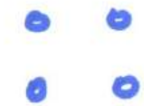
ODD

SOME CLASSICAL COMBINATORIAL OPTIMIZATION PROBLEMS WE WILL FOCUS ON

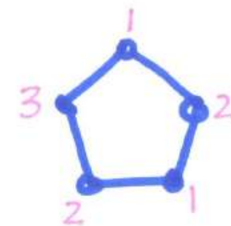
G GRAPH

$\omega(G)$ SIZE OF A LARGEST CLIQUE IN G

$\alpha(G)$ SIZE OF A LARGEST STABLESET IN G



$\chi(G)$ CHROMATIC NUMBER OF G



NP-HARD TO SOLVE IN GENERAL

THEY BECOME P WHEN SOME STRUCTURE IS IMPOSED ON INPUT GRAPH

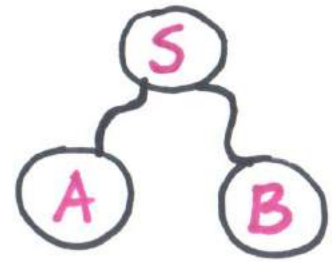
BUT ALSO REMAIN DIFFICULT EVEN WHEN SEEMINGLY QUITE A LOT OF STRUCTURE IS IMPOSED ON INPUT GRAPH

e.g. DETERMENING WHETHER A GRAPH IS 3-COLORABLE IS NP-COMPLETE FOR TRIANGLE-FREE GRAPHS WITH MAX DEGREE 4

GOAL: TO UNDERSTAND STRUCTURAL REASONS THAT ENABLE EFFICIENT ALGORITHMS

DECOMPOSITION METHOD

$S \subseteq V(G) \cup E(G)$ IS A **CUTSET**
IF $G \setminus S$ IS DISCONNECTED



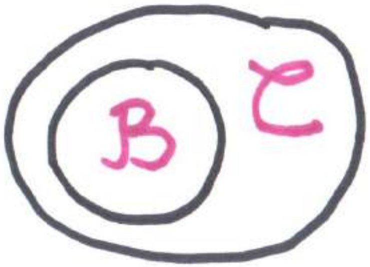
$\mathcal{C}$ CLASS OF GRAPHS

DECOMPOSITION THEOREM

$G \in \mathcal{C} \Rightarrow$ G IS **BASIC**

OR

G HAS **CUTSET** $S \in \mathfrak{S}$



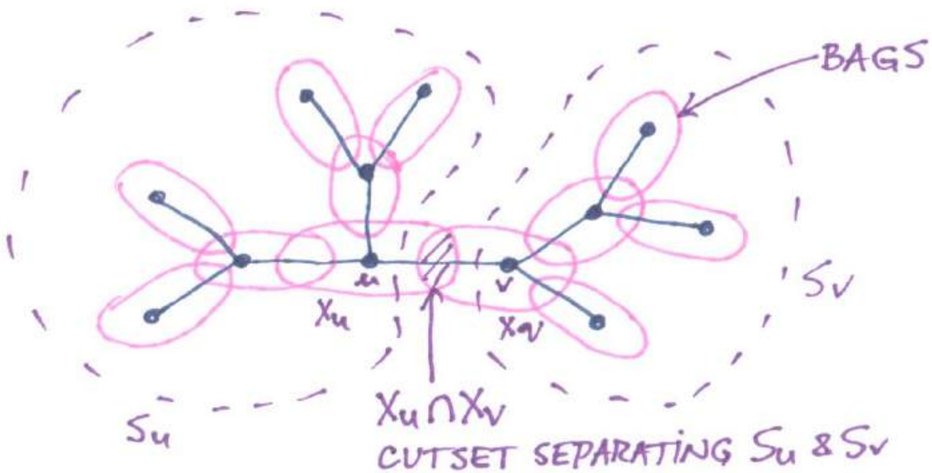
MINOR-CLOSED CLASSES

CLOSED UNDER : DELETING VERTICES
DELETING EDGES
CONTRACTING EDGES

ROBERTSON & SEYMOUR '80 → '04
GRAPH MINOR PROJECT

TREE-DECOMPOSITION (T, X)

T: TREE; X: $\forall v \in V(T)$, X_v



WIDTH $= \max_{v \in V(T)} |X_v| - 1$

$tw(G)$ = MINIMUM WIDTH OVER ALL TREE-DECOMPOSITIONS OF G

TREE-DECOMPOSITION OF WIDTH k CORRESPONDS TO DECOMPOSING A GRAPH INTO PIECES OF SIZE $\leq k+1$ BY A SEQUENCE OF NON-CROSSING CUTSETS OF SIZE $\leq k$

MANY NP-HARD PROBLEMS (IN FACT ALL THAT CAN BE FORMULATED IN MSO LOGIC) CAN BE SOLVED IN LINEAR TIME ON GRAPHS OF BOUNDED TREE-WIDTH

TREE-WIDTH HAS BEEN GENERALIZED TO OTHER WIDTH PARAMETERS e.g. CLIQUE-WIDTH

RANK-WIDTH

BRANCH-WIDTH ...

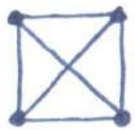
ALL SUITABLE FOR DYNAMIC PROGRAMMING APPROACH

CHORDAL GRAPHS

HOLE = CHORDLESS CYCLE OF LENGTH ≥ 4

G IS CHORDAL IF IT IS HOLE-FREE

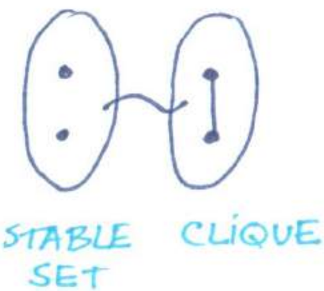
CLIQUE



TREE-WIDTH $(K_n) = n-1$

CLIQUE-WIDTH $(K_n) = 2$

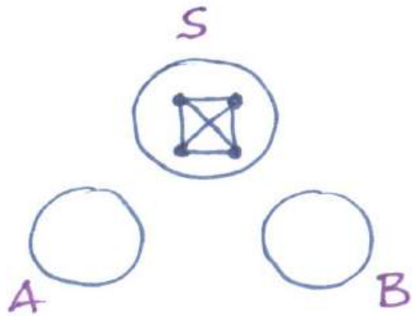
SPLIT GRAPHS



CLIQUE WIDTH UNBOUNDED

(DIRAC '61)

G CHORDAL $\Rightarrow$ G IS A CLIQUE OR
G HAS A CLIQUE CUTSET



$\rightarrow$ EFFICIENT (LINEAR TIME)
ALGORITHMS

(Berge '61)

G IS **PERFECT** IF $\chi(H) = \omega(H)$ $\forall$ INDUCED SUBGRAPH H OF G

- (Lovász '72) PERFECT GRAPH THEOREM

  G perfect $\Leftrightarrow$ $\bar{G}$ perfect

- (Chudnovsky, Robertson, Seymour, Thomas '02)

  STRONG PERFECT GRAPH THEOREM

  G perfect $\Leftrightarrow$ G (odd hole, odd antihole)-free

- (Chudnovsky, Cornuéjols, Liu, Seymour, Vušković '03)

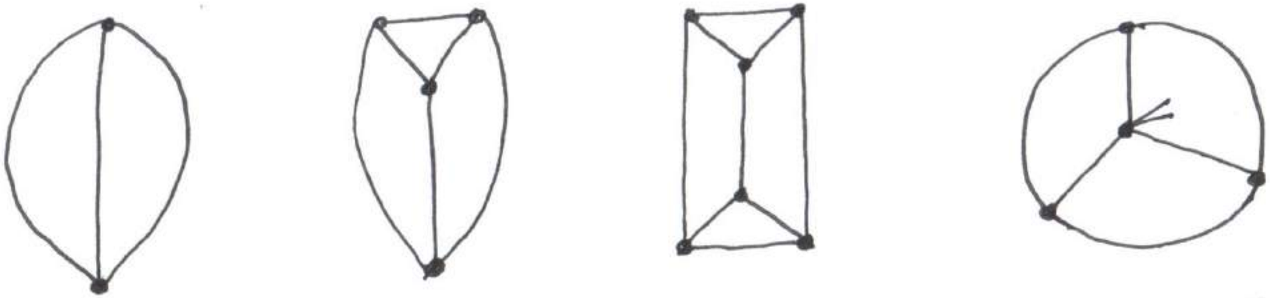
  perfect can be recognized in polynomial time

- (Grötschel, Lovász, Schrijver '81)

  χ, ω P for perfect graphs (ellipsoid method)

OPEN: CAN WE SOLVE χ, ω FOR PERFECT GRAPHS BY GRAPH ALGORITHMS IN POLYNOMIAL TIME

TRUMPER CONFIGURATIONS



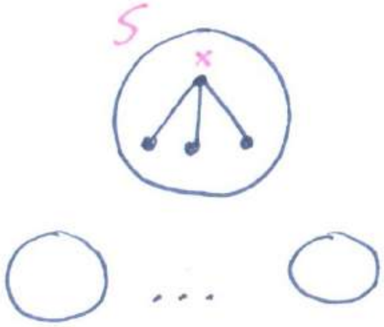
THETA PYRAMID PRISM WHEEL

UNIVERSALLY SIGNABLE GRAPHS ≡
(THETA, PYRAMID, PRISM, WHEEL) - FREE
(Conforti, Cornuéjols, Kapoor, Vušković '96)

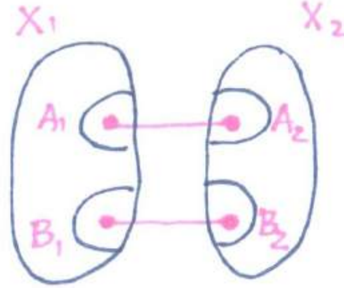
CLIQUE
OR CLIQUE CUTSET
HOLE

				
UNIVERSALLY SIGNABLE CCKV '96	X	X	X	X
CAP-FREE CCKV '96	✓	X	X	SOME
EVEN-HOLE-FREE CCKV '97 dSV '08	X	✓	X	SOME
ODD-HOLE-FREE CCV '01 BERGE CRST '02	✓	X	✓	SOME
CLAW-FREE CS '07	X	X	✓	SOME
BULL-FREE C '10	✓	X	ONLY $\overline{C_6}$	SOME
ISK4-FREE LMT '13	✓	X	✓	SOME
CHORDLESS GRAPHS MdFT '13	✓	X	X	X
ONLY-PYRAMID DRTV	X	✓	X	X
ONLY-PRISM DRTV	X	X	✓	X
(THETA, WHEEL)-FREE RTV	X	✓	✓	X

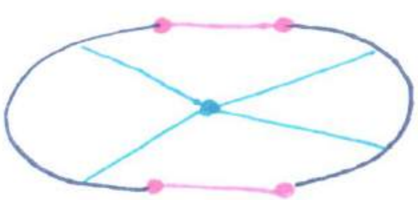
COMMONLY APPEARING CUTSETS IN DECOMPOSITIONS OF COMPLEX HEREDITARY CLASSES

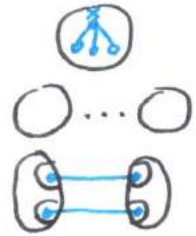
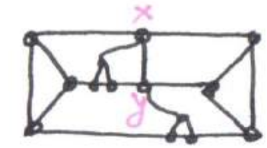
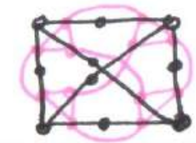
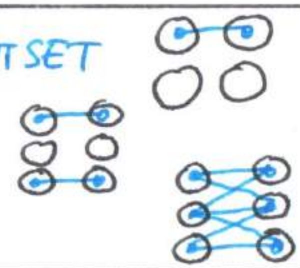
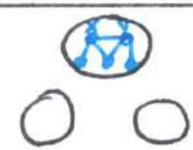


STAR CUTSET



2- JOIN



CLASSES	CUTSETS	BASIC CLASSES
EVEN-HOLE-FREE (Conforti, Cornuéjols, Kapoor, Vušković '97) (da Silva, Vušković '08)	STAR CUTSET 2-JOIN 	EXTENDED CLIQUE TREE 
C_4-FREE BERGE (Conforti, Cornuéjols, Vušković '01)	STAR CUTSET 2-JOIN	BIPARTITE GRAPH LINE GRAPH OF BIPARTITE 
BERGE (Chudnovsky, Robertson, Seymour, Thomas '02)	SKEW CUTSET 2-JOIN $\overline{\text{2-JOIN}}$ 	BIPARTITE LINE GRAPH OF BIPARTITE DOUBLE SPLIT + COMPLEMENTS
ODD-HOLE-FREE (Conforti, Cornuéjols, Vušković '01)	DOUBLE STAR CUTSET 2-JOIN 	Δ-FREE LINE GRAPH OF Δ-FREE COMPLEMENT OF LINE GRAPH OF Δ-FREE

(RADOVANOVIĆ, TROTIGNON, VUŠKOVIĆ '17)

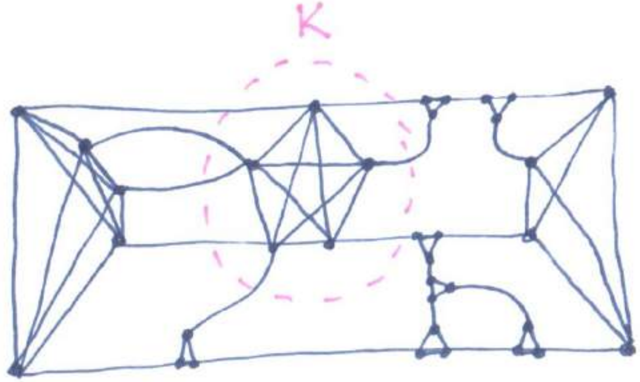
G (THETA, WHEEL)-FREE ⇒

FOR SOME CLIQUE K

G \ K IS

LINE GRAPH OF Δ-FREE

CHORDLESS GRAPH



CLIQUE CUTSET OR 2-JOIN

PROBLEM	PERFECT	OHF	EHF
RECOGNITION	P [2]	P [3]	P [1]
ω	P [4]	NPH [6]	P [5]
α	P [4]	?	?
χ	P [4]	NPH [7]	?

1 (Conforti, Cornuéjols, Kapoor, Vušković '97)

2 (Chudnovsky, Cornuéjols, Liu, Seymour, Vušković '03)

3 (Chudnovsky, Scott, Seymour, Spirkl '18)

4 (Grötschel, Lovász, Schrijver '81)

5 C_4-free $\Rightarrow$ polynomially many maximal cliques

6 ω NPH on (long hole $_{\geq 5}$, $K_{2,3}$, $\overline{C_6}$)-free

$\cap|$

long-hole-free

$\cap|$

odd-hole-free

TRUEMPER CONFIGURATIONS

7 χ NPH on ( , C_5)-free

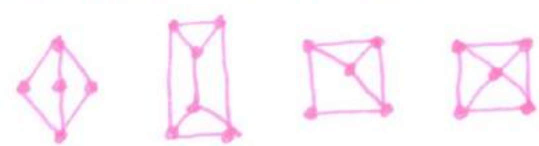
$\cap|$

long-hole-free

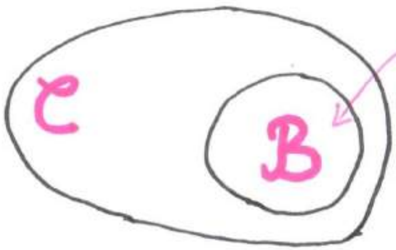
$\cap|$

odd-hole-free

TRUEMPER CONFIGURATIONS



DECOMPOSITION BASED ALGORITHMS



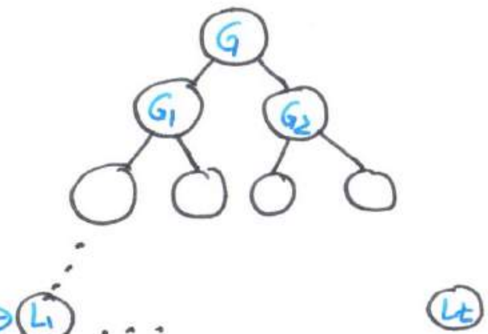
BASIC
efficient
algorithms
known

(DT) Decomposition Theorem

$G \in \mathcal{C} \Rightarrow G \in \mathcal{B}$ or
G has CUTSET $S \in \mathcal{S}$

$G \in \mathcal{C}$ with cutset $S \in \mathcal{S}$ $\longrightarrow$ construct BLOCKS OF DECOMPOSITION $G_1, G_2 \in \mathcal{C}$

decompose G with cutsets from $\mathcal{S}$ $\longrightarrow$ DECOMPOSITION TREE

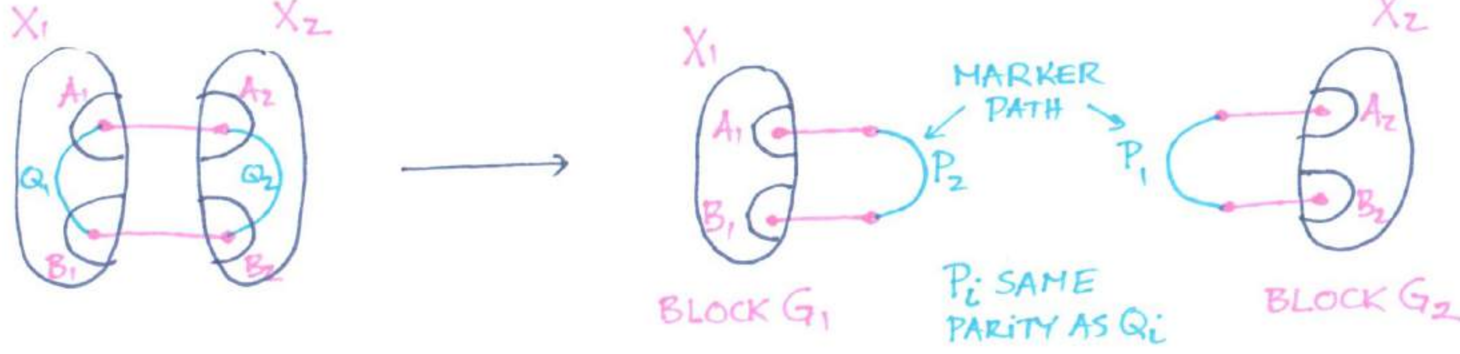


leaves have no cutset in $\mathcal{S}$ $\rightarrow$

$\downarrow$

by (DT): $G \in \mathcal{C} \Rightarrow L_1, .., L_t \in \mathcal{B}$

2-JOINS AND RECOGNITION

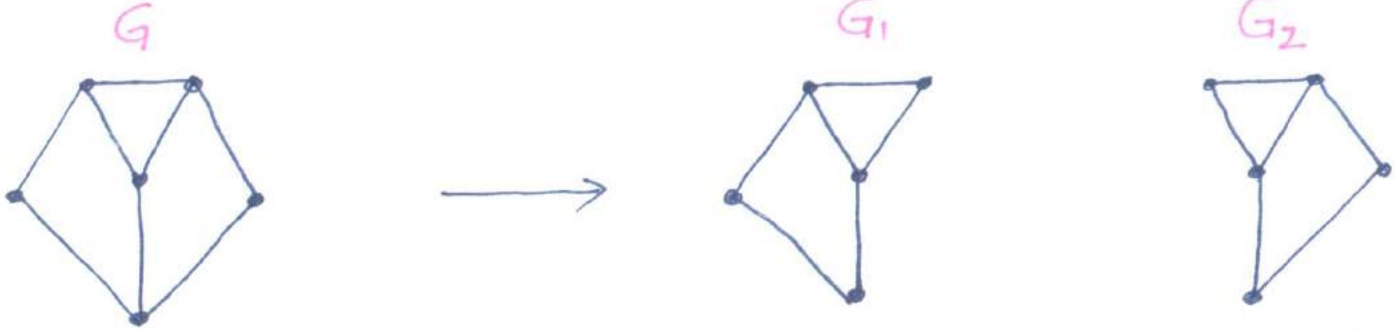


G E-H-F ⟺ G_1 AND G_2 E-H-F

O-H-F O-H-F

PERFECT PERFECT

STAR CUTSETS AND RECOGNITION



NOT O-H-F (PERFECT)

BLOCKS BOTH O-H-F (PERFECT)

SO THIS WAY OF DECOMPOSING IS NOT CLASS-PRESERVING

CAN ADD MORE TO BLOCKS TO MAKE THEM CLASS-PRESERVING,
BUT THEN THE SIZE OF THE DECOMPOSITION TREE BLOWS UP

CLEANING

First used by Conforti and Rao '93 in recognition of linear balanced matrices
Key to the following recognition algorithms:

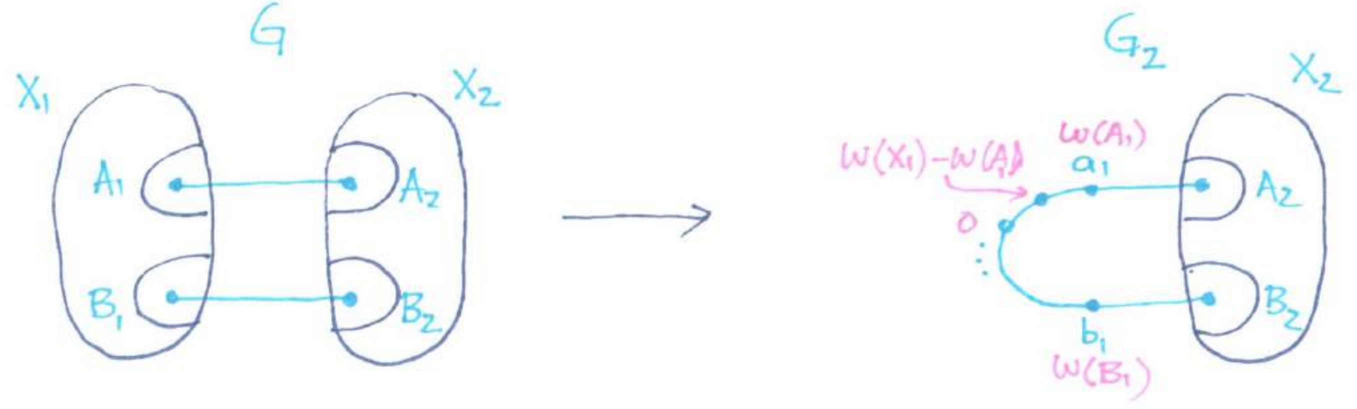
(Conforti, Cornuéjols, Rao '90) **BALANCED MATRICES**
(Conforti, Cornuéjols, Kapoor, Vušković '94) **BALANCEABLE MATRICES**
(Conforti, Cornuéjols, Kapoor, Vušković '97) **EHF GRAPHS**
(Chudnovsky, Cornuéjols, Liu, Seymour, Vušković '03) **PERFECT GRAPHS**
(Chudnovsky, Scott, Seymour, Spirkl '18) **OHF GRAPHS**

1. $G \longrightarrow$ CLEAN G'

   $G \in \mathcal{C} \Leftrightarrow G' \in \mathcal{C}$

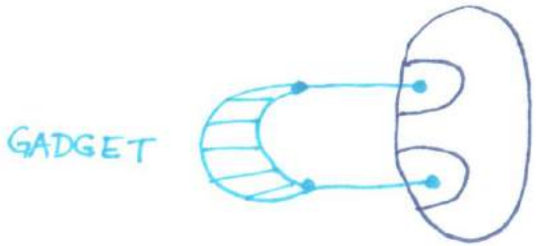
   DECOMPOSITIONS ARE CLASS-PRESERVING FOR G'
2. APPLY DECOMPOSITION BASED RECOGNITION ALGORITHM TO G'

ω & 2-JOINS

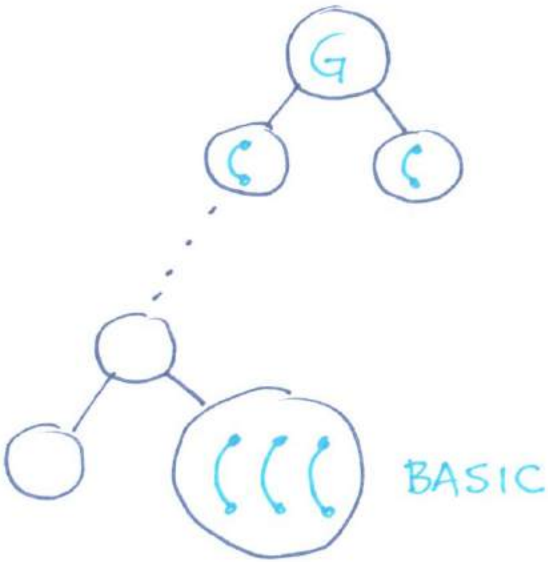


$$\omega(G) = \omega(G_2)$$

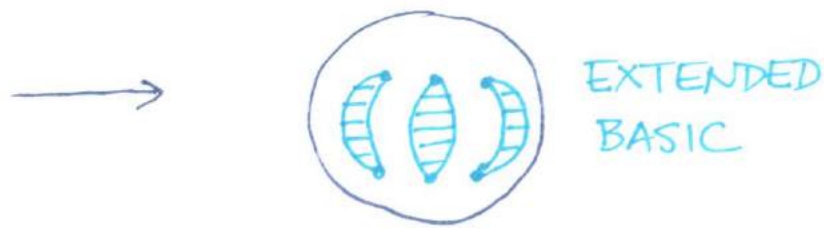
α & 2-JOINS



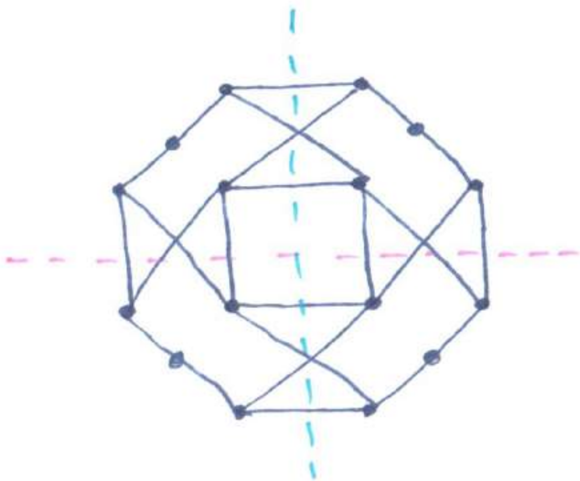
NOT CLASS-PRESERVING

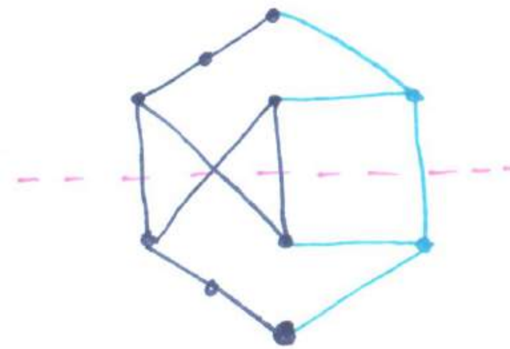


ESSENTIAL TO DECOMPOSE WITH A SEQUENCE OF NON-CROSSING 2-JOINS



(TROTIGNON, VUŠKOVIĆ '09)

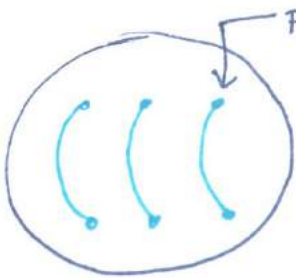




NO EXTREME 2-JOIN

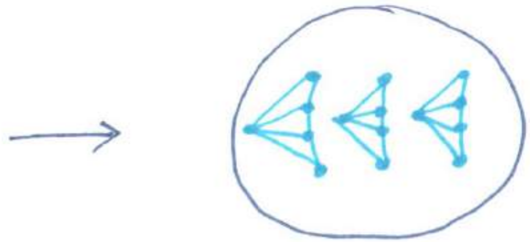
NO STAR CUTSET ⇒ EXTREME 2-JOIN

+ CAN DECOMPOSE BY SEQUENCE OF NON-CROSSING 2-JOINS

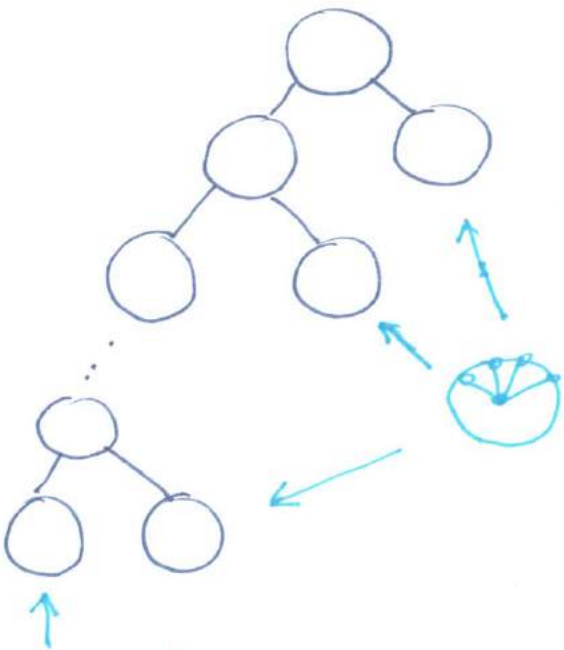


PATHS OF LENGTH ≥3 WHOSE INTERIOR VERTICES ARE OF DEGREE 2, AND WHOSE ENDS HAVE NO COMMON NEIGHBOR

2-CONNECTED BIPARTITE



$\mathcal{C}$



BIPARTITE

CAN DECOMPOSE BY SEQUENCE OF NON-CROSSING EXTREME 2-JOINS

(NAVES, TROTIGNON, VUŠKOVIĆ '09)

α NPC FOR $\mathcal{C}$

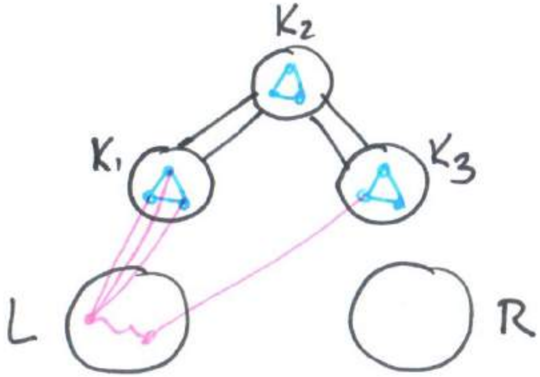
(Chudnovsky, Lo, Maffray, Trotignon, Vušković 2015)

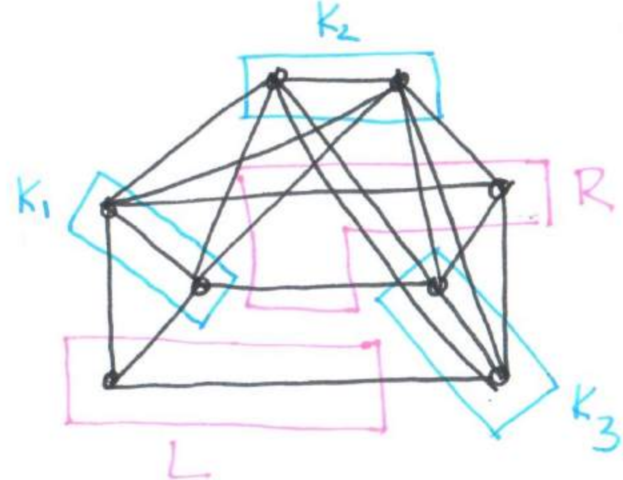
C_4-FREE BERGE

PURELY GRAPH THEORETIC COLORING ALGORITHM ($O(n^9)$ TIME)

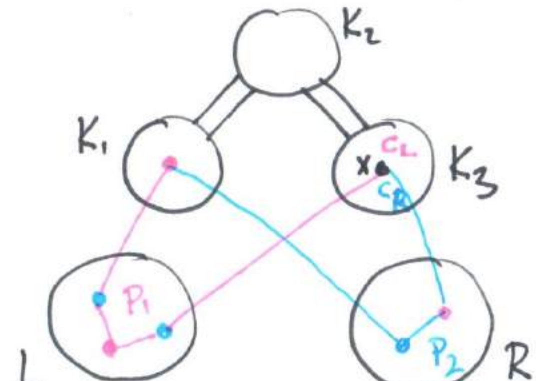
decompose prisms with

GOOD PARTITIONS

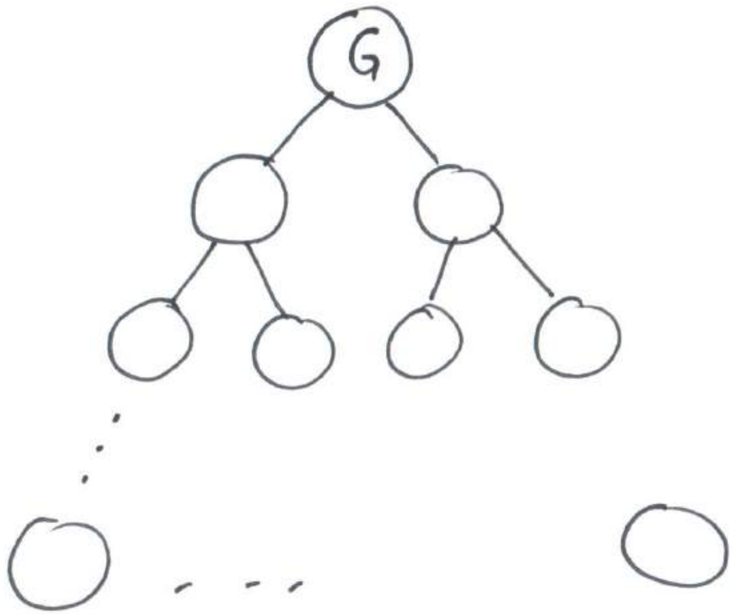




color blocks G_L, G_R
recolor them to make
the 2 colorings agree
on $K_1 \cup K_2 \cup K_3$



$P_1 + P_2$
ODD HOLE



decompose using good partitions

leaves in
$\mathcal{A}$ = (odd hole, antihole of length ≥ 6,
prism) - free

(Maffray, Trotignon 2005)
can color graphs in $\mathcal{A}$ by even pair contractions in $O(n^6)$ time

α & C_4-FREE BERGE GRAPHS

work in progress, joint with

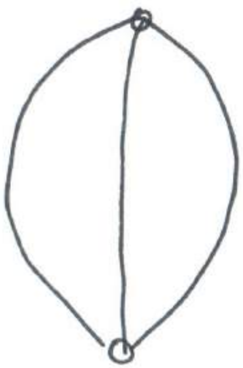
TARA ABRISHAMI
BOGDAN ALECU
MARIA CHUDNOVSKY
CEMIL DIBEK

C_4-FREE

EVEN-SIGNABLE = (PYRAMID, ODD WHEEL)-FREE

X

X WHEELS WITH ODD # OF Δs



THETA

PYRAMID

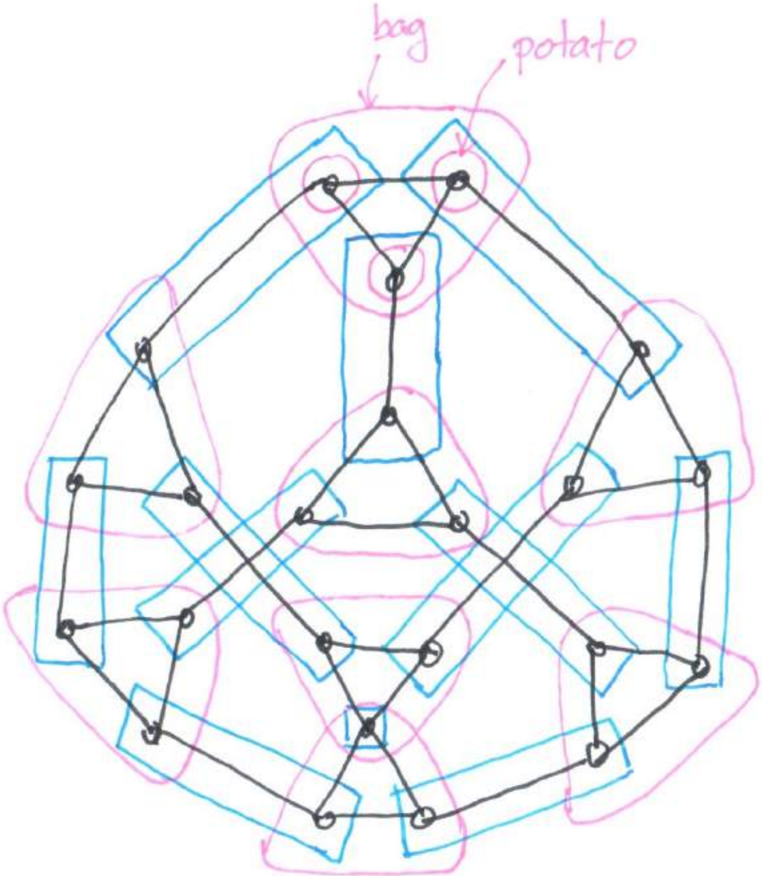
PRISM

WHEEL

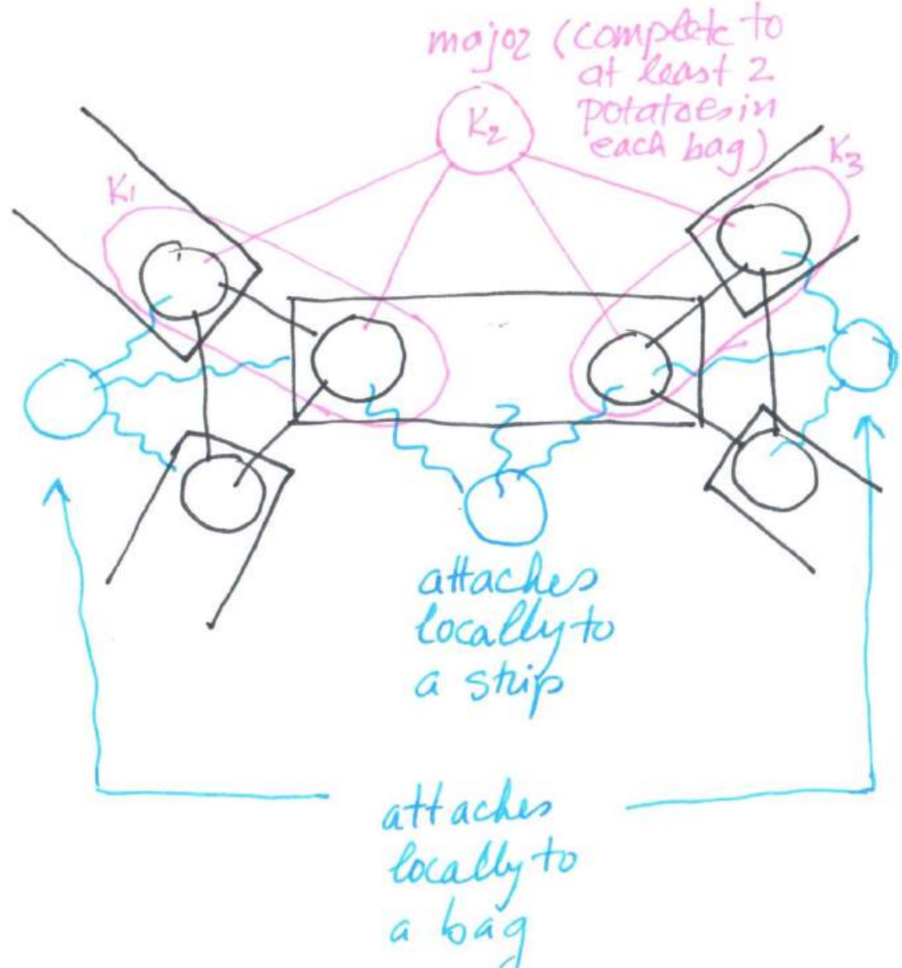
Δ-FREE

LINE GRAPH OF Δ-FREE

maximal strip system



line graph of bipartite graph blown into strip system

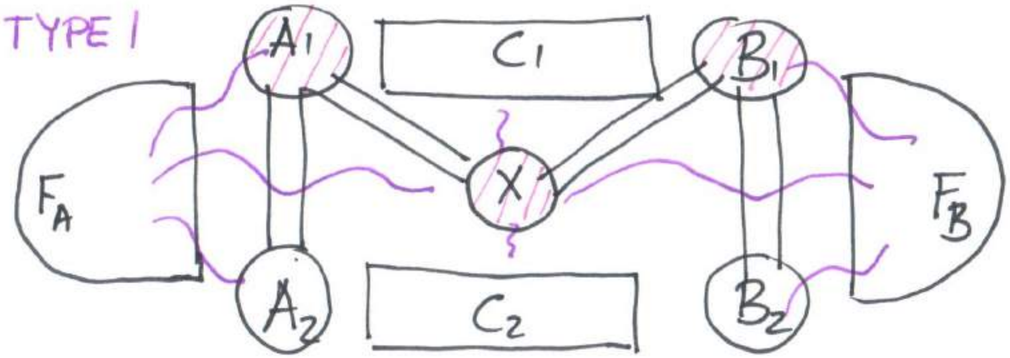


$K_1 \cup K_2$, $K_2 \cup K_3$ cliques

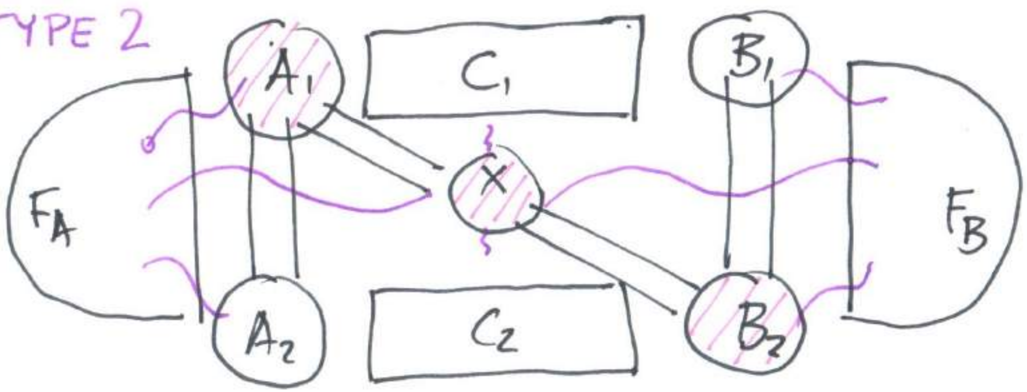
EXTENDED 2-JOIN PARTITIONS

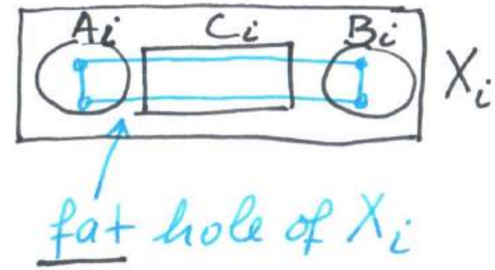

clique

TYPE 1



TYPE 2

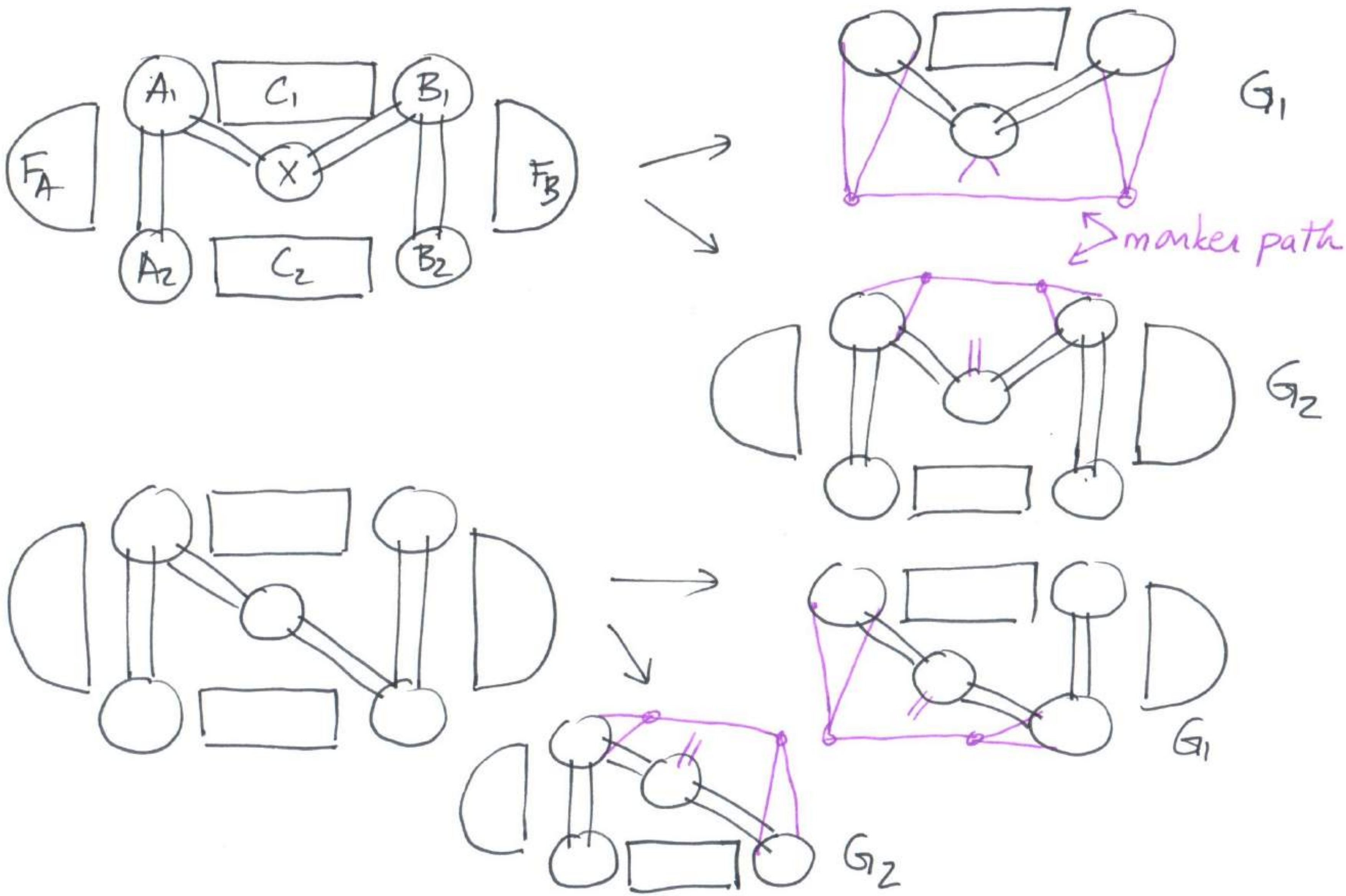




$\exists i \ \forall v \in A_i \cup B_i$
v is contained on a fat hole of X_i

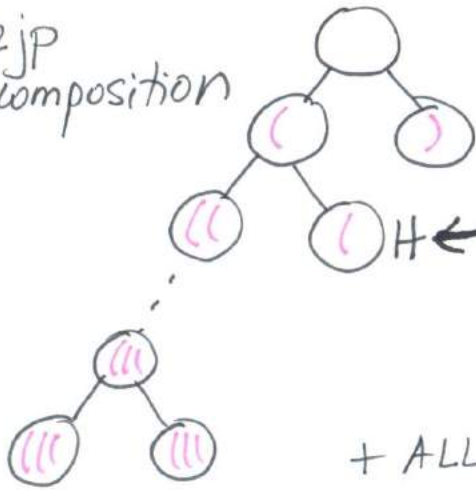
+ ALMOST TYPE 2

BLOCKS OF DECOMPOSITION



C_4-FREE BERGE NO CLIQUE CUTSET	$\Longrightarrow$	MINIMALLY-SIDED E2JP $\Rightarrow$ EXTREME E2JP

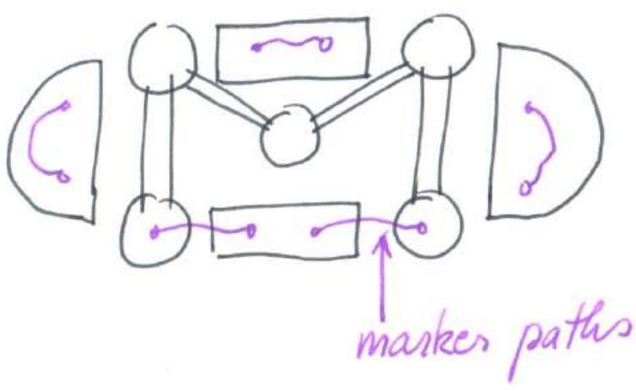
e2jp decomposition

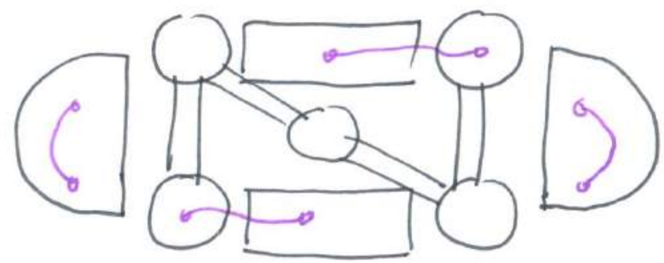


$\forall$ LEAF H
$\exists$ CLIQUE K s.t. $H \setminus K$ is

PRISM-FREE
OR
LINE GRAPH OF BIPARTITE MULTIGRAPH

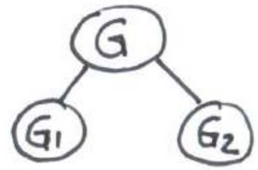
+ ALL MARKER PATH ARE VERTEX DISJOINT

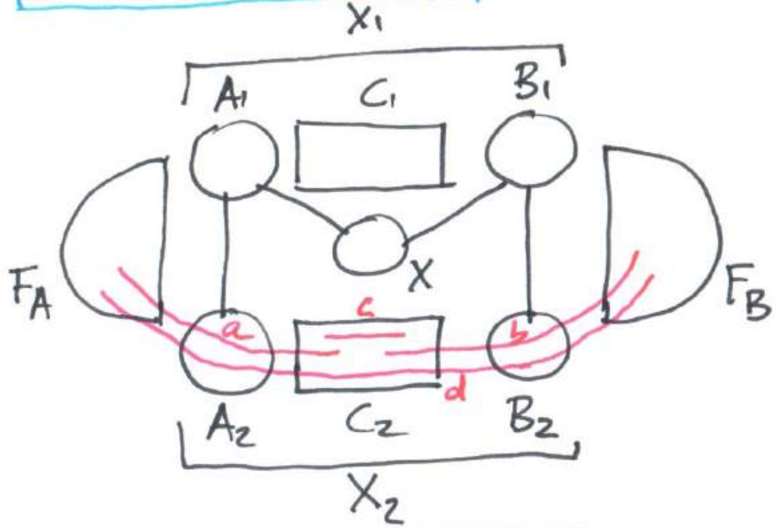




GADGETS

Type 1 even
G_2 basic





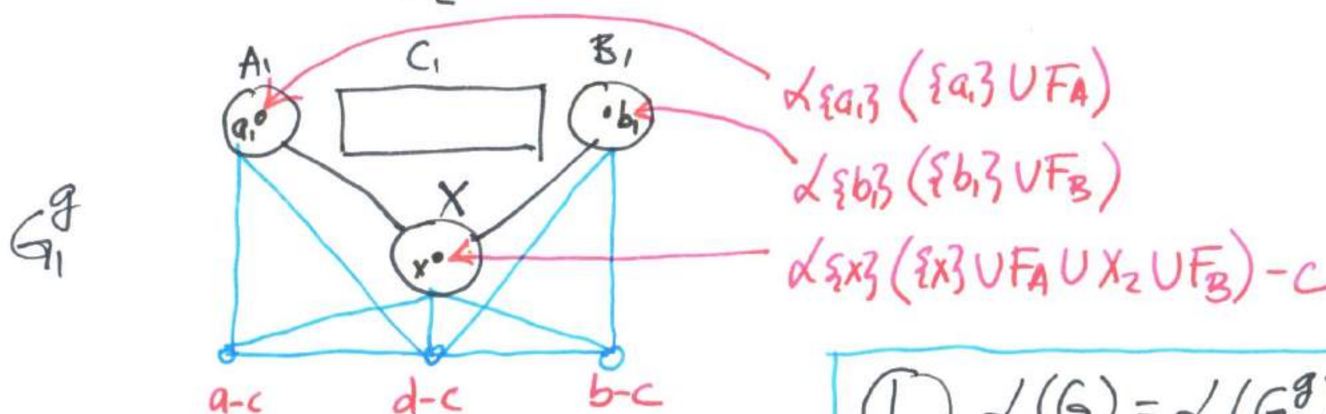
$a = \alpha(F_A \cup A_2 \cup C_2)$

$b = \alpha(F_B \cup B_2 \cup C_2)$

$c = \alpha(C_2)$

$d = \alpha(F_A \cup X_2 \cup F_B)$

$\boxed{\text{(L)}\ a + b \leq c + d}$

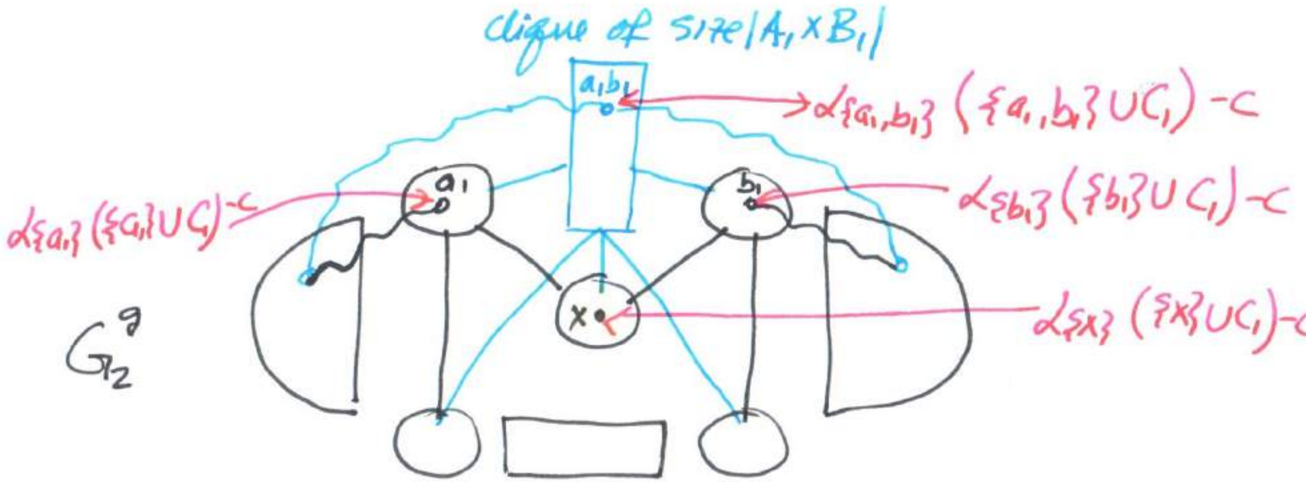


 weights change

$\boxed{\text{(L)}\ \alpha(G) = \alpha(G_1^g) + c}$

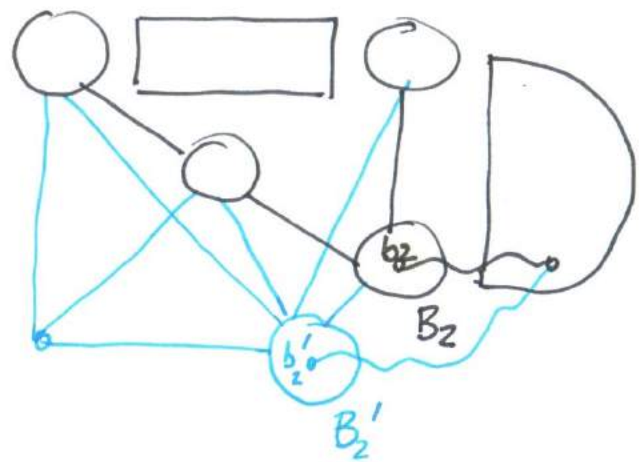
Type 1 even
G_1 basic

G_2^g



Type 2 even
G_2 basic

G_1^g



α for **PRISM-FREE** case inspired by the **CENTRAL BAG METHOD** developed in

(Abrishami, Chudnovsky, Vušković '20)

$\forall \delta \geq 0 \ \exists k$ s.t. G even-hole-free + max degree $\delta \Rightarrow tw(G) \leq k$

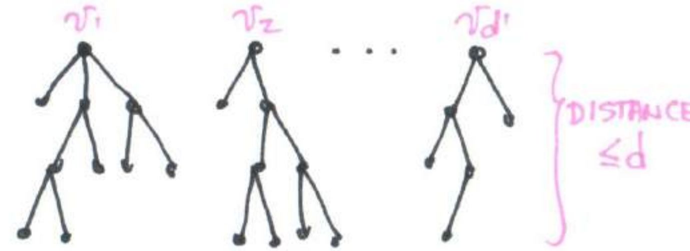
(Conjectured by: Aboulker, Adler, Kim, Sintiari, Trotignon)

(Korhonen '22)

$\forall \delta, t \ \exists k$ s.t. G (K_t, $K_{t,t}$, all subdivisions of $(t \times t)$-walls, line graphs of all subdivisions of $(t \times t)$-walls)-free + max degree δ $\Rightarrow tw(G) \leq k$

$S \subseteq V(G)$ IS **d-BOUNDED** IF $\exists\, v_1, \dots, v_{d'}$, $d' \leq d$ S.T.

$S \subseteq N^d[v_1] \cup \dots \cup N^d[v_{d'}]$



e.g. STAR 

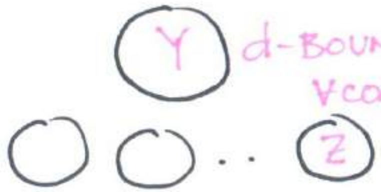
IS 1-BOUNDED

DOUBLE STAR 

IS 2-BOUNDED

GRAPH, WEIGHT FUNCTION $w: V(G) \to [0,1]$ S.T. $w(G) = 1$

$Y \subseteq V(G)$ is **(w,c,d)-BALANCED SEPARATOR** OF G IF



d-BOUNDED

$\forall$ CONNECTED COMPONENT Z OF $G \setminus Y$

$w(Z) \leq c$

(L1) G MAXIMUM DEGREE δ, $\delta \leq d$, $c \in [\frac{1}{2}, 1)$, $\Delta(d) = \underbrace{d + d\delta + \dots + d\delta^d}_{\text{MAXIMUM SIZE OF } d\text{-BOUNDED SET}}$

IF $\forall\, w: V(G) \to [0,1]$ WITH $w(G) = 1$ AND $w^{max} < \frac{1}{\Delta(d)}$

G HAS (w,c,d)-BALANCED SEPARATOR

THEN $\mathrm{Sep}_c^*(G) \leq \Delta(d)$ AND HENCE $tw(G) \leq \frac{1}{1-c} \Delta(d)$

$\mathcal{C}$ = C_4-FREE ODD-SIGNABLE GRAPHS

$\mathcal{C}^{\leq \delta}$ = GRAPHS FROM $\mathcal{C}$ WITH MAXIMUM DEGREE δ

(T1) $G \in \mathcal{C}^{\leq \delta}$, $w : V(G) \to [0,1]$ S.T. $w(G) = 1$

$c \in [\frac{1}{2}, 1)$, d INTEGER S.T.

$d \geq 49\delta + 4(2(\delta+1)^2+1)\delta + 4(2(\delta+1)^2+1)$ &

$(1-c) + [w^{max} + (2(\delta+1)^2+2)\delta 2^{\delta}(1-c) + (2(\delta+1)^2+1)2^{\delta}(1-c)](\delta+\delta^2) < \frac{1}{2}$

$\Rightarrow$ G HAS (w,c,d)-BALANCED SEPARATOR

(L1) & (T1) $\Rightarrow$

(T2) $G \in \mathcal{C}^{\leq \delta} \Rightarrow \exists\, c \in [\frac{1}{2}, 1)$ & INTEGER d S.T.

$$tw(G) \leq \frac{1}{1-c}\underbrace{(d + d\delta + d\delta^2 + \dots + d\delta^d)}_{\Delta(d)}$$

IDEA OF PROOF OF T1

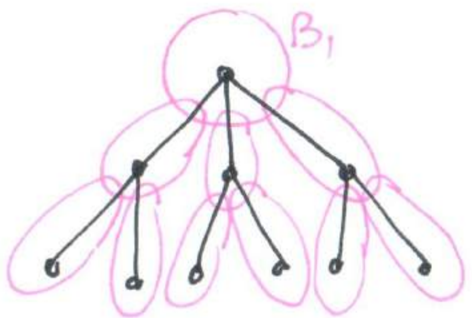
- COLLECTION X OF DECOMPOSITIONS THAT ARISE FROM DECOMPOSITION THEOREM IS VERY FAR FROM BEING NON-CROSSING

- BUT, DUE TO BOUND ON MAX DEGREE X CAN BE PARTITIONED INTO BOUNDED # OF $\overbrace{\text{NON-CROSSING}}^{\text{LAMINAR}}$ COLLECTIONS $X_1, \ldots, X_p$ (WHERE p DEPENDS ON δ)

- DECOMPOSITIONS IN X ARE FORCED BY PRESENCE OF CERTAIN INDUCED SUBGRAPHS CALLED "FORCERS"

- TO PROVE T1 WMA THAT FOR CERTAIN c & d G HAS NO (w, c, d)-BALANCED SEPARATOR

- DECOMPOSE G USING $X_1 \rightarrow (T_{X_1}, \mathcal{X}_{X_1})$ TREE DECOMPOSITION CORRESPONDING TO X_1



$\exists$ CENTRAL BAG β_1 S.T.

β_1 HAS NO (w_1, c, d_1)-BALANCED SEPARATOR
(FOR CERTAIN w_1, d_1 THAT DEPEND OF w, d)

$+$ β_1 HAS NO FORCERS RESPONSIBLE FOR X_1

- DECOMPOSE β_1 WITH $X_2 \rightarrow \beta_2$

$\vdots$

- WE REACH β_p S.T. β_p HAS NO (w_p, c, d_p)-BALANCED SEPARATOR

$$\underbrace{+ \ \beta_p \text{ HAS NO "FORCERS"}}$$

$\downarrow$

β_p HAS (w_p, c, d_p)-BALANCED SEPARATOR

$\times$

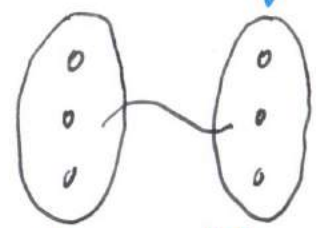
$\mathcal{C}$

α for (C4, PRISM)-FREE BERGE with MAX DEGREE δ

(Abrishami, Chudnovsky, Dibek, Vušković '21)

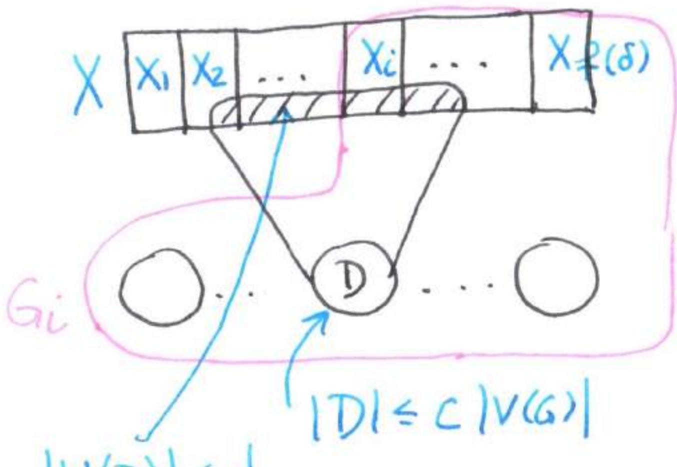
$X \subseteq V(G)$ **EVEN SET** if $\forall a,b \in X$ a,b is an **EVEN PAIR** (all induced a,b-paths are even)

even set



bipartite

ITERATED (c,d)-EVEN SET SEPARATOR



$\forall i$ X_i even set in G_i
+
$\forall$ component D of $G \setminus X$

Ⓣ $\exists c,d$ s.t. $G \in \mathcal{C} \Rightarrow$ G has iterated (c,d)-even set separator.

STAR CUTSET FORCERS

CENTRAL BAG

$\alpha \to$ MINIMIZING A SUBMODULAR FUNCTION