On Self-duality of Branchwidth in Graphs of Bounded Genus [⋆]

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Abstract

A graph parameter is *self-dual* in some class of graphs embeddable in some surface if its value does not change in the dual graph more than a constant factor. Self-duality has been examined for several width-parameters, such as branchwidth, pathwidth, and treewidth. In this paper, we give a direct proof of the self-duality of branchwidth in graphs embedded in some surface. In this direction, we prove that $\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2g - 4$ for any graph G embedded in a surface of Euler genus g.

Key words: graphs on surfaces, branchwidth, duality, polyhedral embedding.

1 Preliminaries

Our main reference for graphs on surfaces is the monograph by Mohar and Thomassen [10]. A *surface* is a connected compact 2-manifold without boundaries. A surface Σ can be obtained, up to homeomorphism, by adding $\mathbf{eg}(\Sigma)$ *crosscaps* to the sphere, and $\mathbf{eg}(\Sigma)$ is called the *Euler genus* of Σ. We denote by (G, Σ) a graph G embedded in a surface Σ, that is, drawn in Σ without edge crossings. A subset of Σ meeting the drawing only at vertices of G is called *G-normal*. An *O-arc* on Σ is a subset that is homeomorphic to a cycle. If an O-arc is G-normal, then we call it a *noose*. A noose N is *contractible* if it is the boundary of some disk on Σ and is *surface separating* if $\Sigma \setminus N$ is

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disconnected. The *length* of a noose is the number of the vertices it meets. *Representativity*, or *face-width*, is a parameter that quantifies local planarity and density of embeddings. The representativity $\mathbf{rep}(G, \Sigma)$ of a graph embedding (G, Σ) is the smallest length of a non-contractible noose in Σ. We call an embedding (G, Σ) *polyhedral* if G is 3-connected and $\mathbf{rep}(G, \Sigma) \geq 3$.

For a given embedding (G, Σ), we denote by (G^*, Σ) its dual embedding. Thus G^* is the geometric dual of G. Each vertex v (resp. face r) in (G, Σ) corresponds to some face v^* (resp. vertex r^*) in (G^*, Σ). Also, given a set $X \subseteq E(G)$, we denote as X^* the set of the duals of the edges in X.

Let $\mathcal{G}$ be a class of graphs embeddable in a surface Σ. We say that a graph parameter $\mathbf{p}$ is (c, d)-*self-dual* on $\mathcal{G}$ if for every graph $G \in \mathcal{G}$ and for its geometric dual G^*, $\mathbf{p}(G^*) \leq c \cdot \mathbf{p}(G) + d$. Self-duality of treewidth, pathwidth, or branchwidth (defined in Section 2) has played a fundamental role in the proof of the celebrated Graph Minors Theorem [13], as well as being useful for finding polynomial-time approximation algorithms for these parameters [2].

Most of the research concerning self-duality of graph parameters has been devoted to treewidth. Lapoire proved [7], using algebraic methods, that treewidth is $(1, 1)$-self-dual in planar graphs, settling a conjecture stated by Robertson and Seymour [11]. Bouchitté *et al.* [3] gave a much shorter proof of this result, exploiting the properties of minimal separators in planar graphs.

Fomin and Thilikos [5] proved that pathwidth is $(6, 6g-2)$-self-dual in graphs polyhedrically embedded in surfaces of Euler genus at most g. This result was improved for planar graphs by Amini *et al.* [1], who proved that pathwidth is $(3, 2)$-self-dual in 3-connected planar graphs and $(2, 1)$-self-dual in planar graphs with a Hamiltonian path.

Concerning branchwidth, Seymour and Thomas [14] proved that it is $(1, 0)$-self-dual in planar graphs that are not forests (for more direct proofs, see also [9] and [6]). In this note, we give a short proof that branchwidth is $(6, 2g-4)$-self-dual in graphs of Euler genus at most g. We also believe that our result can be considerably improved. In particular, we conjecture that branchwidth is $(1, g)$-self-dual.

2 Self-duality of branchwidth

Given a graph G and a set $X \subseteq E(G)$, we define $\partial X = (\bigcup_{e \in X} e) \cap (\bigcup_{e \in E(G) \setminus X} e)$, where edges are naturally taken as pairs of vertices (notice that $\partial X = \partial(E(G) \setminus X))$. A *branch decomposition* (T, μ) of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices are of degree three) and a bijection

$\mu : L \to E(G)$ from the set L of leaves of T to the edge set of G. For every edge $f = \{t_1, t_2\}$ of T we define the *middle set* $\mathbf{mid}(e) \subseteq V(G)$ as follows: Let L_1 be the leaves of the connected component of $T \setminus \{e\}$ that contain t_1. Then $\mathbf{mid}(e) = \partial\mu(L_1)$. The *width* of (T, μ) is defined as $\max\{|\mathbf{mid}(e)| \colon e \in T\}$. An optimal branch decomposition of G is defined by a tree T and a bijection μ which give the minimum width, called the *branchwidth* of G, and denoted by $\mathbf{bw}(G)$.

If (G, Σ) is a polyhedral embedding, then the following proposition follows by an easy modification of the proof of [5, Theorem 1].

Proposition 1 *Let (G, Σ) and (G^*, Σ) be dual polyhedral embeddings in a surface of Euler genus g. Then $\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2g - 4$.*

In the sequel, we focus on generalizing Proposition 1 to arbitrary embeddings. For this, we first need some technical lemmata, whose proofs are easy or well known, and omitted in this short note. Note that the removal of a vertex in G corresponds to the contraction of a face in G^*, and viceversa (the contraction of a face is the contraction of all the edges incident to it to a single vertex).

Lemma 1 *Branchwidth is closed under taking of minors, i.e., the branchwidth of a graph is no less than the branchwidth of any of its minors.*

Lemma 2 *The removal of a vertex or the contraction of a face from an embedded graph decreases its branchwidth by at most 1.*

Lemma 3 (Fomin and Thilikos [4]) *Let G_1 and G_2 be graphs with one edge or one vertex in common. Then $\mathbf{bw}(G_1 \cup G_2) \leq \max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2), 2\}$.*

We need a technical definition before stating our main result. Suppose that G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G_i' $(i = 1, 2)$ be obtained from G_i by deleting some (possibly none) of the edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a *clique-sum* $G_1 \oplus G_2$ of G_1 and G_2 to be the graph obtained from the union of G_1' and G_2' by identifying w with $h(w)$ for all $w \in W_1$.

Theorem 1 *Let (G, Σ) be an embedding with $g = \mathbf{eg}(\Sigma)$. Then $\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2g - 4$.*

Proof. The proof uses the following procedure that applies a series of cutting operations to decompose G into polyhedral pieces plus a set of vertices whose size is linearly bounded by $\mathbf{eg}(\Sigma)$. The input is the graph G and its dual G^* embedded in Σ.

1. Set $\mathcal{B} = \{G\}$, and $\mathcal{B}^* = \{G^*\}$ (we call the members of $\mathcal{B}$ and $\mathcal{B}^*$ *blocks*).

2. If (G, Σ) has a minimal separator S with $|S| \leq 2$, let $C_1, \ldots, C_\rho$ be the connected components of $G[V(G) \setminus S]$ and, for $i = 1, \ldots, \rho$, let G_i be the graph obtained by $G[V(C_i) \cup S]$ by adding an edge with both endpoints in S in the case where $|S| = 2$ and such an edge does not already exist (we refer to this operation as *cutting* G along the separator S). Notice that a separator S of G with $|S| = 1$ corresponds to a separator S^* of G^* with $|S^*| = 1$, given by the vertex of G^* corresponding to the external face of G. Also, to a separator S of G with $|S| = 2$ we can associate a separator S^* of G^* with $|S^*| = 2$, given by the vertex of G^* corresponding to the external face of G and a vertex of G^* corresponding to a face of G containing both vertices in S. Let $G_i^*, i = 1, \ldots, \rho$ be the graphs obtained by cutting G^* along the corresponding separator S^*. We say that each G_i (resp G_i^*) is a *block* of G (resp. G^*) and notice that each G and G^* is the clique sum of its blocks. Therefore, from Lemma 3,

$$\mathbf{bw}(G^*) \leq \max\{2, \max\{\mathbf{bw}(G_i^*) \mid i = 1, \ldots, \rho\}\}. \tag{1}$$

Observe now that for each $i = 1, \ldots, \rho$, G_i and G_i^* are embedded in a surface Σ_i such that G_i is the dual of G_i^* and $\mathbf{eg}(\Sigma) = \sum_{i=1,\ldots,\rho} \mathbf{eg}(\Sigma_i)$. Notice also that

$$\mathbf{bw}(G_i) \leq \mathbf{bw}(G), i = 1, \ldots, \rho, \tag{2}$$

as the possible edge addition does not increase the branchwidth, since each block of G is a minor of G and Lemma 1 applies. We set $\mathcal{B} \leftarrow \mathcal{B} \setminus \{G\} \cup \{G_1, \ldots, G_\rho\}$ and $\mathcal{B}^* \leftarrow \mathcal{B}^* \setminus \{G^*\} \cup \{G_1^*, \ldots, G_\rho^*\}$.

3. If (G, Σ) has a non-contractible and non-surface-separating noose meeting a set $S \subseteq V(G)$ with $|S| \leq 2$, let $G' = G[V(G) \setminus S]$ and let F be the set of faces in G^* corresponding to the vertices in S. Observe that the obtained graph G' has an embedding to some surface Σ' of Euler genus *strictly* smaller than Σ that, in turn, has some dual G'^* in Σ'. Therefore $\mathbf{eg}(\Sigma') < \mathbf{eg}(\Sigma)$. Moreover, G'^* is the result of the contraction in G^* of the $|S|$ faces in F. From Lemma 2,

$$\mathbf{bw}(G^*) \leq \mathbf{bw}(G'^*) + |S|. \tag{3}$$

Set $\mathcal{B} \leftarrow \mathcal{B} \setminus \{G\} \cup \{G'\}$ and $\mathcal{B}^* \leftarrow \mathcal{B}^* \setminus \{G^*\} \cup \{G'^*\}$.

4. As long as this is possible, apply (recursively) Steps 2–4 for each block $G \in \mathcal{B}$ and its dual.

We now claim that before each recursive call of Steps 2 and 3, it holds that $\mathbf{bw}(G^*) \leq 6 \cdot \mathbf{bw}(G) + 2\mathbf{eg}(\Sigma) - 4$. The proof uses descending induction on the the distance from the root of the recursion tree of the above procedure.

Notice that all embeddings of graphs in the collections $\mathcal{B}$ and $\mathcal{B}^*$ constructed by the above algorithm are polyhedral (except from the trivial cases that they have size at most 3). Then the theorem follows directly from Proposition 1.

Suppose that G (resp. G^*) is the clique sum of its blocks $G_1, \ldots, G_\rho$ (resp. $G_1^*, \ldots, G_\rho^*$) embedded in the surfaces $\Sigma_1, \ldots, \Sigma_\rho$ (Step 2). By induction, we have that $\mathbf{bw}(G_i^*) \leq 6 \cdot \mathbf{bw}(G_i) + 2\mathbf{eg}(\Sigma_i) - 4, i = 1, \ldots, \rho$ and the claim follows from Relations (1) and (2) and the fact that $\mathbf{eg}(\Sigma) = \sum_{i=1,\ldots,\rho} \mathbf{eg}(\Sigma)$.

Suppose now (Step 3) that G (resp. G^*) occurs from some graph G' (resp. G'^*) embedded in a surface Σ' where $\mathbf{eg}(\Sigma') < \mathbf{eg}(\Sigma)$ after adding the vertices in S (resp. S^*). From the induction hypothesis, $\mathbf{bw}(G'^*) \leq 6 \cdot \mathbf{bw}(G') + 2\mathbf{eg}(\Sigma') - 4 \leq 6 \cdot \mathbf{bw}(G') + 2\mathbf{eg}(\Sigma) - 2 - 4$ and the claim follows directly from Relation (3) as $|S| \leq 2$ and $\mathbf{bw}(G') \leq \mathbf{bw}(G)$. ∎

3 Recent results and a conjecture

Recently, Mazoit [8] proved that treewidth is a $(1, g+1)$-self-dual parameter in graphs embeddable in surfaces of Euler genus g, using completely different techniques. Since the branchwidth and the treewidth of a graph G, with $|E(G)| \geq 3$, satisfy $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$ [12], this implies that $\mathbf{bw}(G^*) \leq \frac{3}{2}\mathbf{bw}(G) + g + 2$, improving the constants of Theorem 1. We believe that an even tighter self-duality relation holds for branchwidth and hope that the approach of this paper will be helpful to settle the following conjecture.

Conjecture 1 *If G is a graph embedded in some surface Σ, then $\mathbf{bw}(G^*) \leq \mathbf{bw}(G) + \mathbf{eg}(\Sigma)$.*

References

[1] O. Amini, F. Huc, and S. Pérennes. On the Pathwidth of Planar Graphs. *SIAM Journal on Discrete Mathematics*, 23(3):1311–1316, 2009.

[2] H. L. Bodlaender and F. V. Fomin. Approximation of path-width of outerplanar graphs. *Journal of Algorithms*, 43(2):190–200, 2002.

[3] V. Bouchitté, F. Mazoit, and I. Todinca. Chordal embeddings of planar graphs. *Discrete Mathematics*, 273(1-3):85–102, 2003.

[4] F. V. Fomin and D. M. Thilikos. Dominating Sets in Planar Graphs: Branch-Width and Exponential Speed-Up. *SIAM Journal on Computing*, 36(2):281–309, 2006.

[5] F. V. Fomin and D. M. Thilikos. On self duality of pathwidth in polyhedral graph embeddings. *Journal of Graph Theory*, 55(1):42–54, 2007.

[6] I. V. Hicks and N. B. McMurray, Jr. The branchwidth of graphs and their cycle matroids. *Journal of Combinatorial Theory, Series B*, 97(5):681–692, 2007.

[7] D. Lapoire. Treewidth and duality for planar hypergraphs. `http://www.labri.fr/perso/lapoire/papers/dual_planar_treewidth.ps`, 1996 (manuscript).

[8] F. Mazoit. Tree-width of graphs and surface duality. *Electronic Notes in Discrete Mathematics*, 32:93–97, 2009. Proceedings of *DIMAP workshop on Algorithmic Graph Theory (AGT)*, Warwick, U.K., March 2009.

[9] F. Mazoit and S. Thomassé. Branchwidth of graphic matroids. In *Surveys in combinatorics 2007*, volume 346 of *London Math. Soc. Lecture Note Ser.*, pages 275–286. Cambridge Univ. Press, Cambridge, 2007.

[10] B. Mohar and C. Thomassen. *Graphs on surfaces.* John Hopkins University Press, 2001.

[11] N. Robertson and P. Seymour. Graph Minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984.

[12] N. Robertson and P. Seymour. Graph Minors. X. Obstructions to Tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.

[13] N. Robertson and P. D. Seymour. Graph Minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004.

[14] P. Seymour and R. Thomas. Call routing and the ratcatcher. *Combinatorica*, 14(2):217–241, 1994.