Optimization in Graphs Under Degree Constraints.

Application to Telecommunication Networks

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Mascotte – MA4

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October 16, 2009

Outline of the talk

Traffic grooming

Degree-constrained subgraph problems

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- Motivation
- Overview of the results

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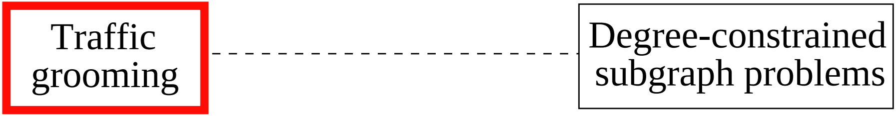

General idea

- WDM (Wavelength Division Multiplexing) networks
  - 1 wavelength (or frequency) = up to 40 Gb/s
  - 1 fiber = hundreds of wavelengths = Tb/s

- Traffic grooming consists in packing low-speed traffic flows into higher speed streams

  ⟶ we allocate the same wavelength to several low-speed requests (TDM, Time Division Multiplexing)

- Objectives:
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Definitions

- **Request** (i, j): two vertices (i, j) that want to exchange (low-speed) traffic

- **Grooming factor** C:

$$C = \frac{\text{Capacity of a wavelength}}{\text{Capacity used by a request}}$$

⋆ Typical values of the grooming factor:
SDH: $4, 16, 64, 256, \ldots$
SONET: $3, 12, 48, \ldots$

Example:
Capacity of one wavelength $= 2.5\ Gb/s$
Capacity used by a request $= 640\ Mb/s$ $\Rightarrow C = 4$

- **load** of an arc in a wavelength: number of requests using this arc in this wavelength ($\leq C$)

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ADM and OADM

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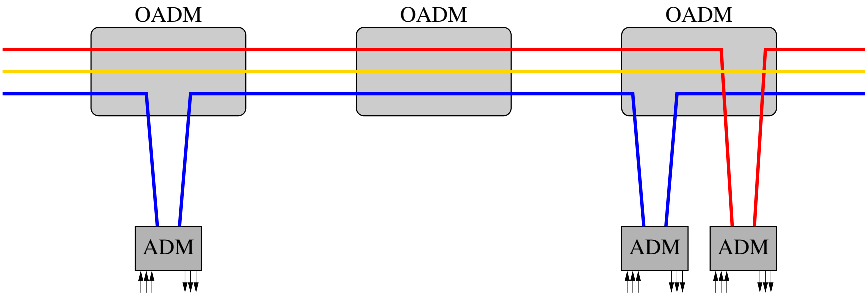
- **ADM** (Add/Drop Multiplexer)= insert/extract an OC/STM (electric low-speed signal) to/from a wavelength



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- We need to use an **ADM only at the endpoints of a request** (lightpaths) in order to save as many ADMs as possible

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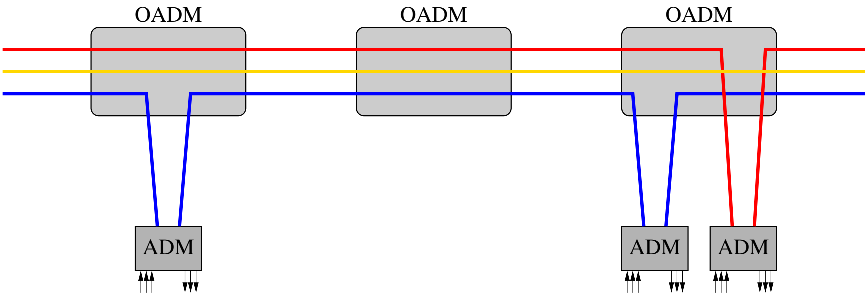

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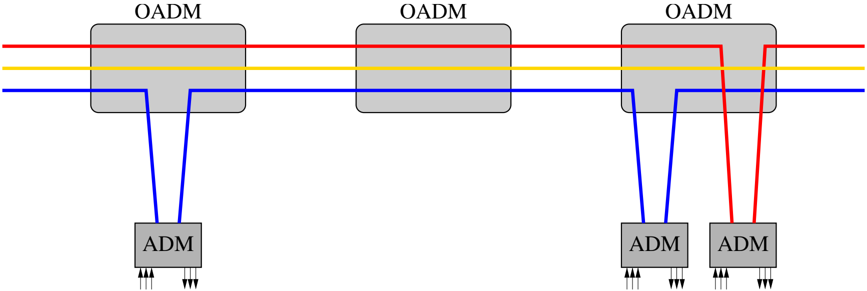

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To fix ideas...

- Model:

Topology	$\longrightarrow$	graph G
Request set	$\longrightarrow$	graph R
Grooming factor	$\longrightarrow$	integer C
Wavelength	$\longrightarrow$	**Subgraph** of R
Requests in a **wavelength**	$\longrightarrow$	edges in a **subgraph** of R
ADM in a wavelength	$\longrightarrow$	vertex in a subgraph of R

- A fundamental case is when $G = \overrightarrow{C}_n$ (**unidirectional ring**)
- It is also natural to consider **symmetric requests**

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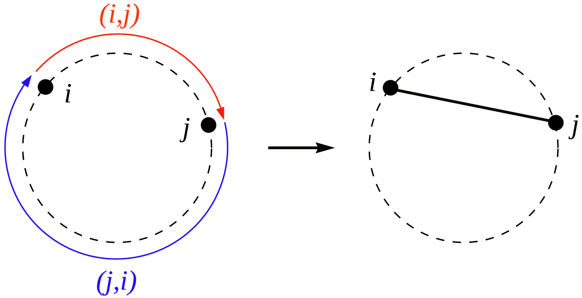
Unidirectional ring with symmetric requests

- **Symmetric requests**: whenever there is the request (i, j), there is also the request (j, i).



- W.l.o.g. requests (i, j) and (j, i) are in the same subgraph
$\rightarrow$ each pair of symmetric requests induces load 1
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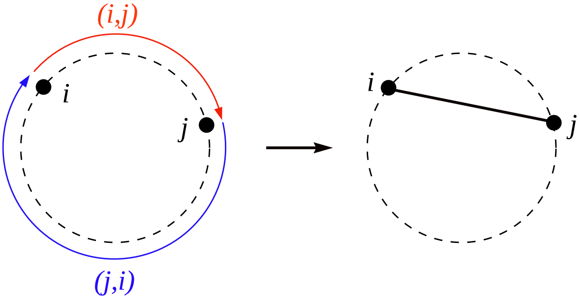

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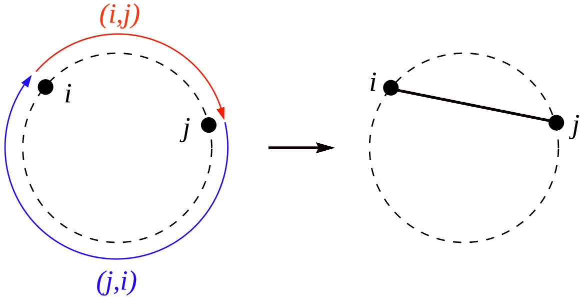

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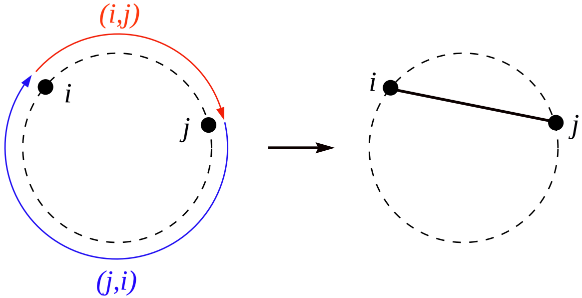

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Statement of the problem

Traffic Grooming in Unidirectional Rings (with symmetric requests)

Input An *undirected* graph R on n nodes (request set);
A grooming factor C.

Output A partition of $E(R)$ into subgraphs
$R_1, \ldots, R_W$ with $|E(R_i)| \leq C$, i=1,...,W.

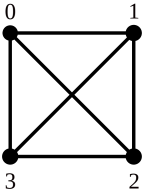
Objective Minimize $\sum_{i=1}^{W} |V(R_i)|$.

Example (unidirectional ring with symmetric requests)

$n = 4$

$R = K_4$

$C = 3$



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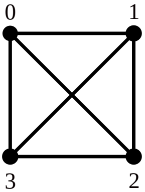

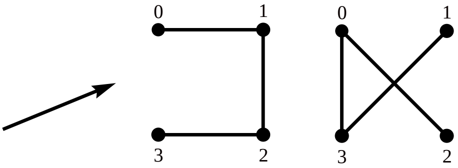

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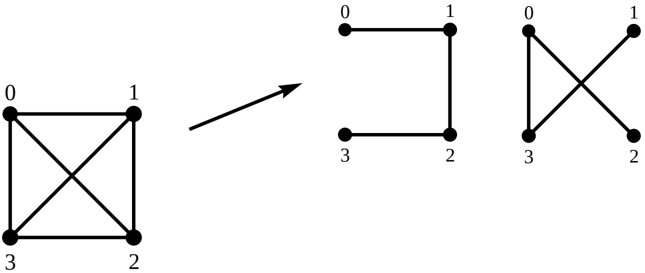

8 ADMs

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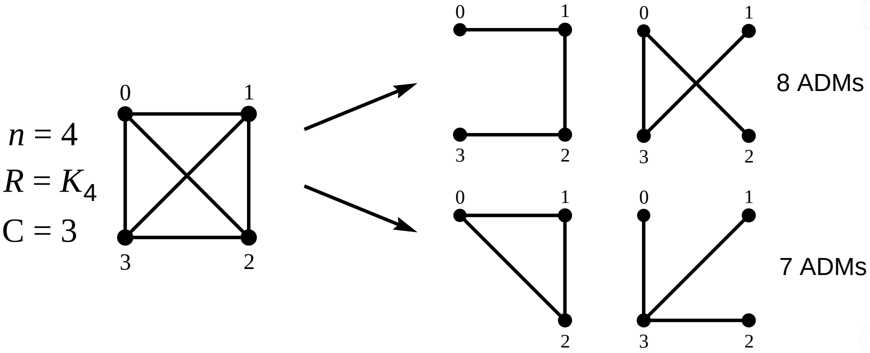

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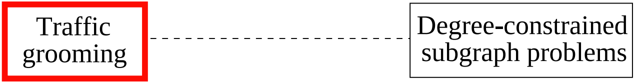

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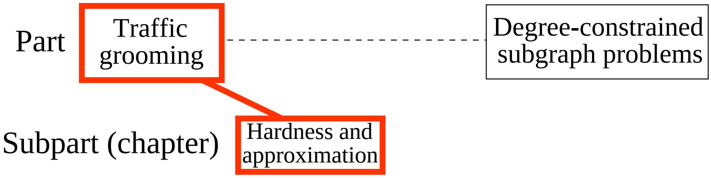

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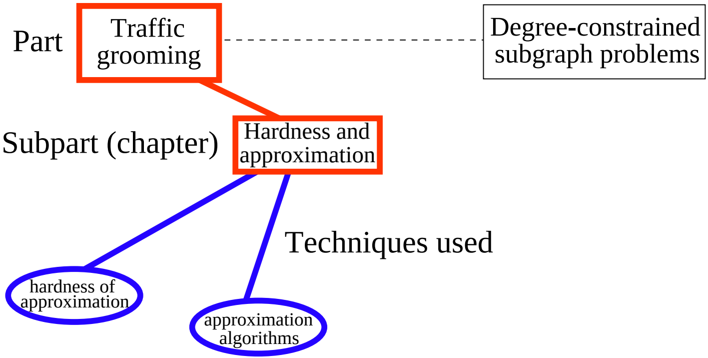

Preliminaries: approximation algorithms

- Given a (typically NP-hard) **minimization** problem Π, ALG is an **α-approximation algorithm** for Π (with $\alpha \geq 1$) if for any instance *I* of Π,

$$ALG(I) \leq \alpha \cdot OPT(I).$$

- **Class APX (Approximable):**

  an NP-hard optimization problem is in APX if it can be approximated within a constant factor.

  Example: MINIMUM VERTEX COVER has a 2-approximation.

- **Class PTAS (Polynomial-Time Approximation Scheme):**

  an NP-hard optimization problem is in PTAS if it can be approximated within a constant factor $1 + \varepsilon$, for all $\varepsilon > 0$
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Hardness of RING TRAFFIC GROOMING

1. **NP-complete** if C is part of the input
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   (the proof assumes a bounded number of wavelengths)
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★ **Open problem:** inapproximability for fixed C?

Conjecture: Not in PTAS for fixed C.

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1. $\sqrt{C}$-approximation is trivial (in poly-time in both n and C)
2. $\mathcal{O}(\log C)$-approximation algorithm, with running time $\mathcal{O}(n^C)$ [Flammini et al. *ISAAC'05, JDA'08*]
3. But in backbone networks, it is usually the case that $C \geq n$.

★ **Open problem:** approximation algorithm in poly-time in **both** C and n, and with approximation factor **independent of** C.

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There is a polynomial-time approximation algorithm that approximates RING TRAFFIC GROOMING *within a factor* $\mathcal{O}(n^{1/3} \log^2 n)$ *for any* $C \geq 1$.

Outline of the algorithm:

1. partition the requests into groups of similar length
2. in each group, extract "dense" subgraphs greedily using an algorithm for the DENSE k-SUBGRAPH problem

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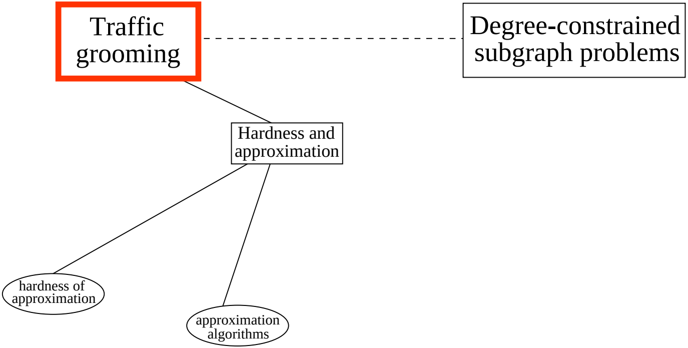

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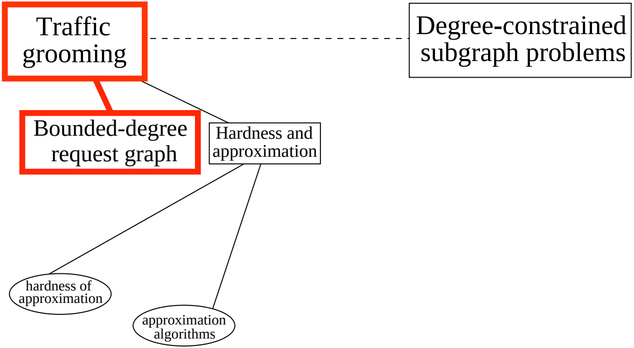

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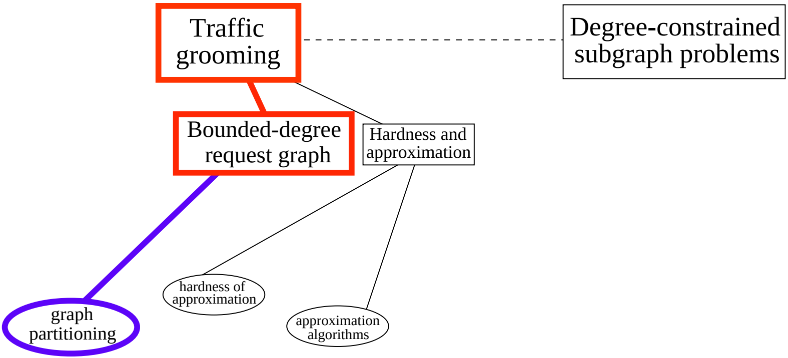

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- In the literature so far:
  place ADMs at nodes for a **fixed** **request graph.**
  → placement of ADMs **a posteriori.**

- **New model** [With Xavier Muñoz]:
  place the ADMs at nodes such that the network can support **any request graph with maximum degree at most** Δ**.**
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- As the network must support any degree-bounded graph, due to symmetry we place the **same number of ADMs at each node.**

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The parameter $M(C, \Delta)$

- Δ-graph: graph with maximum degree at most Δ.
- C-edge partition of G: partition of $E(G)$ into subgraphs with $\leq C$ edges.
- The problem is equivalent to determining the following parameter:
- Therefore, we focus on determining $M(C, \Delta)$.
- W.l.o.g. we can assume that R has **regular degree** Δ.

Proposition (Lower Bound – Muñoz and S.)

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Theorem (Li and S.)

Let $\Delta \geq 2$ be even. Then for any $C \geq 1$, $M(C, \Delta) = \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof.

- We have just seen the lower bound. Construction:
  - Orient the edges of $G = (V, E)$ in an Eulerian tour.
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Question: is the lower bound $\left\lceil \frac{C+1}{C}\frac{\Delta}{2} \right\rceil$ always attained?

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Open cases

Summarizing, we established the value of $M(C, \Delta)$ for “almost” all values of C and Δ, leaving **open** only the case where:

- $\Delta \geq 5$ is odd; and
- $C \geq 4$; and
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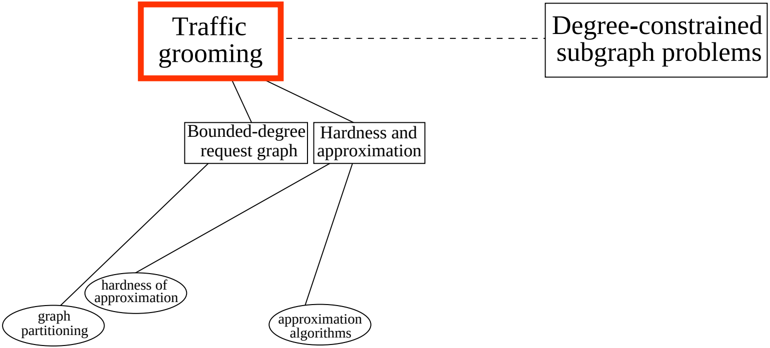

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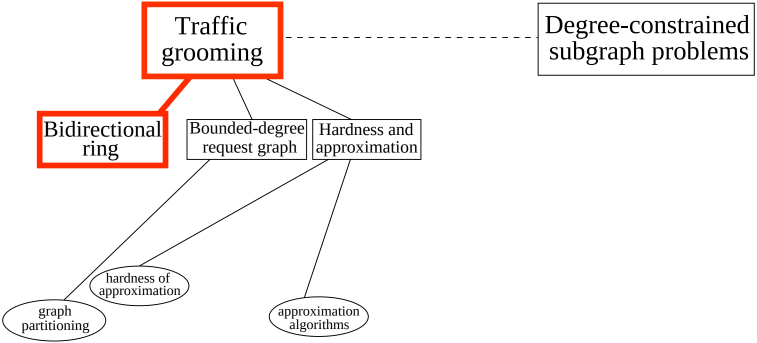

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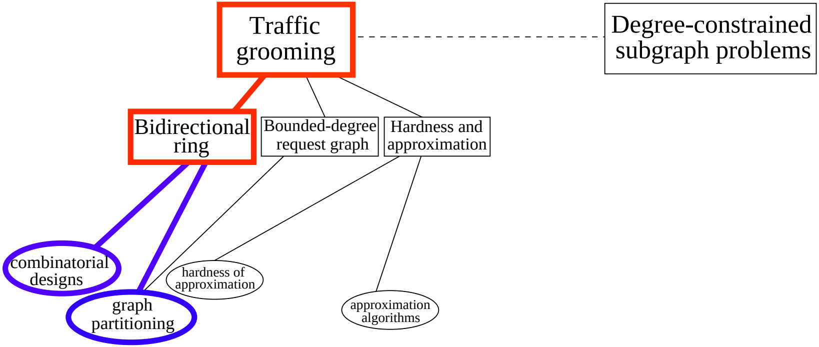

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With Jean-Claude Bermond and Xavier Muñoz

- Most of the research had been done for **unidirectional rings.**
- We consider the bidirectional ring with
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- We provide:
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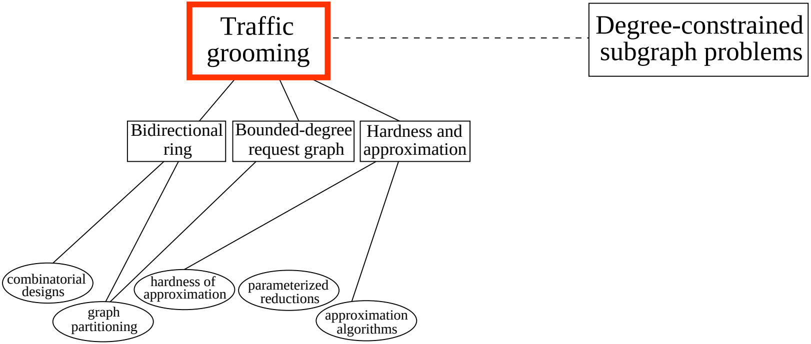

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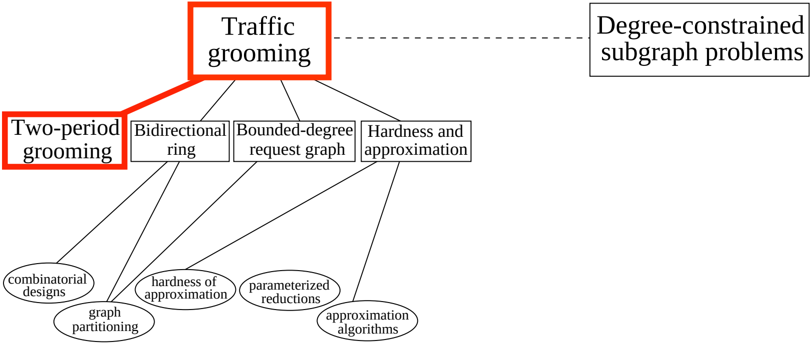

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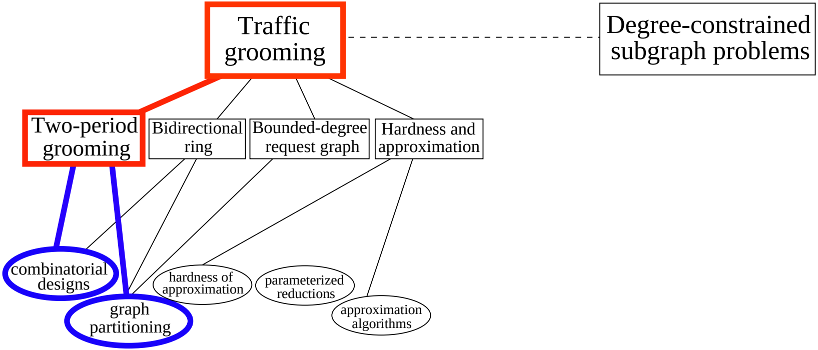

2-period traffic grooming in unidirectional rings

With *J-C. Bermond*, *C.J. Colbourn*, *L. Gionfriddo*, and *G. Quattrocchi*

- We consider a pseudo-dynamic scenario in unidirectional rings:
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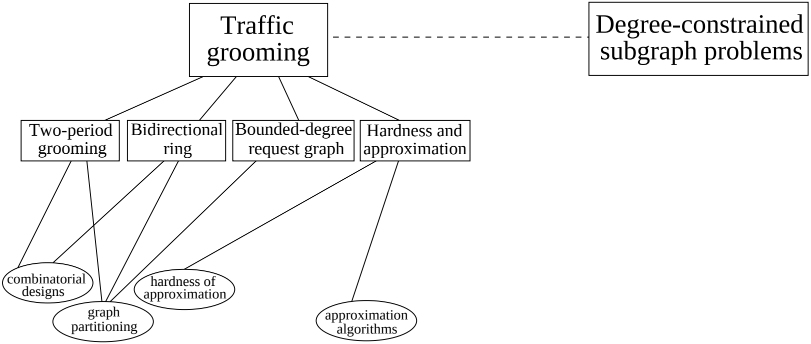

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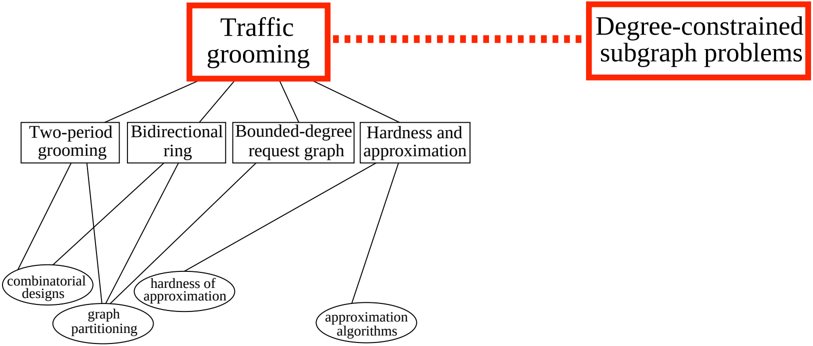

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Input: An undirected graph $G = (V, E)$ and a positive integer k.
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- Summarizing, a β-approximation for the DkS problems yields a $(\beta \cdot \log^2 n)$-approximation for RING TRAFFIC GROOMING.

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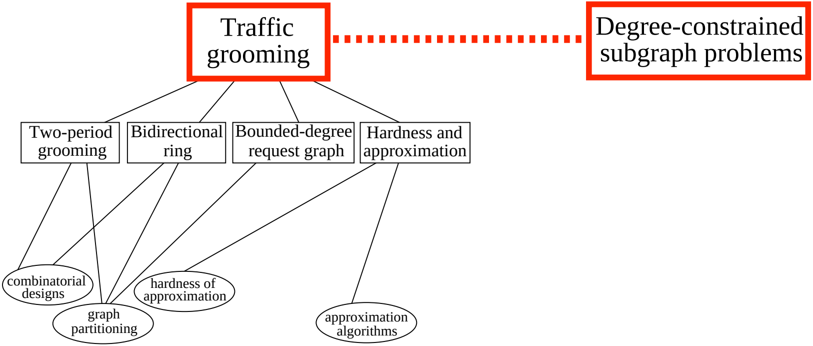

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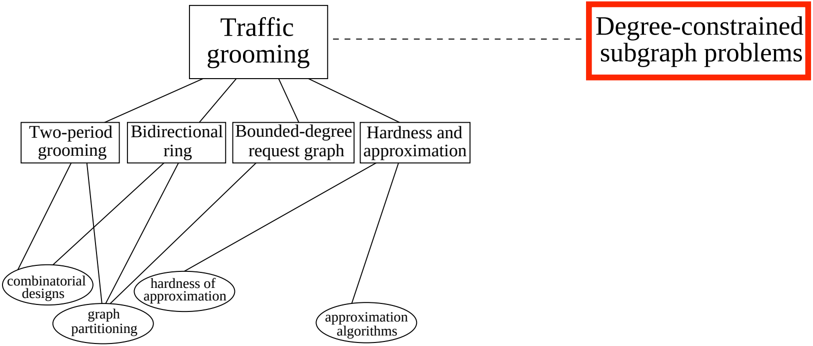

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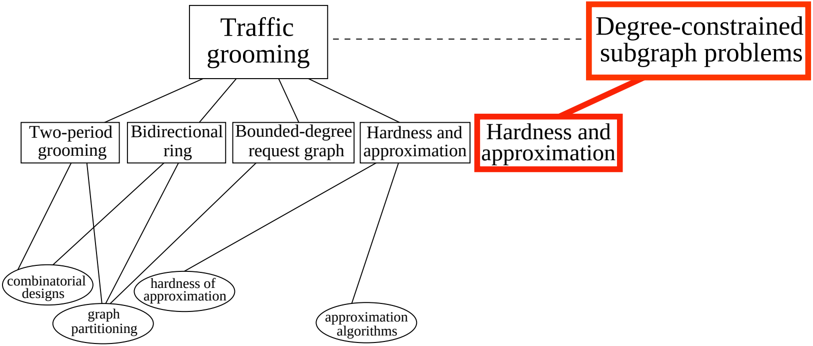

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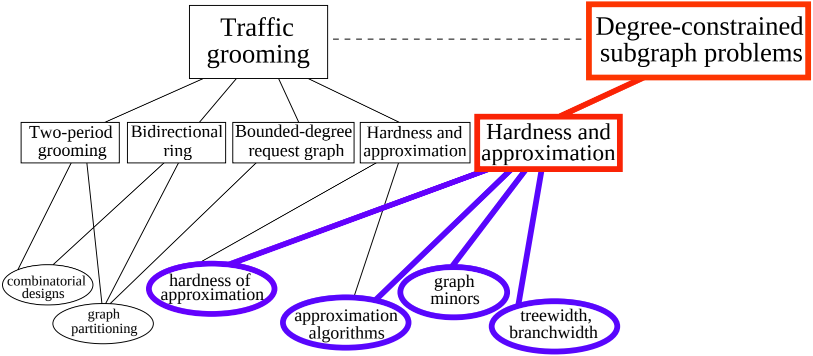

Broad family of problems

A *typical* **DEGREE-CONSTRAINED SUBGRAPH PROBLEM**:

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Output:

- a (*connected*) subgraph H of G,
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- Several problems in this broad family are classical widely studied NP-hard problems.
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Hardness and approximation

With *Omid Amini*, *David Peleg*, *Stéphane Pérennes* and *Saket Saurabh*

1. MSMD_d is not in APX for any $d \geq 3$, using the error amplification technique:
   - first we prove that MSMD_d is not in PTAS (unless P=NP).
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Input:

- an undirected graph $G = (V, E)$,
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- a weight function $\omega : E \to \mathbb{R}^+$.

Output:

a subset of edges $E' \subseteq E$ of **maximum weight**, s.t. $G' = (V, E')$

- is connected (except isolated vertices), and
- satisfies $\Delta(G') \leq d$.

- It is one of the classical **NP**-hard problems of *[Garey and Johnson, Computers and Intractability, 1979]*.
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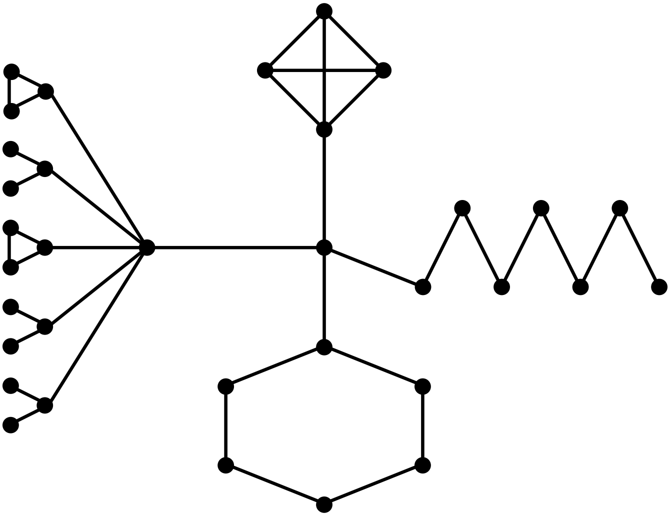
Example with $d = 3$, $\omega(e) = 1$ for all $e \in E(G)$



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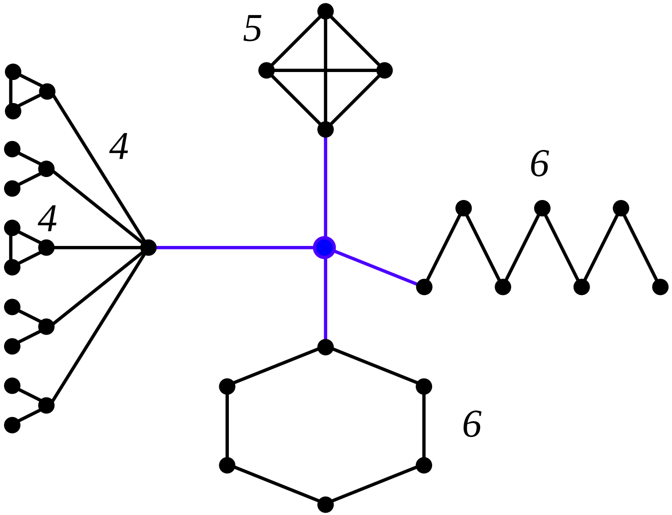

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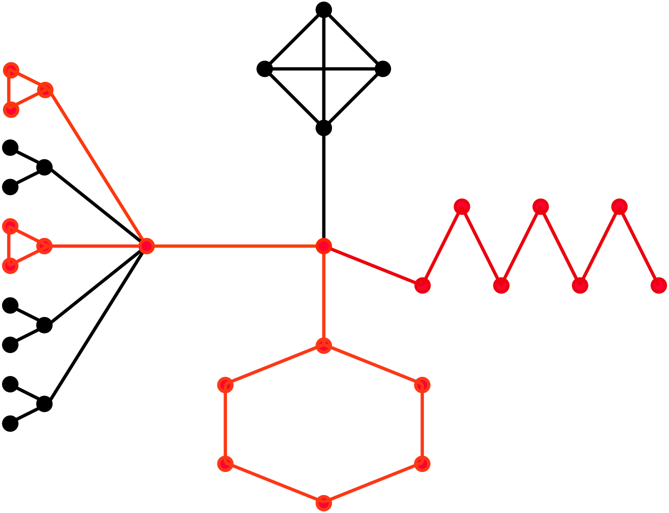

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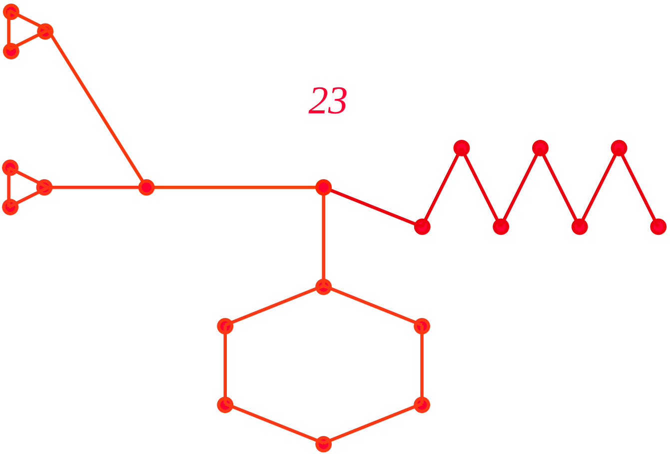

Hardness and approximation

With *Omid Amini*, *David Peleg*, *Stéphane Pérennes* and *Saket Saurabh*

1. not in APX for any fixed $d \geq 2$.
2. if there is a polynomial time algorithm for MDBCS_d, $d \geq 2$, with performance ratio $2^{\mathcal{O}(\sqrt{\log n})}$, then $\text{NP} \subseteq \text{DTIME}(2^{\mathcal{O}(\log^5 n)})$.
3. $\min\{m/\log n,\ nd/(2\log n)\}$-approximation algorithm for unweighted graphs. ($n = |V(G)|$ and $m = |E(G)|$)
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5. if G has a low-degree spanning tree (in terms of d) it can be approximated within a small constant factor.

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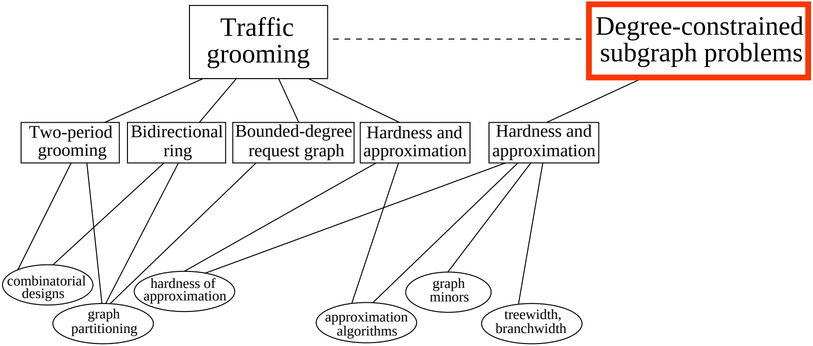

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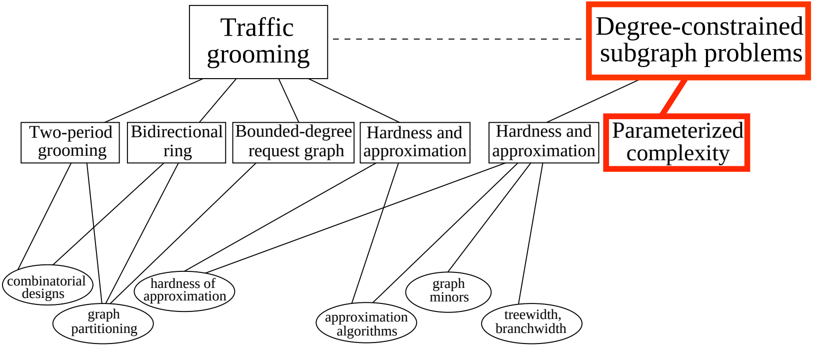

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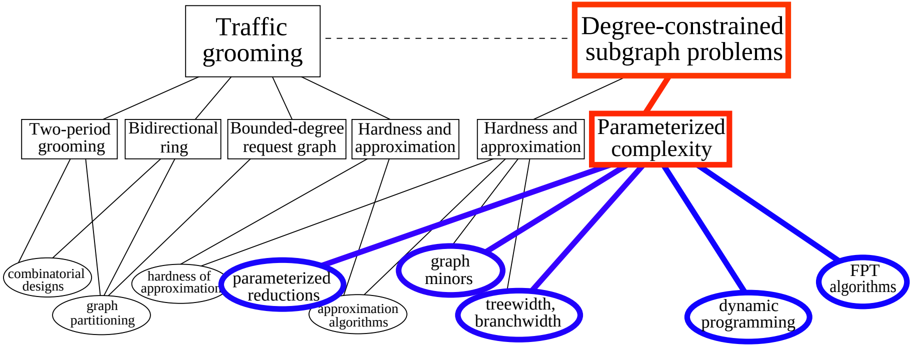

Some words on parameterized complexity

- Idea: given an NP-hard problem, fix one parameter of the input to see if the problem gets more "tractable".

  Example: the size of a VERTEX COVER.

- Given a (NP-hard) problem with input of size n and a parameter k, a fixed-parameter tractable (FPT) algorithm runs in

$$f(k) \cdot n^{\mathcal{O}(1)}, \quad \text{for some function } f.$$

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$$\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq W[3] \subseteq \cdots \subseteq XP$$

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Parameterized complexity of finding degree-constrained subgraphs

With *Omid Amini* and *Saket Saurabh*

- We have studied the parameterized complexity of finding degree-constrained subgraphs, with

  parameter = number of vertices of the desired subgraph

- Namely, given two integers d and k, the problems of finding
  1. a d-regular subgraph (induced or not) with at most $\leq k$ vertices.
  2. a subgraph with at most $\leq k$ vertices and of minimum degree $\geq d$.
- We prove that
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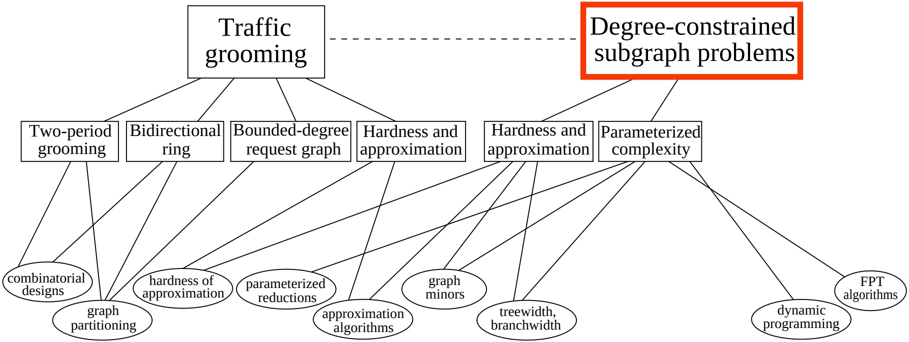

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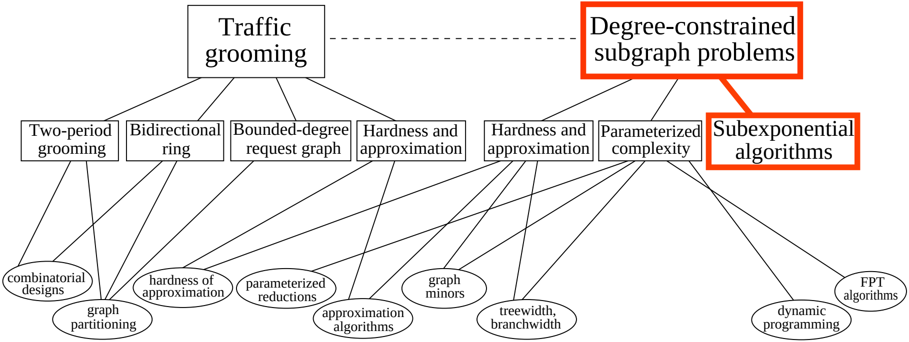

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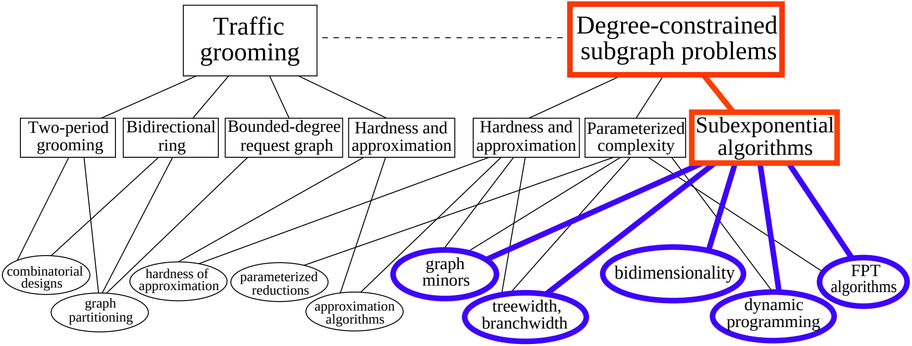

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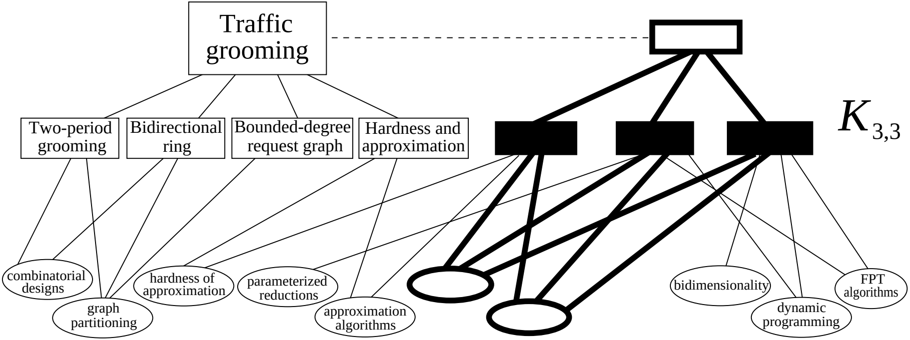

FPT and subexponential algorithms

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Given a parameter **P** defined in a planar graph G, **P**$(G) \leq k$?

First we compute **bw**(G). [Seymour and Thomas. *Combinatorica'94*]

(A) Combinatorial bounds via Graph Minor theorems:

bw(G) is "big" $\Rightarrow$ **P** is also "big" (typically, **P** $= \Omega(\mathbf{bw}^2)$).

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(B) Dynamic programming which uses graph structure:

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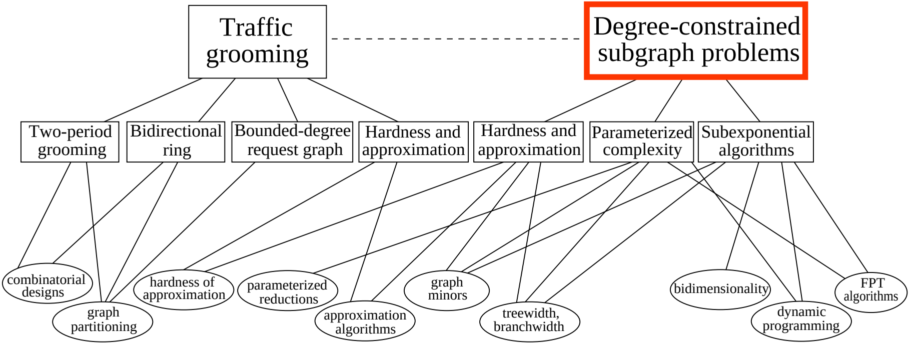

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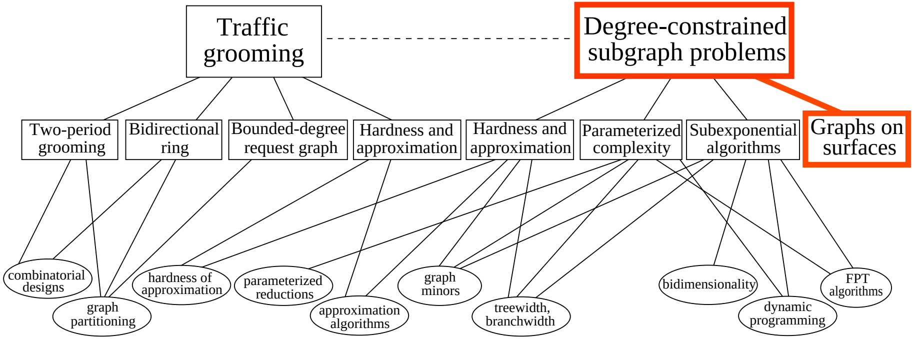

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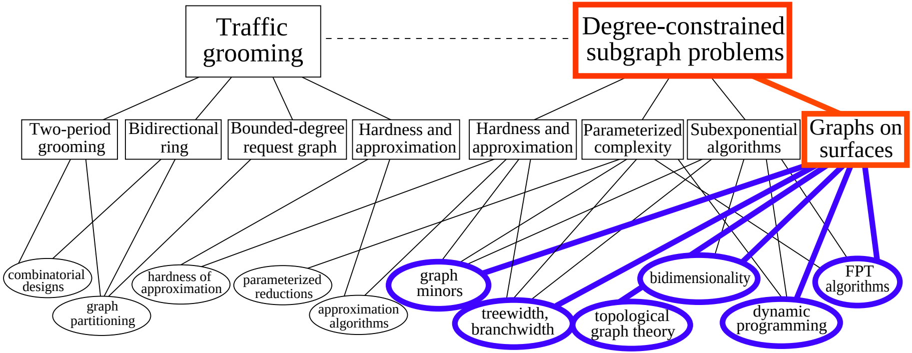
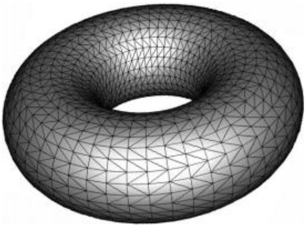
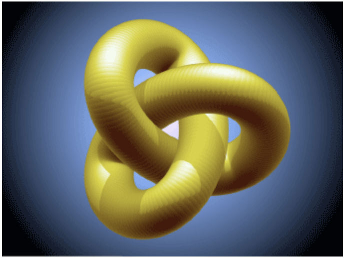


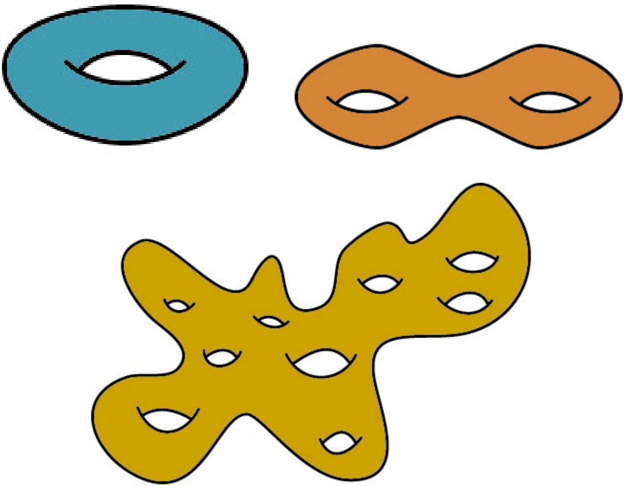
Surfaces

- Surface: connected compact 2-manifold.

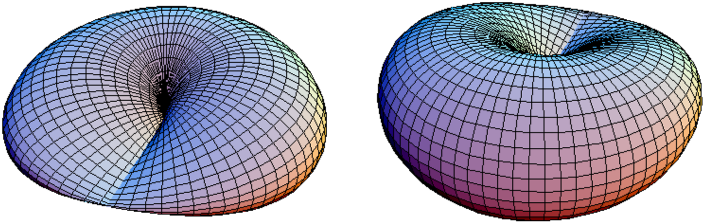




Handles



Cross-caps



Genus of a surface

- The surface classification Theorem: any compact, connected and without boundary surface can be obtained from the sphere $\mathbb{S}^2$ by adding handles and cross-caps.

- Orientable surfaces: obtained by adding $g \geq 0$ *handles* to the sphere $\mathbb{S}^2$, obtaining the g-torus $\mathbb{T}_g$ with Euler genus $\mathbf{eg}(\mathbb{T}_g) = 2g$.

- Non-orientable surfaces: obtained by adding $h > 0$ *cross-caps* to the sphere $\mathbb{S}^2$, obtaining a non-orientable surface $\mathbb{P}_h$ with Euler genus $\mathbf{eg}(\mathbb{P}_h) = h$.

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- An embedding of a graph G on a surface Σ is a drawing of G on Σ without edge crossings.
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Dynamic programming for graphs on surfaces

With *Juanjo Rué* and *Dimitrios M. Thilikos*

- Let G be a graph on n vertices with branchwidth at most k.
- We consider graph problems for which dynamic programming uses tables encoding vertex partitions ("Category (C)").

  For instance, our approach applies to MAXIMUM d-DEGREE-BOUNDED CONNECTED SUBGRAPH, MAXIMUM d-DEGREE-BOUNDED CONNECTED INDUCED SUBGRAPH and several variants, CONNECTED DOMINATING SET, CONNECTED r-DOMINATION, CONNECTED FVS, MAXIMUM LEAF SPANNING TREE, MAXIMUM FULL-DEGREE SPANNING TREE, MAXIMUM EULERIAN SUBGRAPH, STEINER TREE, MAXIMUM LEAF TREE, ...
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- In particular, our results imply and improve all the results in [Dorn, Fomin, and Thilikos. *SWAT'06*]
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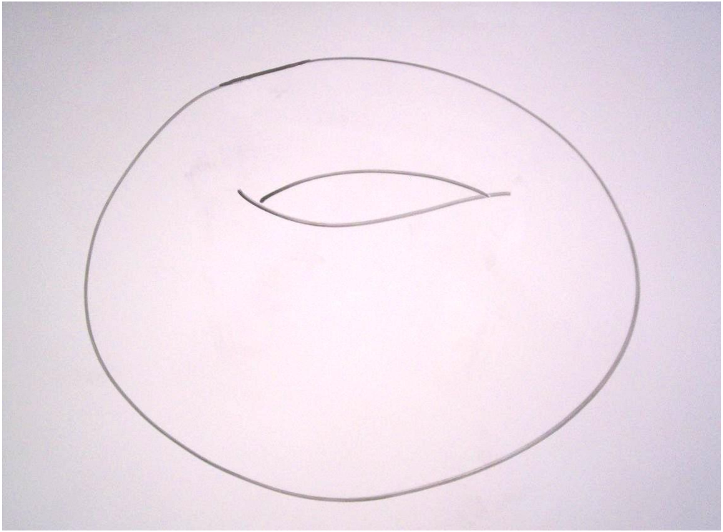

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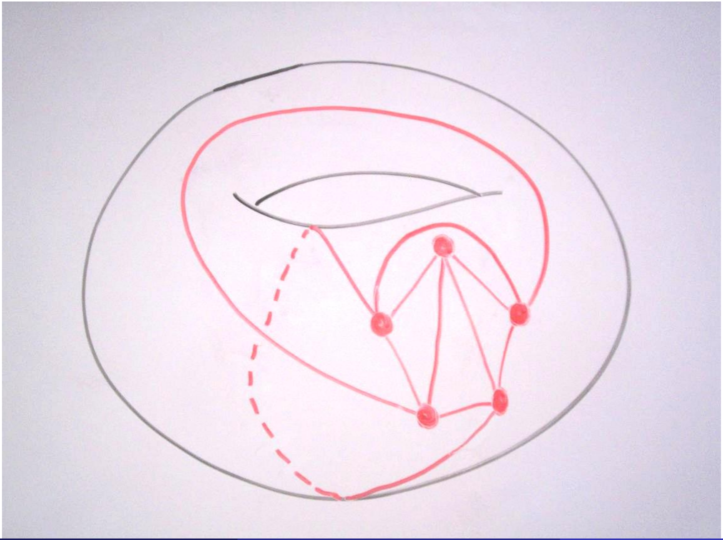

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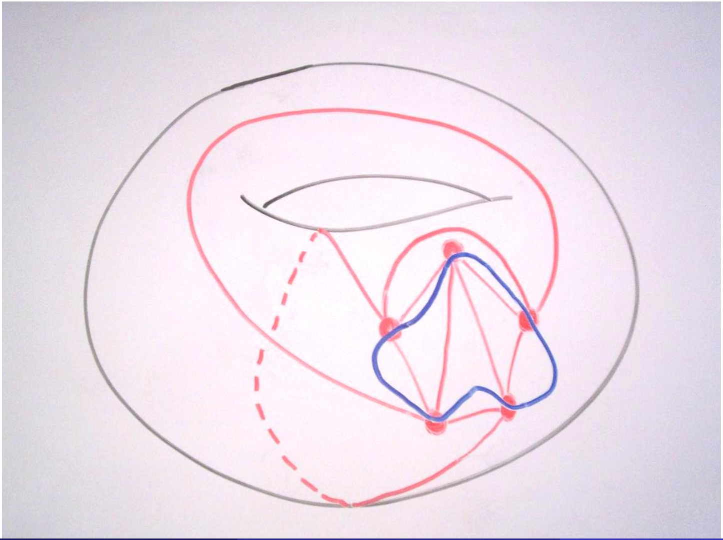

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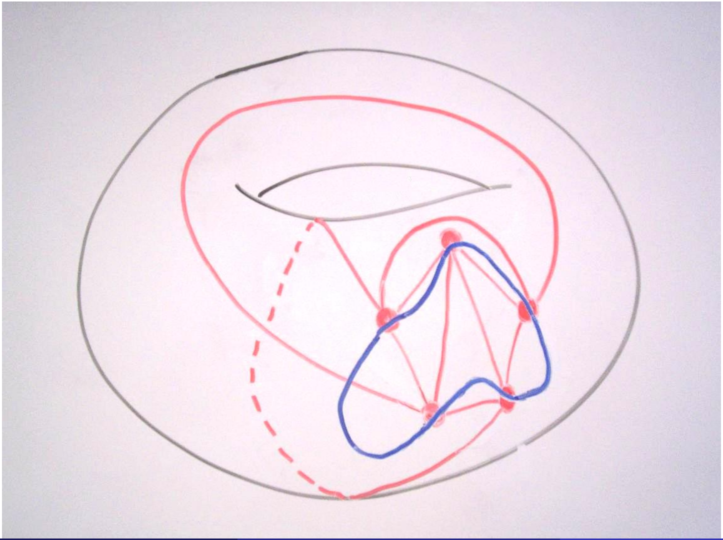

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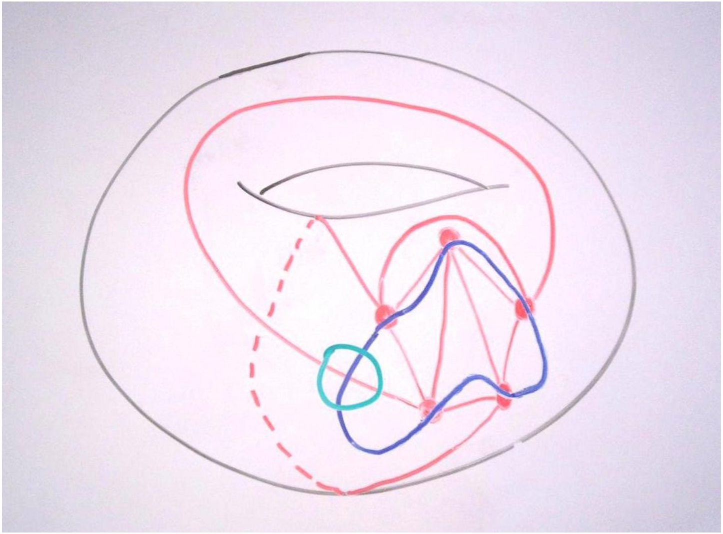

Sphere cut decompositions

- *Sphere cut decomposition*: Branch decomposition where the vertices in each **mid**(e) are situated around a noose.
- The size of the tables of a dynamic programming algorithm depend on how many ways a partial solution can intersect **mid**(e).
- In how many ways we can draw polygons inside a circle such that they touch the circle only on its vertices and they do not intersect?
- Exactly the number of *non-crossing partitions* over ℓ elements, which is given by the ℓ-th Catalan number:

$$\text{CN}(\ell) = \frac{1}{\ell+1}\binom{2\ell}{\ell} \sim \frac{4^\ell}{\sqrt{\pi}\ell^{3/2}} \approx 4^\ell.$$

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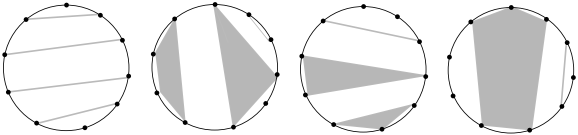

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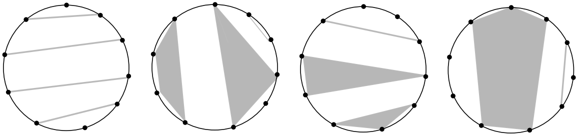

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How to use surface cut decompositions?

Surface cut decompositions can be efficiently computed:

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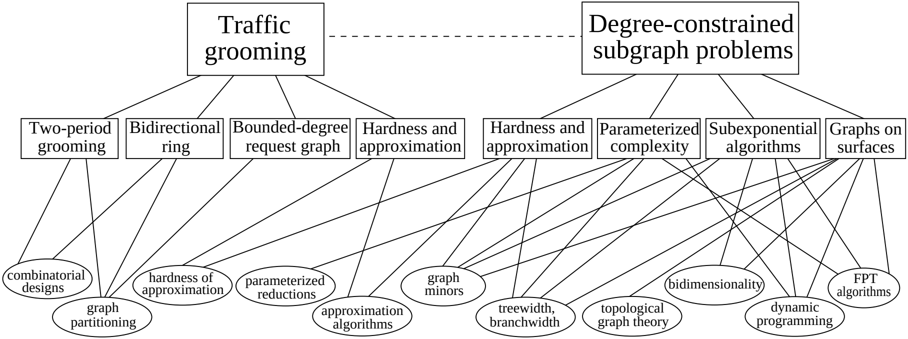

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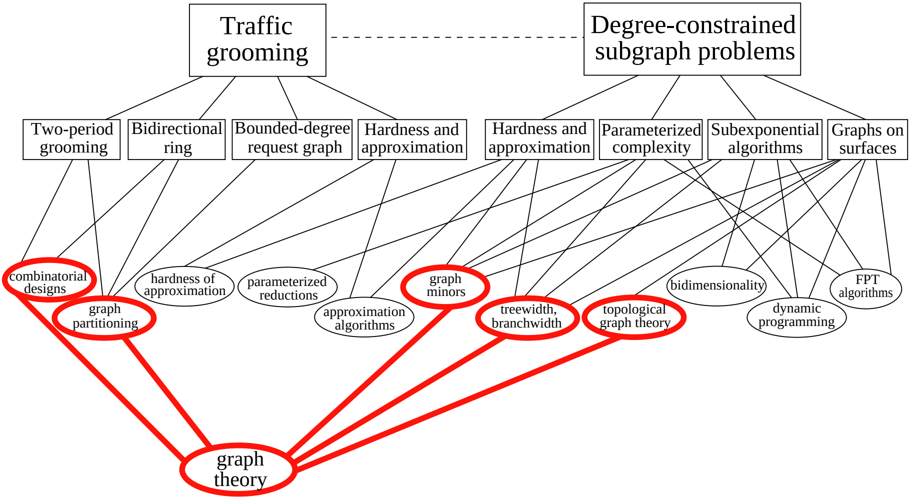

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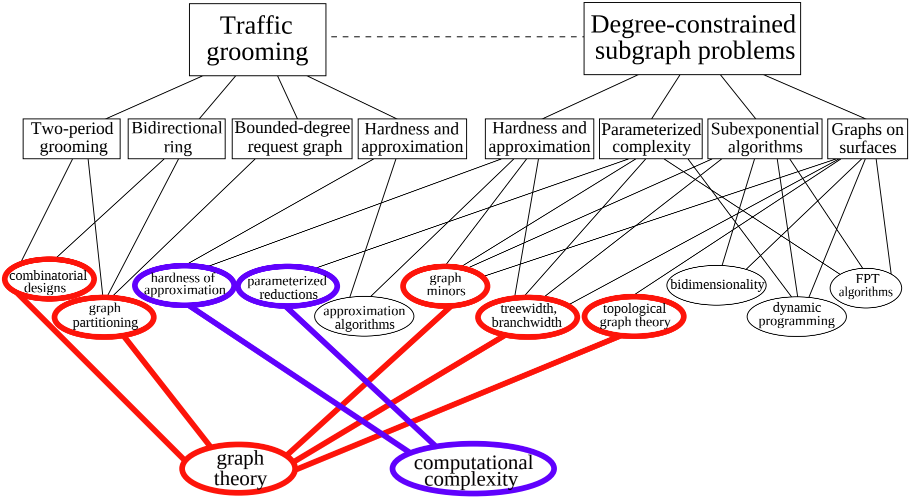

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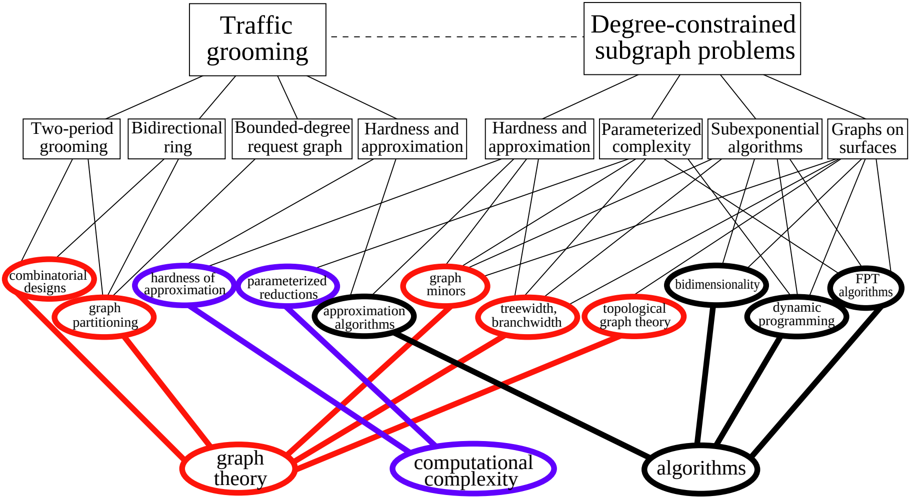

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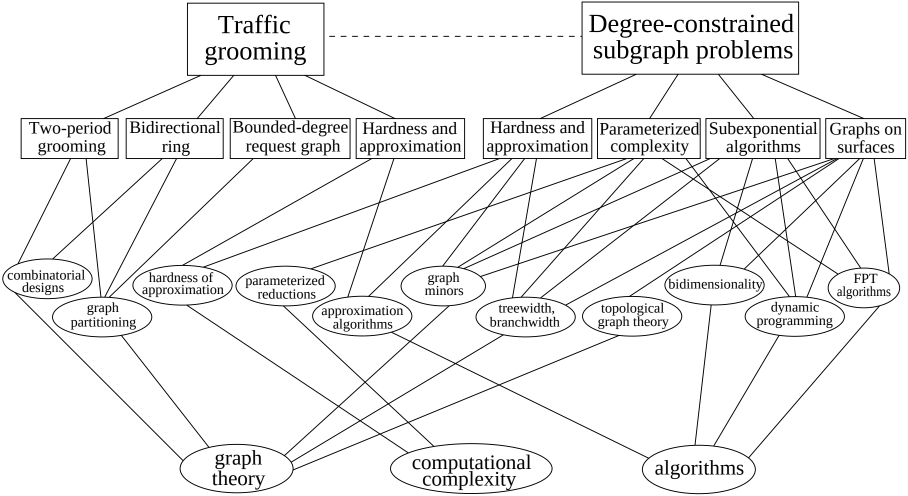

Further research

- Open problems and conjectures in each chapter of the manuscript.
- Traffic grooming:
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Gràcies!