A new Approach for Minimizing Buffer Capacities with Throughput Constraint for Embedded System Design

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Outline

Marked Timed Weighted Event Graph (MTWEG)

Definition

$\mathcal{G} = (T, P, \ell, M_0)$ is a Marked Timed Weighted Event Graph (MTWEG) where

1. $T = \{t_1, \cdots, t_n\}$ transitions;
2. $P = \{p_1, \cdots, p_m\}$ places;
3. $\ell : T \to N$ duration function;
4. $M_0 : P \to N$ initial marking;

Marked Timed Weighted Event Graph (MTWEG)

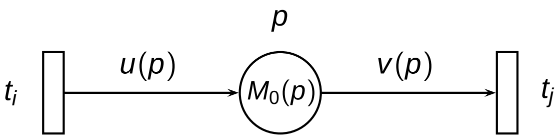

Figure: A place $p = (t_i, t_j)$ of a MTWEG.

1. Each place $p \in P$ is defined between two transitions t_i and t_j;
2. $\forall p \in P$ $u(p)$ and $v(p)$ are integers called the marking functions.

Firing of a transition

$$\mathcal{P}^{+}(t_i) = \{p = (t_i, t_j) \in P, t_j \in T\}$$

$$\mathcal{P}^{-}(t_i) = \{p = (t_j, t_i) \in P, t_j \in T\}$$

if t_i is *fired* at time τ:

1. At time τ, $v(p)$ tokens are removed from every place $p \in \mathcal{P}^{-}(t_i)$.
2. At time $\tau + \ell(t_i)$, $u(p)$ tokens are added to every place $p \in \mathcal{P}^{+}(t_i)$.

$M(\tau, p)$= The instantaneaous marking of a place $p \in P$ at time $\tau \geq 0$

Schedule and Periodic Schedule

Definition

Let $\mathcal{G}$ be a MTWEG. A schedule is a function $s : T \times N^\star \to Q^+$ which associates, with any tuple $(t_i, q) \in T \times N^\star$, the starting time of the qth firing of t_i.

Definition

A schedule s is periodic if there exists a vector $w = (w_1, \ldots, w_n) \in {Q^+}^n$ such that, for any couple $(t_i, q) \in T \times N^\star$, $s(t_i, q) = s(t_i, 1) + (q - 1)w_i$. w_i is then the period of the transition t_i and $\lambda^s(t_i) = \dfrac{1}{w_i}$ its throughput.

A car radio application (Wiggers et al.)

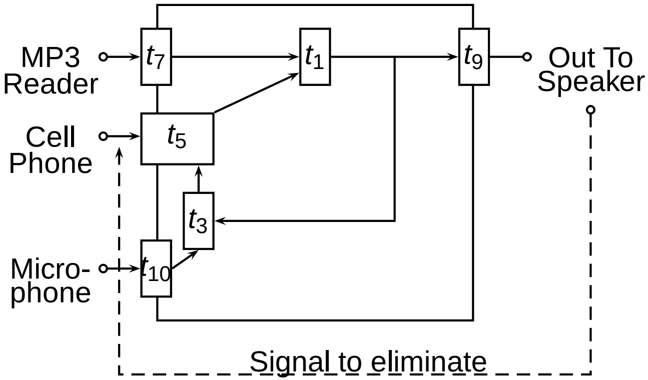

Figure: Block diagram of a car-radio application

Modelling using a MTWEG

1. Transitions corresponds to treatments;
2. Places corresponds to buffered transfers.

But...the size of the buffers should be limited !

Bounded capacity

Definition

A place $p = (t_i, t_j)$ has a bounded capacity $F(p) > 0$ if the number of tokens stored in p can not exceed $F(p)$:

$$\forall \tau \geq 0,\ M(\tau, p) \leq F(p)$$

Definition

A MTWEG $\mathcal{G} = (T, P, M_0, \ell, F)$ is said to be a bounded capacity graph if the capacity of every place $p \in P$ is bounded by $F(p)$.

Bounded capacity (Marchetti, Munier 2009)

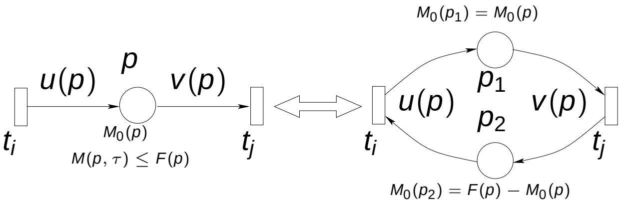

Figure: A place p with limited capacity $F(p)$ and the couple of places $(p_1, p_2)_c$ without capacities that models place p.

Definition

$\mathcal{G}$ is a symmetric MTWEG if every place $p = (t_i, t_j)$ is associated with a backward place $p' = (t_j, t_i)$ modelling the limited capacity.

Modelling of a car radio using a symmetric MTWEG

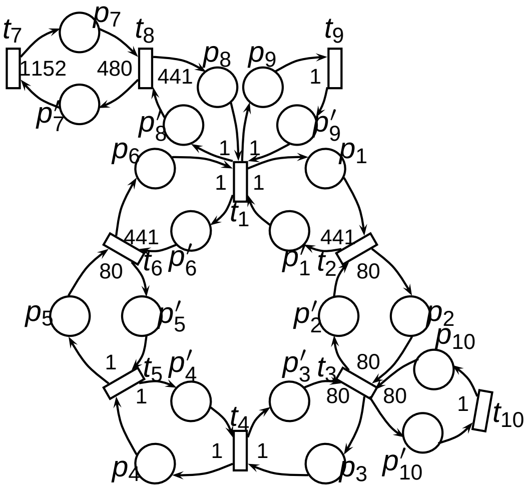

Figure: A MTWEG $\mathcal{G}$ modelling a car-radio application

Problem Formulation

Let $\mathcal{G}$ be a symmetric (non marked) WTEG. $\forall p \in P$, $\theta(p)$ is the size of a data stored in place p.
The problem consists in computing an initial marking M_0 of $\mathcal{G}$ such that:

1. The weighted sum of initial marking $\sum_{p \in P} \theta(p) M_0(p)$ is minimum;
2. There exists a periodic schedule s such that $\lambda^s(t_i) \geq \Delta$, $\forall t_i \in T$.

Unitary MTWEG (Karp, Miller 1966)

Definition

The weight (or gain) of every path μ of a MTWEG $\mathcal{G}$ by

$$W(\mu) = \prod_{p \in P \cap \mu} \frac{u(p)}{v(p)}.$$

Definition

A MTWEG $\mathcal{G}$ is unitary if every circuit c of $\mathcal{G}$ verifies $W(c) = 1$.

Since our symmetric MTWEG must be live, we only consider unitary MTWEG.

Normalized MTWEG (Marchetti, Munier 2009)

Definition

A MTWEG is normalized if all adjacent marking functions of every transition $t_i \in T$ are equal to a single value denoted by Z_i, *i.e.* $\forall p \in \mathcal{P}^+(t_i), u(p) = Z_i$ and $\forall p \in \mathcal{P}^-(t_i), v(p) = Z_i$.

Note that the number of tokens in every circuit of a normalized MTWEG does not vary.

Normalization of a MTWEG

Theorem

Every unitary MTWEG may be transformed into an equivalent normalized MTWEG.

1. For any value $\alpha \in \mathbb{Q}^{+\star}$, the markings functions and the initial markings of any place $p \in P$ may be replaced simultaneously by respectively $\alpha u(p)$, $\alpha v(p)$ and $\alpha M_0(p)$ without any influence on the schedules;
2. Positive integer values $\alpha(p)$, $p \in P$ such that, $\forall t_i \in T$, there exists an integer Z_i with, $\forall p \in \mathcal{P}^+(t_i)$, $\alpha(p)u(p) = Z_i$ and $\forall p \in \mathcal{P}^-(t_i)$, $\alpha(p)v(p) = Z_i$ may be computed in polynomial time.

Normalization of the MTWEG of the car Radio

$$
\left\{
\begin{array}{lcl}
Z_1 & = & \alpha(p_1) = \alpha(p_6) = \alpha(p_8) = \alpha(p_9) \\
Z_2 & = & 441\alpha(p_1) = 80\alpha(p_2) \\
Z_3 & = & 80\alpha(p_2) = 80\alpha(p_3) = 80\alpha(p_{10}) \\
Z_4 & = & \alpha(p_3) = \alpha(p_4) \\
Z_5 & = & \alpha(p_4) = \alpha(p_5) \\
Z_6 & = & 80\alpha(p_5) = 441\alpha(p_6) \\
Z_7 & = & 1152\alpha(p_7) \\
Z_8 & = & 480\alpha(p_7) = 441\alpha(p_8) \\
Z_9 & = & \alpha(p_9) \\
Z_{10} & = & \alpha(p_{10})
\end{array}
\right.
$$

Normalization of the MTWEG of the car Radio

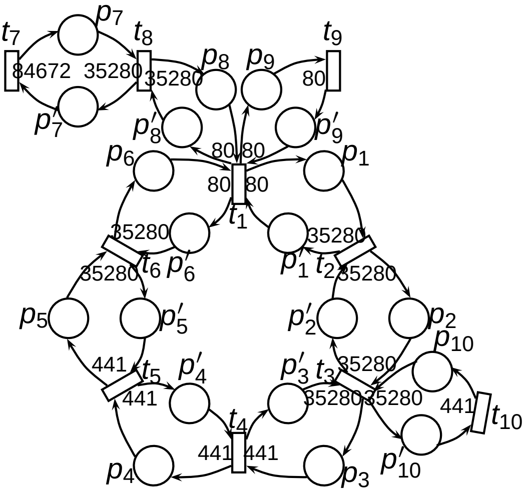

Figure: The normalized MTWEG associated to $\mathcal{G}$.

Precedence constraints induced by a place (Munier 1993)

Let $G = (T, P, \ell, M_0)$ a MGTEG. For any $(t_i, \nu_i) \in T \times N$, (t_i, ν_i) is the ν_ith firing of t_i.

Theorem

A place $p = (t_i, t_j)$ will induce a precedence constraint between (t_i, ν_i) and (t_j, ν_j) iff

$$u(p) - M_0(p) > u(p)\nu_i - v(p)\nu_j \geq \max(u(p) - v(p), 0) - M_0(p)$$

Characterization of a periodic schedule (Benabid et al. 2008)

Theorem

*Let $\mathcal{G}$ be a normalized MTWEG. For any feasible periodic schedule s of $\mathcal{G}$, there exists $K \in \mathbb{Q}^{\star+}$ called the **normalized period** of s such that, for any couple of transitions $(t_i, t_j) \in T^2$, $\frac{w_i}{Z_i} = \frac{w_j}{Z_j} = K$. Moreover, s is feasible iff, for any place $p = (t_i, t_j) \in P$,*

$$s(t_j, 1) - s(t_i, 1) \geq \ell(t_i) + K(Z_j - M_0(p) - gcd_{i,j}).$$

where $gcd_{i,j} = gcd(Z_i, Z_j)$.

Computation of the maximum processing times for the car radio example

- The output (t_9) must have a frequency equal to $44.1kHz$.

$$\ell(t_9) = w_9 = \frac{1}{44.1 \times 10^3}$$

- Thus, $K = \frac{w_9}{Z_9} = 2.83 \times 10^{-4} sec$;
- $\forall t_i \in T - \{t_9\}$, $w_i = Z_i K$ and $\ell(t_i) \leq w_i$.

Table: Upper bound w_i of the processing times, $t_i \in T$ in milliseconds

	t_1, t_9	t_2, t_6, t_8	t_3	t_4, t_5, t_{10}	t_7
ℓ	0.023	10	9.091	0.125	24

Computation of a periodic Schedule

input A normalized MTWEG $\mathcal{G}$;

output Minimum normalized period $K^{\star}$.

Computation of a periodic schedule

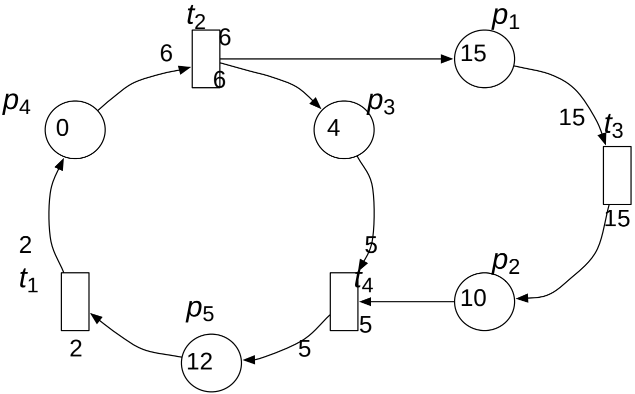

Figure: A MTWEG with $\ell(t_1) = 10$, $\ell(t_2) = 12$, $\ell(t_3) = 6$ and $\ell(t_4) = 5$.

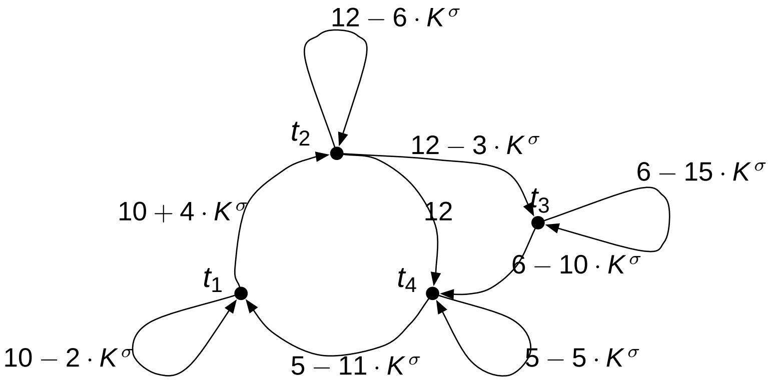

Figure: Valued graph $G = (X, A)$ associated with the normalized Marked WEG pictured by Figure 6.

Computation of a periodic schedule with a minimum period

$$\min \quad K \quad \text{subject to}$$

$$\begin{cases} & s(t_4,1) - s(t_2,1) \geq 12 \\ & s(t_1,1) - s(t_4,1) \geq 5 - 11K \\ & s(t_2,1) - s(t_1,1) \geq 10 + 4K \\ .... & \\ & 0 \geq 10 - 2K \\ & 0 \geq 12 - 6K \\ \forall t_i \in T, & s(t_i,1) \geq 0 \end{cases}$$

Computation of a periodic schedule with a minimum period

$$\begin{array}{l} min \quad K \quad subject\ to \\ \left\{ \begin{array}{ll} \forall p = (t_i, t_j) \in P, & s(t_j, 1) - s(t_i, 1) \geq \ell(t_i) + \\ & K\left(Z_j - M_0(p) - gcd_{i,j}\right) \\ \forall t_i \in T, & s(t_i, 1) \geq 0 \end{array} \right. \end{array}$$

Polynomially solved using Linear Programming or a variant of Bellman-Ford algorithm.

Computation of the minimum size of the buffers under for a given K

$\mathcal{G}$ is a normalized symmetric MTWEG.
System $\Pi(K)$:

$$min \left(\sum_{p\in P} \theta(p)M_0(p)\right) \quad subject\ to$$

$$\begin{cases} \forall p=(t_i,t_j)\in P, & s(t_j,1)-s(t_i,1)\geq \ell(t_i)+ \\ & K(Z_j-M_0(p)-gcd_{i,j}) \\ \forall p=(t_i,t_j)\in P, & M_0(p)=k_{i,j}\cdot gcd_{i,j} \\ \forall p=(t_i,t_j)\in P & k_{i,j}\in\mathbb{N} \\ \forall t_i\in T, & s(t_i,1)\geq 0 \end{cases}$$

Study of a buffer

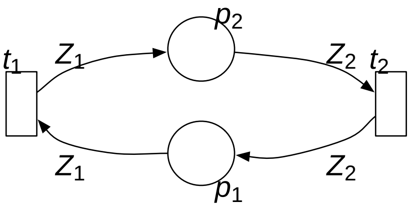

Figure: A unitary WEG with two places. $F(p_1, p_2) = M_0(p_1) + M_0(p_2)$

$$K^{opt} = \max\left\{\frac{\ell(t_1)}{Z_1}, \frac{\ell(t_2)}{Z_2}, \frac{\ell(t_1)+\ell(t_2)}{F(p_1,p_2)+2gcd_{1,2}-(Z_1+Z_2)}\right\}$$

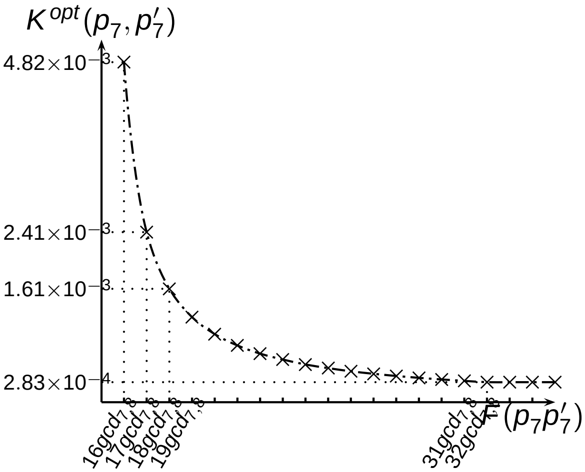

Figure: K^{opt} according to $F(p_7, p'_7) = M_0(p_7) + M_0(p'_7)$.
$F^{min}(p_7, p'_7) = 16gcd_{7,8}$, $K^{min}(p_7, p'_7) = 4.82 \times 10^{-3} ms$,
$F^{max}(p_7, p'_7) = 32gcd_{7,8}$ and $K^{max}(p_7, p'_7) = 2.83 \times 10^{-4} ms$.

An optimal $O(m \log(\max_{i \in \{1,\ldots,n\}} \{Z_i\}))$ algorithm for a symmetric MWTEG without circuits of more than two transitions

Let $K \geq \max_{t_i \in T} \left\{ \frac{\ell(t_i)}{Z_i} \right\}$ and $(p, p')_c$ a couple of places corresponding to a buffer. $F_K(p, p')$ = minimum capacity of buffer p to achieve a normalized period K.

1. If $K \leq K^{max}(p, p')$, set $M_0(p) = F_K(p, p')$ and $M_0(p') = 0$;
2. Else, $K > K^{max}(p, p')$. Set $M_0(p) = F^{min}(p, p')$ and $M_0(p') = 0$.

This solution is clearly minimum for every buffer. So, it minimizes the overall weighted capacity of $\mathcal{G}$.

Example for the car radio

Let $\mathcal{G}'$ limited to the transitions $T' = \{t_1, t_5, t_6, t_7, t_8, t_9\}$ corresponding to the mixing of the sounds coming from the MP3 reader and the cell phone to the output.
The corresponding undirected graph defined as $G' = (T', \{\{t_7, t_8\}, \{t_5, t_6\}, \{t_6, t_1\}, \{t_8, t_1\}, \{t_1, t_9\}\})$ is clearly a tree (see Figure 10).

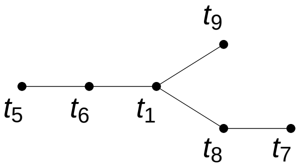

Figure: The undirected graph G' is a tree.

Example for the car radio

Table: Optimal initial markings of the subgraph $\mathcal{G}'$ for different processing time of t_8.

$\ell(t_8) =$	10	7.5	5	2.5
$\theta(p_6)F(p_6, p'_6)$	882	882	882	882
$\theta(p_7)F(p_7, p'_7)$	3072	2976	2880	2784
$\theta(p_8)F(p_8, p'_8)$	882	772	662	552
$\theta(p_9)F(p_9, p'_9)$	2	2	2	2
Sum	4838	4632	4426	4220

An Approximation Algorithm for the General Case

Let K be a fixed value. Let us consider the Linear Program $\Pi^\star(K)$ obtained from $\Pi(K)$ by replacing the condition $k_{i,j} \in \mathbb{N}$ by $k_{i,j} \in \mathbb{Q}^+$.

1. Compute an optimum solution $M_0^\star(p) \in \mathbb{Q}$, $p \in P$ of $\Pi^\star(K)$. This step can be done in polynomial time;
2. $M_0(p) = \lceil M_0^\star(p) \rceil$ is then a feasible solution of $\Pi(K)$.

An Approximation Algorithm for the General Case

Theorem

The competitive ratio of our algorithm is 2.

Worst case may easily be achieved for a symmetric Timed Non Weighted Event graph (*i.e.* , $Z_i = 1, \forall t_i \in T$).

Application to the car radio

Table: Optimal buffers capacities for the MTWEG pictured by Figure 4

Buffers	$\theta(p_i)a_K(p_i, p'_i)$	$\theta(p_i)F_K^{App}(p_i, p'_i)$
$(p_1, p'_1)_c$	882	882
$(p_2, p'_2)_c$	160	160
$(p_3, p'_3)_c$	153	154
$(p_4, p'_4)_c$	2	2
$(p_5, p'_5)_c$	160	160
$(p_6, p'_6)_c$	882	882
$(p_7, p'_7)_c$	3072	3072
$(p_8, p'_8)_c$	882	882
$(p_9, p'_9)_c$	2	2
$(p_{10}, p'_{10})_c$	153	154
Sum	6348	6350

Conclusion and Perspectives

1. First analytical and polynomial approach to compute the minimum size of the buffers;
2. Implementation in an industrial context;
3. Extension to cyclo-static Synchronous DataFlow Graphs.

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submitted.