PACKING AND COVERING BALLS IN GRAPHS EXCLUDING A MINOR

NICOLAS BOUSQUET, WOUTER CAMES VAN BATENBURG, LOUIS ESPERET, GWENAËL JORET, WILLIAM LOCHET, CAROLE MULLER, AND FRANÇOIS PIROT

Abstract. We prove that for every integer $t \geqslant 1$ there exists a constant c_t such that for every K_t-minor-free graph G, and every set S of balls in G, the minimum size of a set of vertices of G intersecting all the balls of S is at most c_t times the maximum number of vertex-disjoint balls in S. This was conjectured by Chepoi, Estellon, and Vaxès in 2007 in the special case of planar graphs and of balls having the same radius.

1. Introduction

A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where V is the vertex set and $\mathcal{E} \subseteq 2^V$ is the edge set of $\mathcal{H}$. A *matching* in a hypergraph $\mathcal{H}$ is a set of pairwise vertex-disjoint edges, and a *transversal* is a set of vertices that intersects every edge. The *matching number* of a hypergraph $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the maximum number of edges in a matching. The *transversal number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimum size of a transversal of $\mathcal{H}$. We can also consider the linear relaxation of these two parameters: we define the *fractional matching number* $\nu^*(\mathcal{H})$ and the *fractional transversal number* $\tau^*(\mathcal{H})$ as follows.

$$\nu^*(\mathcal{H}) = \max \sum_{e \in \mathcal{E}(\mathcal{H})} w_e$$

$$\text{given that } \begin{cases} \sum_{e \ni v} w_e \leqslant 1 & \text{for every vertex } v \text{ of } \mathcal{H} \\ w_e \geqslant 0 & \text{for every edge } e \text{ of } \mathcal{H}, \end{cases}$$

and the dual of this linear program is

$$\tau^*(\mathcal{H}) = \min \sum_{v \in V(\mathcal{H})} w_v$$

$$\text{given that } \begin{cases} \sum_{v \in e} w_v \geqslant 1 & \text{for every edge } e \text{ of } \mathcal{H} \\ w_v \geqslant 0 & \text{for every vertex } v \text{ of } \mathcal{H}. \end{cases}$$

By the strong duality theorem, $\nu(\mathcal{H}) \leqslant \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leqslant \tau(\mathcal{H})$ for every hypergraph $\mathcal{H}$. Given a class $\mathcal{C}$ of hypergraphs, a classical problem in combinatorial optimization is to decide whether there exists a function f such that $\tau(\mathcal{H}) \leqslant f(\nu(\mathcal{H}))$ for every $\mathcal{H} \in \mathcal{C}$. If this is the case the class $\mathcal{C}$ is sometimes said to have the *Erdős-Pósa property*. Classical examples include the family of all cycles of a graph [10] (i.e. given a graph $G = (V, E)$ we consider the hypergraph with vertex set V whose edges are all the cycles of G), and the family of all

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directed cycles of a directed graph [18]. A desirable property is the existence of a constant c such that $\tau(\mathcal{H}) \leqslant c \cdot \tau^*(\mathcal{H})$ or $\nu^*(\mathcal{H}) \leqslant c \cdot \nu(\mathcal{H})$, or (even better) $\tau(\mathcal{H}) \leqslant c \cdot \nu(\mathcal{H})$ for every $\mathcal{H} \in \mathcal{C}$. These properties are often useful in the design of approximation algorithms using a primal-dual approach (see for instance [14, 12]).

Given a graph $G = (V, E)$, an integer $r \geqslant 0$, and a vertex $v \in V$, we denote by $B_r(v)$ the *ball of radius* r in G centered in v, that is

$$B_r(v) := \{u \in V(G) \mid d_G(u, v) \leqslant r\},$$

where $d_G(u, v)$ denotes the distance between u and v in G (we will omit the subscript G when the graph is clear from the context). We say that a hypergraph $\mathcal{H}$ is a *ball hypergraph* of G if $\mathcal{H}$ has vertex set $V = V(G)$ and each edge of $\mathcal{H}$ is a ball $B_r(v)$ in G for some integer r and some vertex $v \in V$. If all the balls forming the edges of $\mathcal{H}$ have the same radius r, we say that $\mathcal{H}$ is an *r-ball hypergraph* of G.

Chepoi, Estellon, and Vaxès [6] proved the existence of a universal constant ρ such that for every $r \geqslant 0$ and every planar graph G of diameter at most $2r$, the vertices of G can be covered with at most ρ balls of radius r. Note that G has diameter at most $2r$ if and only if there are no two disjoint balls of radius r in G. Also, a set of balls of radius r in G cover all of $V(G)$ if and only if their centers intersect all balls of radius r of G. Thus, their result states equivalently the existence of a universal constant ρ such that for every $r \geqslant 0$ and every planar graph G, if the r-ball hypergraph $\mathcal{H}$ consisting of all balls of radius r satisfies $\nu(\mathcal{H}) = 1$, then $\tau(\mathcal{H}) \leqslant \rho$. With this interpretation in mind, Chepoi, Estellon, and Vaxès [2] conjectured the following generalization in 2007 (see also [11]).

Conjecture 1 (Chepoi, Estellon, and Vaxès [2]). *There exists a constant c such that for every integer $r \geqslant 0$, every planar graph G, and every r-ball hypergraph $\mathcal{H}$ of G, we have $\tau(\mathcal{H}) \leqslant c \cdot \nu(\mathcal{H})$.*

If one considers all metric spaces obtained as standard graph-metrics of planar graphs, then Conjecture 1 states that these metric spaces satisfy the so-called *bounded covering-packing property* [5]. Recently, Chepoi, Estellon, and Naves [5] showed that other metric spaces do have this property, including the important case of Busemann surfaces. (Quoting [5], the latter are roughly the geodesic metric spaces homeomorphic to $\mathbb{R}^2$ in which the distance function is convex; they generalize Euclidean spaces, hyperbolic spaces, Riemannian manifolds of global nonpositive sectional curvatures, and CAT(0) spaces.)

Going back to Conjecture 1, let us emphasize that a key aspect of this conjecture is that the constant c is independent of the radius r. If c is allowed to depend on r, then the conjecture is known to be true. In fact, it holds more generally for all graph classes with bounded expansion, as shown by Dvořák [8].

Some evidence for Conjecture 1 was given by Bousquet and Thomassé [1], who proved that it holds with a polynomial bound instead of a linear one. More generally, they proved that for every integer $t \geqslant 1$, there exists a constant c_t such that for every integer $r \geqslant 0$, every K_t-minor free graph G, and every r-ball hypergraph $\mathcal{H}$ of G, we have $\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H})^{2t+1}$.

The main result of this paper is that Conjecture 1 is true, and furthermore it is not necessary to assume that all the balls have the same radius.

Theorem 2 (Main result). *For every integer $t \geqslant 1$, there is a constant c_t such that $\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H})$ for every K_t-minor-free graph G and every ball hypergraph $\mathcal{H}$ of G.*

Our proof of Theorem 2 follows a bootstrapping approach. It relies on the existence of some function f_t such that $\tau(\mathcal{H}) \leqslant f_t(\nu(\mathcal{H}))$, i.e. on the Erdős-Pósa property of the ball hypergraphs of K_t-minor-free graphs, which is used in the proof when $\nu(\mathcal{H})$ is not 'too big'. However, showing this property was an open problem. This was known for r-ball hypergraphs, by the result of Bousquet and Thomassé [1], but their proof method does not extend to the case of balls of arbitrary radii. For this reason, as a first step towards proving Theorem 2, we prove Theorem 3 below establishing said Erdős-Pósa property. We also note that, while the bounding function in Theorem 3 is not optimal, it is a near linear bound of the form $\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H}) \log \nu(\mathcal{H})$ where c_t is a small explicit constant polynomial in t. This is in contrast with the constant c_t in our proof of Theorem 2 which is large, exponential in t. Thus, the bound in Theorem 3 is better for small values of $\nu(\mathcal{H})$. (We note that logarithms in this paper are natural, and the base of the natural logarithm is denoted by e.)

Theorem 3 (Near linear bound)**.** *Let G be a graph with no K_t-minor and such that every minor of G has average degree at most d. Then for every ball hypergraph $\mathcal{H}$ of G,*

$$\tau(\mathcal{H}) \leqslant 2\mathrm{e}(t-1)\, d \cdot \nu(\mathcal{H}) \cdot \log(11\mathrm{e}(t-1)\, d \cdot \nu(\mathcal{H})).$$

In particular, $\tau(\mathcal{H}) \leqslant ct^2\sqrt{\log t} \cdot \nu(\mathcal{H}) \cdot \log(t \cdot \nu(\mathcal{H}))$ for some absolute constant $c > 0$, and if G is planar then $\tau(\mathcal{H}) \leqslant 48\,\mathrm{e} \cdot \nu(\mathcal{H}) \cdot \log(264\,\mathrm{e} \cdot \nu(\mathcal{H}))$.

The proof of Theorem 3 uses known results on the VC-dimension of ball hypergraphs of G when G excludes a minor, together with classical bounds relating $\tau(\mathcal{H})$ and $\tau^*(\mathcal{H})$ when $\mathcal{H}$ has bounded VC-dimension, as well as the following theorem.

Theorem 4. *Let G be a graph and let d be the maximum average degree of a minor of G. Then for every ball hypergraph $\mathcal{H}$ of G, we have $\nu^*(\mathcal{H}) \leqslant \mathrm{e}\, d \cdot \nu(\mathcal{H})$.*
In particular, if G is planar then $\nu^*(\mathcal{H}) \leqslant 6\mathrm{e} \cdot \nu(\mathcal{H})$ and if G has no K_t-minor then $\nu^*(\mathcal{H}) \leqslant c \cdot t\sqrt{\log t} \cdot \nu(\mathcal{H})$, for some absolute constant $c > 0$.

The paper is organized as follows. Sections 2 and 3 are devoted to technical lemmas that will be used in our proofs. Theorems 3 and 4 are proved in Section 4. Theorem 2 is proved in Section 5. Finally, we conclude the paper in Section 6 with a construction suggesting that Theorem 2 does not extend way beyond proper minor-closed classes.

2. Hypergraphs, balls, and minors

We will need two technical lemmas, whose proofs are very similar to the proof of [1, Theorem 4] and [6, Proposition 1]. We start with Lemma 1, which will be used in the proof of Theorem 4.

Lemma 1. *Let G be a graph, let $S = \{B_i = B_{r_i}(s_i)\}_{i \in [n]}$ be a set of n balls in G, with pairwise distinct centers, and let $E_S \subseteq \binom{S}{2}$ be a subset of pairs of intersecting balls $\{B_i, B_j\} \subseteq S$ (with $i < j$), each of which is associated with a vertex $x_{ij} \in B_i \cap B_j$ which is at distance $\left\lfloor \frac{r_i - r_j + d_{ij}}{2} \right\rfloor$ from s_i, and at distance $\left\lceil \frac{r_j - r_i + d_{ij}}{2} \right\rceil$ from s_j, where $d_{ij} = d_G(s_i, s_j) \leqslant r_i + r_j$, and such that the only balls of S containing x_{ij} are B_i and B_j. Then the graph $H = (S, E_S)$ is a minor of G.*

Proof. First note that x_{ij} indeed is in $B_i \cap B_j$ since $\left\lfloor \frac{r_i - r_j + d_{ij}}{2} \right\rfloor \leqslant \left\lfloor \frac{2r_i - r_i - r_j + d_{ij}}{2} \right\rfloor \leqslant r_i$ and $\left\lceil \frac{r_j - r_i + d_{ij}}{2} \right\rceil \leqslant \left\lceil \frac{2r_j - r_i - r_j + d_{ij}}{2} \right\rceil \leqslant r_j$.

Let us fix a total ordering $\prec$ on the vertices of G. In the proof, all distances are in the graph G, so we write $d(u, v)$ instead of $d_G(u, v)$ for the sake of readability. For every pair of balls $B_iB_j \in E_S$ ($i < j$), associated with the vertex x_{ij}, we let P^0_{ij} be a shortest path from s_i to x_{ij}, and we assume that the sequence of vertices from s_i to x_{ij} on the path is minimum with respect to the lexicographic order induced by $\prec$. Similarly, we let P^1_{ij} be a shortest path from s_j to x_{ij} that minimizes the sequence of vertices from s_j to x_{ij} with respect to the lexicographic order. We then define $P^0_{ji} = P^1_{ij}$ and $P^1_{ji} = P^0_{ij}$ (so that $P^0_{k\ell}$ and $P^1_{k\ell}$ are now also defined when $k > \ell$). By the assumptions, we know that $P_{ij} \coloneqq P^0_{ij} \cup P^1_{ij}$ is a shortest path from s_i to s_j. Moreover, by the definition of x_{ij}, for every $B_iB_j \in E_S$ we have $r_j - d(s_j, x_{ij}) \leqslant r_i - d(s_i, x_{ij}) + 1$ (where the $+1$ term is due to the case where $r_j - r_i + d_{ij}$ is odd and $i > j$).

For every $i \in [n]$, we define

$$\mathcal{T}_i \coloneqq \bigcup_{j:B_iB_j \in E_S} P^0_{ij}.$$

Claim 1. For every $i \in [n]$, $\mathcal{T}_i$ is a tree.

Assume for the sake of contradiction that there is a cycle C in $\mathcal{T}_i$. Observe that, by construction, if uv is an edge of $\mathcal{T}_i$ then $|d(s_i, u) - d(s_i, v)| = 1$. Let y be a vertex of C maximizing $d(s_i, y)$, and let z_1, z_2 denote its two neighbors in C. Then $d(s_i, z_1) = d(s_i, z_2) = d(s_i, y) - 1$, and there exist j_1, j_2 such that z_1y is an edge of $P^0_{ij_1}$ and z_2y is an edge of $P^0_{ij_2}$. Let P_1 and P_2 be the subpaths from s_i to y of $P^0_{ij_1}$ and $P^0_{ij_2}$, respectively. Then P_1 and P_2 are two different paths from s_i to y, and one of them is not minimum either in terms of length, or with respect to the lexicographic order induced by $\prec$. This contradicts the definition of $P^0_{ij_1}$ and $P^0_{ij_2}$.

Claim 2. For two pairs of balls $B_iB_k, B_jB_\ell \in E_S$ with $i \neq j$, if P^0_{ik} and $P^0_{j\ell}$ intersect in some vertex y such that $d(y, x_{ik}) \leqslant d(y, x_{j\ell})$, then $i = \ell$, $j = k$ and $y = x_{ij}$.

Note that $d(s_j, x_{ik}) \leqslant d(s_j, y) + d(y, x_{ik}) \leqslant d(s_j, y) + d(y, x_{j\ell}) \leqslant r_j$, which implies that $x_{ik} \in B_j$. By definition, x_{ik} is only contained in the balls B_i and B_k of S and thus $j = k$. Since $P_{ij} = P^0_{ij} \cup P^0_{ji}$ is a shortest path containing vertex y, the s_j–y section of that path (which contains x_{ij}) has the same length as the s_j–y section of $P^0_{j\ell}$. Replacing the latter section by the former, we obtain a shortest path from s_j to $x_{j\ell}$ containing x_{ij}, which we denote $Q^0_{j\ell}$. As a consequence,

$$d(x_{j\ell}, x_{ij}) = d(x_{j\ell}, s_j) - d(s_j, x_{ij}) \leqslant r_j - d(s_j, x_{ij}) \leqslant r_i - d(s_i, x_{ij}) + 1,$$

where the last inequality follows from the definition of x_{ij}. Assume for the sake of contradiction that $y \neq x_{ij}$. We claim that no shortest path from s_i to $x_{j\ell}$ contains x_{ij}. Indeed, if there is one then we may choose its s_i–x_{ij} section to be P^0_{ij} and its x_{ij}–$x_{j\ell}$ section to be that of $Q^0_{j\ell}$; however, vertex y appears twice on this path, a contradiction. Thus, it follows from the inequality above that

$$d(s_i, x_{j\ell}) \leqslant d(s_i, x_{ij}) + d(x_{ij}, x_{j\ell}) - 1 \leqslant r_i.$$

As a consequence $x_{j\ell}$ is contained in B_i, which implies that $i = \ell$ and thus $y = x_{ij}$.

This claim immediately implies that for every $i, j \in [n]$ with $i \neq j$, we have $V(\mathcal{T}_i) \cap V(\mathcal{T}_j) = \{x_{ij}\}$ if $B_iB_j \in E_S$, and $V(\mathcal{T}_i) \cap V(\mathcal{T}_j) = \emptyset$ otherwise. Another consequence is that for every $B_iB_j \in E_S$, the vertex x_{ij} is a leaf in at least one of the two trees $\mathcal{T}_i$ and $\mathcal{T}_j$ (since otherwise there exist $k \neq j$ and $\ell \neq i$ such that $x_{ij} \in P^0_{ik}$ and $x_{ij} \in P^0_{j\ell}$, which readily contradicts the claim).

In the subgraph $\bigcup_{i\in[n]} \mathcal{T}_i$ of G, for each $i \in [n]$ we contract each edge of $\mathcal{T}_i$ except the ones incident to a leaf of $\mathcal{T}_i$. It follows from the paragraph above that the resulting graph is precisely a graph obtained from $H = (S, E_S)$ by subdividing each edge at most once, and thus H is a minor of G. $\square$

The next result has a nearly identical proof, but the setting is slightly different. It will be used twice in the proof of Theorem 2.

Lemma 2. *Let G be a graph and $S = \{B_i = B_{r_i}(s_i)\}_{i\in[n]}$ be a set of n pairwise vertex-disjoint balls in G, and let $E_S \subseteq \binom{S}{2}$ be a subset of pairs of balls $\{B_i, B_j\} \subseteq S$, each of which is associated with a ball $B_{ij} \notin S$ of G which intersects only B_i and B_j in S. Then the graph $H = (S, E_S)$ is a minor of G.*

Proof. Let us fix a total ordering $\prec$ on the vertices of G. As before, all distances are in the graph G, and we write $d(u, v)$ instead of $d_G(u, v)$. For every $B_iB_j \in E_S$ we denote by x_{ij} the center of the ball B_{ij}, and by r_{ij} its radius. We can assume that the centers x_{ij} are chosen so that the radii r_{ij} are minimal (among all balls intersecting only B_i and B_j in S).

We let P^0_{ij} be the shortest path from s_i to x_{ij} which minimizes the sequence of vertices from s_i to x_{ij} with respect to the lexicographic ordering induced by $\prec$. Similarly we let P^1_{ij} be the minimum shortest path from s_j to x_{ij}, with respect to the lexicographic order induced by $\prec$. In order to avoid ambiguity, we assume without loss of generality that $i < j$, and define $P^0_{ji} = P^1_{ij}$ and $P^1_{ji} = P^0_{ij}$. Observe that P^0_{ij} and P^1_{ij} only intersect in x_{ij} (if not, we could replace x_{ij} by a vertex closer from s_i and s_j and reduce the radius r_{ij} accordingly – the new ball B_{ij} would still intersect B_i and B_j, and no other ball of S, and this would contradict the minimality of r_{ij}). We may also assume that $r_i + r_{ij} - 1 \leqslant d(s_i, x_{ij}) \leqslant r_i + r_{ij}$ (otherwise we can again move x_{ij} and decrease r_{ij} accordingly).

For every $i \in [n]$, we define

$$\mathcal{T}_i \coloneqq \bigcup_{j:B_iB_j\in E_S} P^0_{ij}.$$

Claim 1. For every $i \in [n]$, $\mathcal{T}_i$ is a tree.

The proof is the exactly the same as that of Claim 1 in the proof of Lemma 1 (we do not repeat it here).

On the path P^0_{ij}, we let x^0_{ij} be the vertex at distance r_i from s_i, and on the path P^1_{ij}, we let x^1_{ij} be the vertex at distance r_j from s_j (and similarly as before we define $x^0_{ji} = x^1_{ij}$ and $x^1_{ji} = x^0_{ij}$). Note that $r_{ij} - 1 \leqslant d(x^0_{ij}, x_{ij}) \leqslant r_{ij}$ and $r_{ij} - 1 \leqslant d(x^1_{ij}, x_{ij}) \leqslant r_{ij}$, since otherwise we could move x_{ij} and decrease r_{ij} accordingly. In particular, $d(x_{ij}, x^1_{ij}) - 1 \leqslant d(x^0_{ij}, x_{ij})$ and $d(x_{ij}, x^0_{ij}) - 1 \leqslant d(x^1_{ij}, x_{ij})$.

Claim 2. For two pairs of balls $B_iB_k, B_jB_\ell \in E_S$, with $i \neq j$, if P^0_{ik} and $P^0_{j\ell}$ intersect in some vertex y such that $d(y, x^0_{ik}) \leqslant d(y, x^0_{j\ell})$, then $y = x_{ij}$, and $i = \ell$ or $j = k$.

Case 1. Assume first that $d(s_j, x^0_{j\ell}) \leqslant d(s_j, y)$ (i.e. y appears on or after $x^0_{j\ell}$ on $P^0_{j\ell}$). Then we infer that

$$d(x_{j\ell}, x^0_{ik}) \leqslant d(x_{j\ell}, y) + d(y, x^0_{ik}) \leqslant d(x_{j\ell}, y) + d(y, x^0_{j\ell}) = d(x_{j\ell}, x^0_{j\ell}) \leqslant r_{j\ell}.$$

It follows that the ball $B_{j\ell}$ intersects the ball B_i. By the assumption, this means that $i = \ell$, and thus $s_\ell = s_i$ and $x^0_{j\ell} = x^1_{ij}$. If $d(y, x^0_{ik}) = d(y, x^1_{ij})$, then

$$d(x_{ik}, x^1_{ij}) \leqslant d(x_{ik}, y) + d(y, x^1_{ij}) = d(x_{ik}, y) + d(y, x^0_{ik}) \leqslant r_{ik}.$$

This implies that B_j intersects B_{ik}. Thus $j = k$, and $P^0_{ik} = P^0_{ij}$ and $P^0_{j\ell} = P^1_{ij}$, and $y = x_{ij}$ since these two paths have only x_{ij} in common. So we may assume that $d(y, x^0_{ik}) \leqslant d(y, x^1_{ij}) - 1$.

Recall that by definition of x_{ij}, we have $d(x_{ij}, x^0_{ij}) \geqslant d(x_{ij}, x^1_{ij}) - 1$, which implies that

$$d(x^0_{ik}, x_{ij}) \leqslant d(x^0_{ik}, y) + d(y, x_{ij}) \leqslant d(x^1_{ij}, y) - 1 + d(y, x_{ij}) = d(x^1_{ij}, x_{ij}) - 1 \leqslant d(x^0_{ij}, x_{ij}).$$

As a consequence, x^0_{ik} lies on a shortest path from s_i to x_{ij} and since the paths P^0_{ik} and P^0_{ij} are minimal with respect to $\prec$, we have $x^0_{ij} = x^0_{ik}$. But then all the inequalities above are equalities, and in particular $d(x^0_{ik}, x_{ij}) = d(x^0_{ik}, y) + d(y, x_{ij})$. We could then replace B_{ij} by a ball of radius $r_{ij} - d(y, x_{ij})$ centered in y (this ball is contained in B_{ij} and intersects B_i and B_j). By minimality of r_{ij} we have $d(y, x_{ij}) = 0$ and thus $y = x_{ij}$.

Case 2. We may now assume that $d(s_j, x^0_{j\ell}) \geqslant d(s_j, y)$ (i.e. y appears before $x^0_{j\ell}$ on $P^0_{j\ell}$). We then infer that

$$d(s_j, x^0_{ik}) \leqslant d(s_j, y) + d(y, x^0_{ik}) \leqslant d(s_j, y) + d(y, x^0_{j\ell}) = d(s_j, x^0_{j\ell}) \leqslant r_j.$$

So the ball B_{ik} intersects the ball B_j. By the assumption, this means that $j = k$. Note that the ball B centered in y of radius $r_{ij} - d(y, x_{ij})$ intersects B_i, since y is on the path $P^0_{ik} = P^0_{ij}$. The ball B also intersects B_j, because

$$d(s_j, y) \leqslant d(s_j, x^0_{j\ell}) \leqslant r_j \leqslant r_j + r_{ij} - d(y, x_{ij}).$$

The rightmost inequality comes from the fact that $y \in B_j$ (since y appears before $x^0_{j\ell}$ on $P^0_{j\ell}$) and thus $y \notin B_i$ (since $B_i \cap B_j = \varnothing$), and hence $y \in B_{ij}$ (since y is on the path P^0_{ij}), implying $d(y, x_{ij}) \leqslant r_{ij}$. We can then replace the ball B_{ij} by B, which implies that $y = x_{ij}$ by the minimality of r_{ij}. This is in turn implies that B_{ij} intersects B_ℓ, and hence $i = \ell$. This concludes the proof of Claim 2.

As in the proof of Lemma 1, the claim implies that for $i \neq j \in [n]$, $\mathcal{T}_i \cap \mathcal{T}_j = \{x_{ij}\}$ if $B_iB_j \in E_S$ and otherwise the trees $\mathcal{T}_i$ and $\mathcal{T}_j$ are vertex-disjoint. Another direct consequence is that for every $B_iB_j \in E_S$, the vertex x_{ij} is a leaf in at least one of the two trees $\mathcal{T}_i$ and $\mathcal{T}_j$. As before, we can contract the edges of each tree $\mathcal{T}_i$ not incident to a leaf of $\mathcal{T}_i$, and the resulting graph is precisely a graph obtained from $H = (S, E_S)$ by subdividing each edge at most once, and thus H is a minor of G. □

3. Subhypergraphs and density

A *subhypergraph* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph whose vertex set is a subset $X \subseteq V$, and whose edge set is $\{X \cap e \,|\, e \in \mathcal{E}\}$. A *partial hypergraph* of $\mathcal{H}$ is a hypergraph obtained from $\mathcal{H}$ by removing a (possibly empty) subset of the edges. Finally, a *partial subhypergraph* of $\mathcal{H}$ is a partial hypergraph of a subhypergraph of $\mathcal{H}$. The *rank* of $\mathcal{H}$ is the maximum size of an edge of $\mathcal{H}$.

We start with a useful tool, inspired by [13] (see also [3]), itself inspired by the Crossing lemma. Given a graph $G = (V, E)$, we denote by $\mathrm{ad}(G)$ the average degree of G, that is $\mathrm{ad}(G) = 2|E|/|V|$.

Lemma 3. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph of rank at most $k \geqslant 2$ on n vertices, and let $E \subseteq \binom{V}{2}$ be a set of pairs of vertices $\{u, v\}$ of V such that there exists an edge e_{uv} of $\mathcal{H}$ containing u and v. (Note that we allow that $e_{uv} = e_{xy}$ for two different pairs $\{u, v\}$ and $\{x, y\}$.) Then the graph (V, E) contains a subgraph H such that $\mathrm{ad}(H) \geqslant \frac{2|E|}{nek}$ and for every edge uv of H, the corresponding edge e_{uv} of $\mathcal{H}$ contains no vertex from $V(H) - \{u, v\}$.*

Proof. Let $\mathbf{H}$ be the (random) partial subhypergraph of $\mathcal{H}$ obtained by selecting each vertex of $\mathcal{H}$ independently with probability $1/k$, and keeping a single edge (of cardinality 2) between u and v whenever the only selected vertices of e_{uv} are u and v. Then we have

$$\mathbb{E}\left(|V(\mathbf{H})|\right) = \frac{n}{k}, \quad \text{and}$$
$$\mathbb{E}\left(|E(\mathbf{H})|\right) \geqslant |E| \cdot \frac{1}{k^2}\left(1-\frac{1}{k}\right)^{k-2} \geqslant \frac{|E|}{\mathrm{e}k^2},$$

since $k \geqslant 2$. Note that $\mathbf{H}$ is 2-uniform and $\mathbb{E}\left(2|E(\mathbf{H})| - \frac{2|E|}{nek}|V(\mathbf{H})|\right) \geqslant 0$. In particular, there exists a subgraph H of (V,E) such that $\mathrm{ad}(H) \geqslant \frac{2|E|}{nek}$ and for every edge uv of H, the edge e_{uv} of $\mathcal{H}$ contains no vertex from $V(H) - \{u,v\}$, as desired. □

Given a hypergraph $\mathcal{H}$ and a matching $\mathcal{B}$ in $\mathcal{H}$, we define the *packing-hypergraph* $\mathcal{P}(\mathcal{H},\mathcal{B})$ as the hypergraph with vertex set $\mathcal{B}$, in which a subset $\mathcal{B}' \subseteq \mathcal{B}$ is an edge if some edge of $\mathcal{H}$ intersects all the edges in $\mathcal{B}'$ and no other edge of $\mathcal{B}$.

Lemma 4. *Let G be a graph such that each minor of G has average degree at most d, let $\mathcal{H}$ be a ball hypergraph of G, and let $\mathcal{B}$ be a matching of size n in $\mathcal{H}$. For every integer $k \geqslant 2$, the number of edges of cardinality at most k in the packing-hypergraph $\mathcal{P}(\mathcal{H},\mathcal{B})$ is at most*

$$(1+dek)^{k-1} \cdot n.$$

Proof. Let $\mathcal{P}'$ be the partial hypergraph of $\mathcal{P}(\mathcal{H},\mathcal{B})$ induced by the edges of cardinality at most k. Let H be the graph with vertex set $\mathcal{B}$ in which two distinct vertices are adjacent if they are contained in an edge of $\mathcal{P}'$ (i.e. an edge of $\mathcal{P}(\mathcal{H},\mathcal{B})$ of cardinality at most k). Let m be the number of edges of H. Applying Lemma 3 to $\mathcal{P}'$, we obtain a subgraph H' of H of average degree at least $\frac{2m}{nek}$. The vertices of H' correspond to a subset S of pairwise disjoint balls of G (since $\mathcal{B}$ is a matching), and the edges of H' correspond to balls of G intersecting pairs of balls of S.

By Lemma 2, H' is a minor of G, so in particular $\frac{2m}{nek} \leqslant \mathrm{ad}(H') \leqslant d$, and hence $m \leqslant \frac{1}{2}\,dekn$. It follows that H contains a vertex of degree at most dek, and the same is true for every induced subgraph of H (since we can replace $\mathcal{B}$ in the proof by any subset of $\mathcal{B}$). As a consequence, H is $\lfloor dek \rfloor$-degenerate. It is a folklore result that ℓ-degenerate graphs on n vertices have at most $\binom{\ell}{t-1}n$ cliques of size t (see for instance [23, Lemma 18]), and hence there are at most

$$n \cdot \sum_{i=1}^{k} \binom{\lfloor dek \rfloor}{i-1} \leqslant n \cdot (1+dek)^{k-1}$$

cliques of size at most k in H, which is an upper bound on the number of edges of cardinality at most k in $\mathcal{P}(\mathcal{H},\mathcal{B})$. □

4. Fractional packings of balls

We now prove Theorem 4. The proof is inspired by ideas from [17].

Proof of Theorem 4. Let $\mathcal{H}$ be a ball hypergraph of G. Since $\nu^*(\mathcal{H})$ is attained and is a rational number, there exists a multiset $\mathcal{B}$ of p balls of G, such that every vertex $v \in V(G)$ is contained in at most q balls of $\mathcal{B}$, and $\nu^*(\mathcal{H}) = p/q$. We may assume that q is arbitrarily large (by multiplying the multiplicity of all the balls of $\mathcal{B}$ by an arbitrarily large constant), so in particular we may assume that $q \geqslant 2$. We may also assume that G contains at least one edge (i.e. $d \geqslant 1$), otherwise the result clearly holds. Enumerate all the balls in $\mathcal{B}$ as

$B_1, B_2, \dots, B_p$ (keeping multiplicities). We may assume that there is no pair of balls B_i, B_j such that $B_i \subsetneq B_j$ (otherwise we can replace B_j by B_i in $\mathcal{B}$, and we still have a fractional matching). Assume that each ball B_i is centered in s_i and has radius r_i, for each $i \in \{1, \dots, p\}$, and set $d_{ij} = d_G(s_i, s_j)$ for every distinct $i, j \in \{1, \dots, p\}$. Since we cannot have $B_i \subsetneq B_j$ and $B_j \subsetneq B_i$, if B_i and B_j intersect and $B_i \neq B_j$, then $s_i \neq s_j$, and $r_j < r_i + d_{ij}$ and $r_i < r_j + d_{ij}$, and thus $\left\lfloor \frac{r_i - r_j + d_{ij}}{2} \right\rfloor \geqslant 0$ and $\left\lceil \frac{r_j - r_i + d_{ij}}{2} \right\rceil \geqslant 1$. In this case, assuming $i < j$, we define x_{ij} as a vertex on a shortest path between s_i and s_j, at distance $\left\lfloor \frac{r_i - r_j + d_{ij}}{2} \right\rfloor$ from s_i and $\left\lceil \frac{r_j - r_i + d_{ij}}{2} \right\rceil$ from s_j. Note that it follows from this definition that x_{ij} is in $B_i \cap B_j$.

We let $\mathcal{G}$ be the intersection graph of the balls in $\mathcal{B}$, that is $V(\mathcal{G}) = \mathcal{B}$ and two vertices $B_i, B_j \in \mathcal{B} = V(\mathcal{G})$ with $i \neq j$ are adjacent in $\mathcal{G}$ if and only if $B_i \cap B_j \neq \varnothing$. (In particular, there is an edge linking B_i and B_j when B_i and B_j are two copies of the same ball.) Let m be the number of edges of $\mathcal{G}$. Let $\mathcal{B}^*$ denote the multi-hypergraph with vertex set $\mathcal{B}$, where for every vertex of G of the form x_{ij} there is a corresponding edge consisting of the balls in $\mathcal{B}$ that contain x_{ij}. Note that two distinct such vertices could possibly define the same edge, which is why edges in $\mathcal{B}^*$ could have multiplicities greater than 1. The multi-hypergraph $\mathcal{B}^*$ has rank at most q and contains p vertices. Note moreover that the number of pairs of vertices B_i, B_j of $\mathcal{B}^*$ with $i \neq j$ such that there exists an edge of $\mathcal{B}^*$ containing B_i and B_j is precisely m.

Since Lemma 3 clearly holds for multi-hypergraphs as well, we may apply Lemma 3 to $\mathcal{B}^*$, yielding a graph $H = (S, E_S)$ satisfying the following properties:

- $S \subseteq \mathcal{B}$;
- for each edge $B_iB_j \in E_S$ such that the two balls B_i and B_j do not coincide, x_{ij} is contained in B_i and B_j but in no other ball from S, and
- $\text{ad}(H) \geqslant \frac{2m}{peq}$.

We would like to apply Lemma 1 to H but this is not immediately possible, since some balls of S might coincide (recall that $\mathcal{B}$ is a multiset), and therefore the centers of the balls of S might not be pairwise distinct. However, observe that if two balls of S coincide, then by definition the two corresponding vertices of H have degree either 0 or 1 in H (and in the latter case the two vertices are adjacent in H). Indeed, if two balls B_i, B_j of S coincide and B_i is adjacent to B_k in H with $B_k \neq B_i$, then the only balls of S containing x_{ik} are B_i and B_k, contradicting the fact that x_{ik} is also in B_j.

Let $S_1 \subseteq S$ be the subset of balls of S having multiplicity 1 in S. Since no ball of $\mathcal{B}$ is a strict subset of another ball of $\mathcal{B}$, the centers of the balls of S_1 are pairwise distinct. As a consequence of the previous paragraph, if we consider the subgraph H_1 of H induced by S_1, then $\text{ad}(H) \leqslant \max(1, \text{ad}(H_1))$.

By Lemma 1 applied to the set of balls S_1 in G, we obtain that H_1 is a minor of G and thus $\text{ad}(H_1) \leqslant d$. It follows that $\frac{2m}{peq} \leqslant \text{ad}(H) \leqslant \max(1, d) \leqslant d$ (since $d \geqslant 1$). This implies that the average degree $2m/p$ of $\mathcal{G}$ is at most $\mathrm{e}\, dq$. By the Caro-Wei inequality [4, 22] (or Turán's theorem [20]), it follows that $\mathcal{G}$ contains an independent set of size at least

$$\frac{|V(\mathcal{G})|}{\text{ad}(\mathcal{G}) + 1} \geqslant \frac{p}{\mathrm{e}\, dq + 1} = \frac{\nu^*(\mathcal{H})}{\mathrm{e}\, d + 1/q}.$$

An independent set in $\mathcal{G}$ is precisely a matching in $\mathcal{H}$, and thus $\nu(\mathcal{H}) \geqslant \frac{1}{\mathrm{e}\, d + 1/q} \cdot \nu^*(\mathcal{H})$ and $\nu^*(\mathcal{H}) \leqslant (\mathrm{e}\, d + 1/q) \cdot \nu(\mathcal{H})$. Since we can assume that q is arbitrarily large, it follows that $\nu^*(\mathcal{H}) \leqslant \mathrm{e}\, d \cdot \nu(\mathcal{H})$, as desired.

The rest of the result follows from well known results on the average degree of graphs. On the one hand, an easy consequence of Euler's formula is that planar graphs have average degree at most 6. On the other hand, it was proved by Kostochka [15] and Thomason [19] that every K_t-minor-free graph has average degree $O(t\sqrt{\log t})$. □

The *VC-dimension* of a hypergraph $\mathcal{H}$ is the cardinality of a largest subset X of vertices such that for every $X' \subseteq X$, there is an edge e in $\mathcal{H}$ such that $e \cap X = X'$. Bousquet and Thomassé [1] proved the following result.

Theorem 5. *If G has no K_t-minor, then every ball hypergraph $\mathcal{H}$ of G has VC-dimension at most $t-1$.*

A classical result is that for hypergraphs of bounded VC-dimension, $\tau = O(\tau^* \log \tau^*)$. We will use the following precise bound of Ding, Seymour, and Winkler [7].

Theorem 6. *If a hypergraph $\mathcal{H}$ has VC-dimension at most δ, then*

$$\tau(\mathcal{H}) \leqslant 2\delta\tau^*(\mathcal{H}) \log(11\tau^*(\mathcal{H})).$$

Combining Theorems 4, 5, and 6, and using that $\nu^*(\mathcal{H}) = \tau^*(\mathcal{H})$, we obtain Theorem 3 as a direct consequence.

5. Linear bound

In this section we prove Theorem 2. Recall that by Theorem 3, there is a (monotone) function f_t such that $\tau(\mathcal{H}) \leqslant f_t(\nu(\mathcal{H}))$ for every ball hypergraph $\mathcal{H}$ of a K_t-minor-free graph. In the proof, we write d_t for the supremum of the average degree of G taken over all graphs G excluding K_t as a minor. Recall that $d_t = O(t\sqrt{\log t})$ [15, 19].

Let $t \geqslant 1$ be an integer and let $c_t := 2 \cdot (1 + \frac{3}{2}d_t^2\mathrm{e})^{3d_t/2} \cdot f_t(\frac{3}{2}d_t)$. We will prove that every ball hypergraph $\mathcal{H}$ of a K_t-minor-free graph satisfies $\tau(\mathcal{H}) \leqslant c_t \cdot \nu(\mathcal{H})$.

Proof of Theorem 2. We prove the result by induction on $k := \nu(\mathcal{H})$. The result clearly holds if $k = 0$ so we may assume that $k \geqslant 1$. If $k \leqslant \frac{3}{2}d_t$ then by the definition of f_t we have $\tau(\mathcal{H}) \leqslant f_t(\frac{3}{2}d_t) \leqslant c_t \leqslant c_t \cdot k$, as desired.

Assume now that $k \geqslant \frac{3}{2}d_t$ and for every ball hypergraph $\mathcal{H}'$ of a K_t-minor-free graph with $\nu(\mathcal{H}') < k$, we have $\tau(\mathcal{H}') \leqslant c_t \cdot \nu(\mathcal{H}')$. Let G be a K_t-minor-free graph and $\mathcal{H}$ be a ball hypergraph of G with $\nu(\mathcal{H}) = k$. Our goal is to show that $\tau(\mathcal{H}) \leqslant c_t \cdot k$. Note that we can assume that $\mathcal{H}$ is *minimal*, in the sense that no edge of $\mathcal{H}$ is contained in another edge of $\mathcal{H}$ (otherwise we can remove the larger of the two from $\mathcal{H}$, this does not change the matching number nor the transversal number).

Consider a maximum matching $\mathcal{B}$ (of cardinality k) in $\mathcal{H}$. Let $\mathcal{E}_1$ be the set consisting of all the edges of $\mathcal{H}$ that intersect at most $\frac{3}{2}d_t$ edges of $\mathcal{B}$. By Lemma 4, the packing-hypergraph $\mathcal{P}(\mathcal{H}, \mathcal{B})$ contains at most $(1 + \frac{3}{2}d_t^2\mathrm{e})^{3d_t/2} \cdot k$ edges of cardinality at most $\frac{3}{2}d_t$. For each such edge e of $\mathcal{P}(\mathcal{H}, \mathcal{B})$, consider the corresponding subset $\mathcal{B}_e$ of at most $\frac{3}{2}d_t$ edges of $\mathcal{B}$, and the subset $\mathcal{E}_e$ of edges of $\mathcal{H}$ that intersect each ball of $\mathcal{B}_e$, and no other ball of $\mathcal{B}$. Denoting by $\mathcal{H}_e$ the partial hypergraph of $\mathcal{H}$ with edge set $\mathcal{E}_e$, observe that by the maximality of the matching $\mathcal{B}$ we have $\nu(\mathcal{H}_e) \leqslant \frac{3}{2}d_t$ (since in $\mathcal{B}$, replacing the edges of $\mathcal{B}_e$ by a matching of $\mathcal{E}_e$ again gives a matching of $\mathcal{H}$). It follows that $\tau(\mathcal{H}_e) \leqslant f(\frac{3}{2}d_t)$. And thus, if we denote by $\mathcal{H}_1$ the partial hypergraph of $\mathcal{H}$ with edge set $\mathcal{E}_1$, we have

$$\tau(\mathcal{H}_1) \leqslant (1 + \tfrac{3}{2}d_t^2\mathrm{e})^{3d_t/2} \cdot f(\tfrac{3}{2}d_t) \cdot k = \tfrac{1}{2}c_t \cdot k.$$

Consider now the subset $\mathcal{E}_2$ consisting of all the edges of $\mathcal{H}$ that intersect more than $\frac{3}{2}d_t$ edges of $\mathcal{B}$, and let $\mathcal{H}_2$ be the partial hypergraph of $\mathcal{H}$ with edge set $\mathcal{E}_2$. Note that $\mathcal{E}_1$ and $\mathcal{E}_2$

partition the edge set of $\mathcal{H}$ and thus $\tau(\mathcal{H}) \leqslant \tau(\mathcal{H}_1) + \tau(\mathcal{H}_2)$. Let $\mathcal{B}_2$ be a maximum matching in $\mathcal{H}_2$, and let $\ell = \nu(\mathcal{H}_2) = |\mathcal{B}_2|$. Let H be the (bipartite) intersection graph of the edges of $\mathcal{B} \cup \mathcal{B}_2$, i.e. each vertex of H corresponds to an edge of $\mathcal{B} \cup \mathcal{B}_2$, and two vertices are adjacent if the corresponding edges intersect. (The graph is bipartite because $\mathcal{B}$ and $\mathcal{B}_2$ are matchings.)

Note that since H is bipartite, for every two distinct edges BB' and CC' of H, the sets $B \cap B'$ and $C \cap C'$ are disjoint. Moreover, no ball of $\mathcal{B} \cup \mathcal{B}_2$ is a subset of another ball of $\mathcal{B} \cup \mathcal{B}_2$. So, enumerating the balls in $\mathcal{B} \cup \mathcal{B}_2$ as $B_1, B_2, \ldots, B_n$, we can choose, for each edge B_iB_j of H with $i < j$, a vertex $x_{ij} \in B_i \cap B_j$ which is at distance $\left\lfloor \frac{r_i - r_j + d_{ij}}{2} \right\rfloor$ from s_i, and at distance $\left\lceil \frac{r_j - r_i + d_{ij}}{2} \right\rceil$ from s_j, where $d_{ij} = d_G(s_i, s_j) \leqslant r_i + r_j$, and $B_i = B_{r_i}(s_i)$ and $B_j = B_{r_j}(s_j)$, and such that the only balls of $\mathcal{B} \cup \mathcal{B}_2$ containing x_{ij} are B_i and B_j. By Lemma 1, H is a minor of G and thus has average degree at most d_t. On the other hand, the vertices of H corresponding to the edges of $\mathcal{B}_2$ have degree at least $\frac{3}{2}d_t$ in H, and thus

$$\tfrac{3}{2}d_t \cdot \ell \leqslant \tfrac{1}{2}\,\mathrm{ad}(H)(k+\ell) \leqslant \tfrac{1}{2}d_t \cdot (k+\ell),$$

where the central term counts the number of edges of H. It follows that $\nu(\mathcal{H}_2) = \ell \leqslant \frac{k}{2}$, and thus by the induction hypothesis we have $\tau(\mathcal{H}_2) \leqslant c_t \cdot \nu(\mathcal{H}_2) \leqslant c_t \cdot \frac{k}{2}$. As a consequence,

$$\tau(\mathcal{H}) \leqslant \tau(\mathcal{H}_1) + \tau(\mathcal{H}_2) \leqslant \tfrac{1}{2}c_t \cdot k + c_t \cdot \tfrac{k}{2} = c_t \cdot k,$$

which concludes the proof of Theorem 2. □

6. Conclusion

The proof of Theorem 2 gives a bound of the order of $\exp(t \log^{3/2} t)$ for the constant c_t. It would be interesting to improve this bound to a polynomial in t.

It is also natural to wonder whether Theorem 2 remains true in a setting broader than proper minor-closed classes. Natural candidates are graphs of bounded maximum degree, graphs excluding a topological minor, k-planar graphs, classes with polynomial growth (meaning that the size of each ball is bounded by a polynomial function of its radius, see e.g. [16]), and classes with strongly sublinear separators (or equivalently, classes with polynomial expansion [9]). We now observe that in all these cases, the associated ball hypergraphs do not satisfy the Erdős-Pósa property, even if all the balls have the same radius. That is, we can find r-ball hypergraphs in these classes with bounded ν and unbounded τ. Our construction shows that this is true even in the seemingly simple case of subgraphs of a grid with all diagonals (i.e. strong products of two paths).

Fix two integers k, ℓ with $k \geqslant 3$, and ℓ sufficiently large compared to k and divisible by $2(\binom{k}{2} - 1)$. Given k vertices $v_0, v_1, \ldots v_{k-1}$, an *ℓ-broom with root v_0 and leaves $v_1, \ldots, v_{k-1}$* is a tree T of maximum degree 3 with root v_0 and leaves $v_1, \ldots, v_{k-1}$ such that

(1) each leaf is at distance ℓ from the root v_0,
(2) the ball of radius $\ell/2$ centered in v_0 in T is a path (called the *handle* of the broom), and
(3) the distance between every two vertices of degree 3 in T is sufficiently large compared to k.

We now construct a graph $G_{k,\ell}$ as follows. We start with a set X of k vertices $x_1, \ldots, x_k$, and a path of $\binom{k}{2}$ vertices with vertex set $Y = \{y_{\{i,j\}} \mid 1 \leqslant i < j \leqslant k\}$, disjoint from X. We then subdivide each edge of the latter path $\frac{\ell}{2}\frac{1}{\binom{k}{2}-1} - 1$ times, so that the subdivided path has length $\ell/2$. Finally, for each $1 \leqslant i \leqslant k$, we add an ℓ-broom T_i with root x_i and leaves $Y = \{y_{\{i,j\}} \mid j \neq i\}$.

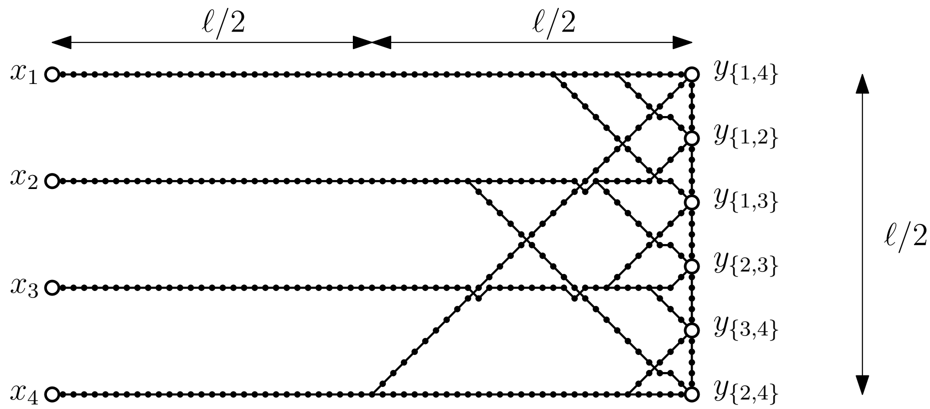

FIGURE 1. An embedding of the graph $G_{4,\ell}$ in the 2-dimensional grid with all diagonals (the grid itself is not depicted for the sake of clarity).

We first claim that $G_{k,\ell}$ is a subgraph of the 2-dimensional grid with all diagonals (i.e. the strong products of two paths). To see this, place X on a single column on the left, and Y on another column on the right (in the sequence given by the path), at distance ℓ from the column of X, then draw each of the brooms in the plane (with crossings allowed). Since the distance between two vertices of degree 3 in a broom is sufficiently large compared to k, we can safely embed each topological crossing in the strong product of two edges (see Figure 1 for an example).

Let $\mathcal{H}_{k,\ell}$ be the ℓ-ball hypergraph of $G_{k,\ell}$ obtained by considering all the balls of radius ℓ in $G_{k,\ell}$. We first observe that $\nu(\mathcal{H}_{k,\ell}) = 1$: this follows from the fact that each ball of radius ℓ centered in a vertex that does not belong to the handle of a broom contains all the vertices of Y, while every two vertices on the handles of two brooms T_i and T_j are at distance at most ℓ from $y_{\{i,j\}}$. Finally, for every two vertices x_i and x_j of X, note that $y_{\{i,j\}}$ is the unique vertex of $G_{k,\ell}$ lying at distance at most ℓ from x_i and x_j, and thus $\tau(\mathcal{H}_{k,\ell}) \geqslant \frac{k}{2}$. It follows that there is no function f such that $\tau(\mathcal{H}) \leqslant f(\nu(\mathcal{H}))$ for every ball hypergraph of a subgraph of the strong product of two paths (even when all the balls in the ball hypergraph have the same radius).

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Laboratoire G-SCOP, CNRS, Univ. Grenoble Alpes, Grenoble, France.
Email address: nicolas.bousquet@grenoble-inp.fr

Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium.
Email address: wcamesva@ulb.ac.be

Laboratoire G-SCOP, CNRS, Univ. Grenoble Alpes, Grenoble, France.
Email address: louis.esperet@grenoble-inp.fr

Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium.
Email address: gjoret@ulb.ac.be

Algorithms Research Group, University of Bergen, Bergen, Norway.
Email address: william.lochet@gmail.com

Département de Mathématique, Université Libre de Bruxelles, Brussels, Belgium.
Email address: camuller@ulb.ac.be

Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium, and Laboratoire G-SCOP, CNRS, Univ. Grenoble Alpes, Grenoble, France.
Email address: francois.pirot@grenoble-inp.fr