A Polynomial Time Algorithm for the k-Disjoint Shortest Paths Problem

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Abstract

The disjoint paths problem is a fundamental problem in algorithmic graph theory and combinatorial optimization. For a given graph G and a set of k pairs of terminals in G, it asks for the existence of k vertex-disjoint paths connecting each pair of terminals. The proof of Robertson and Seymour [JCTB 1995] of the existence of an n^3 algorithm for any fixed k is one of the highlights of their Graph Minors project. In this paper, we focus on the version of the problem where all the paths are required to be shortest paths. This problem, called the disjoint shortest paths problem, was introduced by Eilam-Tzoreff [DAM 1998] where she proved that the case $k = 2$ admits a polynomial time algorithm. This problem has received some attention lately, especially since the proof of the existence of a polynomial time algorithm in the directed case when $k = 2$ by Bérczi and Kobayashi [ESA 2017]. However, the existence of a polynomial algorithm when $k = 3$ in the undirected version remained open since 1998.

In this paper we show that for any fixed k, the disjoint shortest paths problem admits a polynomial time algorithm. In fact for any fixed C, the algorithm can be extended to treat the case where each path connecting the pair (s, t) has length at most $d(s, t) + C$.

1 Introduction

Given a graph G and a set of pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, the *Vertex-Disjoint Paths Problem* asks whether there exists a set of vertex-disjoint paths $P_1, \ldots, P_k$ such that every P_i is an (s_i, t_i) path. This is a classical NP-hard [8] problem in graph theory with applications to VLSI-layouts and networks problem [10] which has been extensively studied. The proof of the existence of a $O(n^3)$ algorithm for any fixed k by Robertson and Seymour [9] is one of the highlights from the Graph Minors project, and the running time has later been improved to $O(n^2)$ [1]. In the directed case, the problem is NP-hard even for $k = 2$ [7], but some results are known for special classes of digraphs like acyclic digraphs [7], planar digraphs [11] or tournaments [5].

One natural question is, given an instance of the Vertex-Disjoint Path Problem, to find a solution which minimises the sum of the lengths of the P_i. This problem appears to be much harder, as only the case $k = 2$ was recently solved by Björklund and Husfeldt [4]. In fact, even deciding if the problem admits an optimal solution, i.e where every P_i is a shortest path between s_i and t_i is open for $k \geq 3$.

This problem was first considered by Eilam-Tzoreff [6] 20 years ago. In the same paper, she gave an algorithm for the case $k = 2$ and conjectured that a polynomial algorithm exists for any fixed k, both in the directed and undirected setting. This problem has received some attention lately. In particular Bérczi and Kobayashi [3] gave a nice proof that the directed version admits a polynomial time algorithm when $k = 2$. They also show that the algorithm of Schrijver [11] can be used to prove Eilam-Tzoreff's conjecture when the input (di)graph is planar. The goal of this paper is to solve this problem for any k in the undirected case.

Theorem 1. *For any fixed integer k, there exists an algorithm running in $n^{O(k^{5^k})}$ time that decides, given a graph G and k pairs of vertices $(s_1, t_1), \dots (s_k, t_k)$, of the existence of k internally vertex-disjoint paths $P_1, \dots, P_k$ such that each P_i is a shortest (s_i, t_i)-path.*

We also show that this problem is $W[1]$-hard parameterized by k, which means that we cannot hope to remove the dependency in k in the exponent. However, we can extend the previous result to the case where paths are not required to be shortest paths, but of length at most $d(s_i, t_i) + C$, where C is a fixed constant.

Corollary 2. *For any fixed integers k and C , there exists an algorithm running in $n^{O((Ck)^{5^k})}$ that decides, given a graph G and k pairs of vertices $(s_1, t_1), \dots (s_k, t_k)$, of the existence of k internally vertex-disjoint paths $P_1, \dots, P_k$ such that each P_i is a path of length at most $d(s_i, t_i) + C$ between s_i and t_i.*

2 Preliminaries

For any integer k, $[k]$ denotes the set of integers between 1 and k, and for any integer $j \leq k$, $[j..k]$ denotes the set of integers between j and k. The *length* of a path P correspond to the number of edges in the path, and given two vertices x and y in the same connected component, $d(x, y)$ denotes the minimal length of a path between x and y.

A graph G is said to be a *k-shortest* graph if there exists k partitions of G $(V^1_1, \dots, V^1_{l_1})$, $(V^2_1, \dots, V^2_{l_2})$, $\dots$, $(V^k_1, \dots, V^k_{l_k})$ such that xy is an edge of G implies that there exists $i \in [k]$ and $j \in [l_i - 1]$ such that $x \in V^i_j$ and $y \in V^i_{j+1}$. Moreover, if $x \in V^i_j$ and $y \in V^i_l$, for $i \in [k]$, $j, l \in [l_i]$ and $|j - l| > 1$ then $xy \notin E(G)$. Intuitively, one way to obtain a k-shortest graph is to start from any graph, doing k breadth-first searches from different vertices and removing all the edges which are not between two consecutive levels of at least one of the BFS. For a k-shortest graph G, we associate naturally k colours to the edges of G as follows: any edge between $x \in V^i_j$ and $y \in V^i_{j+1}$ is said to be of colour i. Note that the same edge can be of different colours. Moreover, each colour i defines a partial order on the vertices of G $\leq_i$ as follows: $x \leq_i y$ if $x \in V^i_j$ and $y \in V^i_r$ with $j \leq r$. This naturally defines an orientation of the edges of colour i. Note that the same edge can have two different orientations for two different colours. Let G be a k-shortest graph and r and i be two indices. We say that a path $P_i = x_1, \dots, x_r$ is a *path of colour i* if for every $j \in [r - 1]$, $x_j x_{j+1}$ is an edge of colour i and $x_j \leq_i x_{j+1}$. Note that, since whenever $u \in V^i_j$ and $v \in V^i_l$ with $|j - l| > 1$ there is no edge between u and v, any path of colour i between x and y is also a shortest path in G. Moreover, concatenating two paths of colour i also gives a path of colour i. By convention, we consider the paths of colour i to be oriented from the endpoint which belongs to the part of $(V^i_1, \dots, V^i_{l_i})$ with the lowest index to the endpoint with the largest one. In particular, an *(x, y)-path of colour i* is a path of colour i between x and y oriented from x to y. For a directed path P and two vertices x and y belonging to this path, $P[x, y]$ denotes the subpath of P from x to y. By convention, if y is before x along P, then $P[x, y]$ will be the empty path. The *length* of a path is its number of edges. A *path-partition* of a path P is a set of internally vertex-disjoint subpaths of

P such that concatenation of all the paths gives P. Let Q_1, Q_2 be two different path partitions of the same path P, the *intersection* of Q_1 and Q_2 is the path partition of P obtained as follows: If S is the set of vertices which are endpoints of paths of either Q_1 or Q_2, then the intersection of Q_1 and Q_2 consists of all the subpaths of P between vertices of S which are consecutive along P. Note that the number of paths in the intersection of Q_1 and Q_2 is at most the sum of the number of paths in Q_1 and Q_2. Moreover, every path in the intersection is a subpath of some path in Q_1 and some path in Q_2. For an oriented edge $e = xy$, x is called the *tail* of e and is denoted as $t(e)$ and y the *head*, denoted as $h(e)$.

Let G be a k-shortest graph, $(s_1, t_1), \dots (s_l, t_l)$ a set of l pairs of vertices and c a function from $[l]$ to $[k]$, the *k-DSP* problem defined by G, the (s_i, t_i) and c is the problem of finding a set of internally vertex-disjoint paths $P_1, \dots, P_l$ such that for any $i \in [l]$, P_i is a path of colour $c(i)$ between s_i and t_i. The (s_i, t_i) will be referred to as *requests*. The following lemma shows that we can reduce Eilam-Tzoreff's question to solving an instance of k-DSP.

Lemma 3. *Let G be a graph and $(s_1, t_1), \dots, (s_k, t_k)$ be a set of k pairs of vertices in G. Let G' be the k-shortest graph obtained from G by taking for each i $(V_1^i, \dots, V_{l_i}^i)$ the partition obtained by doing a breath-first-search from s_i, and removing the edges which are not between two consecutive levels of some BFS. There exists a set of internally vertex-disjoint paths $P_1, \dots, P_k$ such that for every $i \in [k]$, P_i is a shortest path between s_i and t_i if and only if the k-DSP problem defined by G', the (s_i, t_i) and the identity function $c : [k] \to [k]$ has a solution.*

Proof. The proof follows from the fact that, for every i and l, the set V_l^i corresponds to the set of vertices at distance l from s_i. Therefore, a shortest path in G between s_i and some vertex $x \in V_l^i$ is a path of colour i from s_i to x in G' and vice versa. □

The main contribution of this paper is to prove the following result, which implies Theorem 1

Theorem 4. *Let G be a k-shortest graph, $(s_1, t_1), \dots, (s_l, t_l)$ a set of l pairs of vertices and c a function from $[l]$ to $[k]$. There exists an algorithm running in time $n^{O(l^{5^k})}$ deciding if the problem of k-DSP defined by G, the (s_i, t_i) and c has a solution.*

In fact, by noting that a path of length at most $d(s_i, t_i) + C$ between s_i and t_i uses at most C edges which are not edges between consecutive levels of the BFS starting in s_i, trying all $\binom{n^2}{kC}$ choices for these edges allows us to reduce the problem of Corollary 2 to a k-DSP problem with at most $k \cdot C$ pairs.

An interesting case is when $k = 1$. The problem then reduces to the problem of directed disjoint paths in acyclic digraphs by orienting all the edges of G from each set V_i to V_{i+1}. Therefore, the algorithm of Fortune et al. [7] gives a solution in $n^{O(l)}$. As noted in [2], we can also reduce the problem of directed-disjoint paths in acyclic digraphs to 1-DSP, which implies the following theorem:

Theorem 5. *The* 1*-DSP problem is* $W[1]$*-hard parameterized by the number of requests.*

Proof. Consider an instance D and $(s_1, t_1), \dots, (s_k, t_k)$ of disjoint paths in acyclic digraphs and $v_1, \dots, v_n$ a topological ordering of the vertices of D. Let D' be the digraph obtained from D by subdividing each arc (v_i, v_j) $j - i - 1$ times. By doing so, every (v_i, v_j)-path in D' now has length exactly $i - j - i$, which means that every path is a shortest path. The underlying graph G' obtained by forgetting the orientation of the arcs in D' is a 1-shortest graph, as all the edges appear in the

BFS starting from v_1, and a shortest path between v_i and v_j still corresponds to a (v_i, v_j)-path in D. This means that solving the 1-DSP problem defined by G' and the (s_i, t_i) gives a solution to the original instance of disjoint paths in D, which ends the proof as disjoint paths in acyclic digraphs is $W[1]$-hard [12][1]. □

The rest of the paper is devoted to the proof of Theorem 4. The main idea behind the proof is to reduce to a set of $O(l^{5^k})$ requests such that, roughly speaking, for each pair of requests of different colours, no pair of shortest paths solving these requests can intersect. Once we have achieved this, it means that the only potential conflicts arise for pairs of requests of the same colour. However, the edges of each colour class can be seen as an acyclic digraph, and we can adapt the algorithm of Fortune et al. to that case. The main difficulty lies in reducing to these $O(l^{5^k})$ requests. To achieve this, we need to look at a potential solution to the original k-DSP problem and say that, for each pair of paths of different colours in this solution, there is a way to partition each of these paths into a finite number of subpaths, such that the endpoints of each pair of subpaths now correspond to requests that can never intersect. The next two sections are devoted to this task. In particular, it is devoted to the understanding of the structure of bi-coloured edges.

3 Bi-coloured components

Let G be a k-shortest graph and i, j two integers in $[k]$. Consider $G^+_{i,j}$ (resp. $G^-_{i,j}$), the graph induced by the edges xy of G of colour both i and j such that $x \leq_i y$ and $x \leq_j y$ (resp. $x \leq_i y$ and $y \leq_j x$). A *bi-coloured* component of colours i, j is a connected component of $G^+_{i,j}$ or $G^-_{i,j}$. Note that $G^+_{i,j}$ and $G^-_{i,j}$ play identical roles, as reversing the order of the partition $(V^j_1, \ldots, V^j_{l_j})$ transforms the k-shortest graph G into a k-shortest graph G' where every component of $G^-_{i,j}$ becomes a component of $G^+_{i,j}$ and vice-versa. Bi-coloured components will play an essential role in order to decompose a k-DSP problem into a set of $O(l^{5^k})$ requests such that the intersections between request of different colours behave nicely[2]. The first property we need to prove is that for any path P_i of colour i and bi-coloured component C of colour i and j, then $P_i \cap C$ is a subpath of P_i. Let us first show some properties of the bi-coloured components.

Lemma 6. *Let G be a k-shortest graph, i, j two indices in $[k]$, and S some component of $G^+_{i,j}$. There exists a constant C_S such that for any vertex $x \in S$, if $x \in V^i_r$, then $x \in V^j_{r+C_S}$.*

Proof. Let x be any vertex belonging to S. Let r and t be the constants such that $x \in V^i_r$ and $x \in V^j_t$ and define $C_S = t - r$. Let y be another vertex of S. By definition of S, there exists a path P in $G^+_{i,j}$ between x and y. Let s_1 be the number of edges of P which are used positively for the order induced by the colour i when going from x to y, and s_2 the number of edges used negatively. By definition of this order, we have that $y \in V^i_{r+s_1-s_2}$.

Because the orders induced by the colours i and j are the same on S, we also have that $y \in V^j_{t+s_1-s_2}$, which ends the proof. □

Let us now show the following properties of paths of colour i.

[1] The hardness proof of Slivkins, while stated for the arc-disjoint version works also for the vertex-disjoint one.

[2] We will define what we mean by nicely in the next section.

Proposition 7. *Let G be a k-shortest graph. Suppose $x \in V_r^i$ and $y \in V_t^i$ for some $i \in [k]$ and $r, t \in [l_i]$ with $r > t + 1$. If there exists a path in G of length $r - t$ between x and y, then this path is a path of colour i from y to x.*

Proof. Let $P = x_1, \ldots, x_s$ with $x_1 = y$, $x_s = x$ and $s = r - t + 1$ be a path of length $r - t$ between x and y. For every $j \in [s]$, let i_j be the integer such that $x_j \in V_{i_j}^i$. We know that for any $j \in [2..s]$, $i_j \leq i_{j-1} + 1$ as x_j and x_{j-1} are adjacent. However, $i_1 = t$, $i_s = r$ and $s = r - t + 1$. This means that $i_j = i_{j-1} + 1$ for every $j \in [2..s]$ and all the edges of P are edges of colour i. □

Proposition 8. *Let G be a k-shortest graph, i, j two indices of $[k]$ and x and y two vertices of G. If there exists a path P_i of colour i between x and y and a path P_j of colour j between x and y, then P_j is also a path of colour i and P_i is also a path of colour j.*

Proof. We know that P_i and P_j are shortest paths between x and y, and in particular have the same length. The result follows by applying Proposition 7 to the paths P_j and P_i. □

We are ready to prove the following lemma, which shows how paths of colour i interact with $G_{i,j}^+$.

Lemma 9. *Let G be a k-shortest graph, i, j two indices in $[k]$, P_i a path of colour i and S some bi-coloured component of colours i, j. The intersection of P_i and S is a subpath of P_i.*

Proof. Let S be a component of $G_{i,j}^+$ and suppose that P_i does not intersect S along a single subpath. This means that we can find a subpath P' of P_i of colour i between two vertices x, y of S such that P' uses no edge of S. Suppose x is the first endpoint of this path, y the last and l denote the length of P'. Then $x \in V_r^i$ and $y \in V_{r+l}^i$. However, by Lemma 6, we know that there exists a constant C_S such that, since both x and y belong to S, $x \in V_{r+C_S}^j$ and $y \in V_{r+l+C_S}^j$. By Proposition 7, this implies that P' is also a path of colour j, and thus $P' \in S$. The case where S is a component of $G_{i,j}^-$ is very similar and thus omitted. □

Let G be a k-shortest graph, i, j two indices in $[k]$, P_i a path of colour i and P_j a path of colour j. We say that P_i and P_j are in *conflict* if there exists a bi-coloured component S of colour i, j such that the intersection of P_i with S is a (s_1', t_1')-path and the intersection of P_j with S is an (s_2', t_2')-path for some $s_1', s_2', t_1', t_2' \in S$, with the property that there exists a (s_1', t_1')-path P_i' of colour i and an (s_2', t_2')-path P_j' of colour j using at least one vertex outside of s_1', s_2', t_1', t_2' in common. The component S will be called a *conflicting component* for P_i and P_j. By convention, two paths of the same colour are never conflicting. The following lemma is the main ingredient of our Algorithm. It shows that for two paths of colour i and j, there is at most one conflicting component.

Lemma 10. *Let G be a k-shortest graph, i, j two indices in $[k]$, P_i a path of colour i and P_j a path of colour j. Suppose S is a conflicting component for the paths P_i and P_j, then P_i and P_j do not have any vertex in common outside S.*

Proof. Again, we can assume that S is a component of $G_{i,j}^+$ by potentially reversing the order i. Suppose that the intersection of P_i with S is an (s_1', t_1')-path and the intersection of P_j with S is an (s_2', t_2')-path for some $s_1', s_2', t_1', t_2' \in S$. We prove the lemma by contradiction, distinguishing several cases depending on which part of P_i and P_j (compared to S) the intersection lies on.

Suppose first that the intersection of P_i and P_j lies after S for both paths. By definition of conflicting components, we know that there exists a vertex $x \in S$ such that there exists a path P_1 of colour i from x to t'_1 and a path P_2 of colour j from x to t'_2. Let z be the first vertex belonging to the intersection of $P_i \cap P_j$ after S. By applying Proposition 8 to the path obtained by concatenating P_1 and $P_i[t'_1, z]$ and the one obtained by concatenating P_2 and $P_j[t'_2, z]$, we get that these two paths are both of colour i and j. In particular this implies that the edges of these paths belong to $G^+_{i,j}$ and $z \in S$, which is a contradiction.

Suppose now that the intersection of P_i and P_j lies before S on P_i and after S on P_j. By definition of conflicting components, we know that there exists a vertex $x \in S \setminus \{s'_1, s'_2, t'_1, t'_2\}$ such that there exists a path P_1 of colour i from s'_1 to x and a path P_2 of colour j from x to t'_2. Because $x \notin \setminus\{s'_1, s'_2, t'_1, t'_2\}$, the two paths P_1 and P_2 have both length at least 1. Moreover, they are both paths of colour i and j, and with the same orientation associated to these colours. Let z denote the first vertex in the intersection of P_i and P_j before S on P_i and after S on P_j and consider the path $H_1 = P_i[z, s'_1]P_1P_2$. H_1 is a path of colour i between z and t'_2, and thus $|H_1| = |P_j[t'_2, z]|$. Likewise, we can show that $|P_i[z, s'_1]P_1| = |P_2P_j[t'_2, z]|$, which gives us a contradiction.

The other cases are symmetrical. □

Let us now explain how the previous lemma will be used. Remember that our goal is to reduce an original instance of k-DSP with l requests to one with $O(l^{5^k})$ requests such that for every pair of requests of different colours, no pair of shortest paths solving these requests can intersect. Suppose P_i and P_j are two paths of different colours in a solution of the original k-DSP which are in conflict. Let S denote the conflicting component for P_i and P_j. Because of Lemma 9, we know that the intersection of P_i and P_j with S are subpaths. For every $a \in \{i, j\}$, consider the path partition (P^1_a, P^2_a, P^3_a) of P_a, where P^2_a is the subpath of P_a on S, P^1_a the part of P_a before this component, and P^3_a the part after. What Lemma 10 roughly says is that the endpoints of P^1_i, P^1_j, P^3_i and P^3_j correspond to requests such that no pair of shortest paths solving these requests can intersect, which is exactly what we wanted. This is not true for the requests associated to P^2_j and P^2_i, however since they both belong to a bi-coloured component S, these two requests can be considered of the same colour.

Surprisingly, the case where P_i and P_j are not in conflict is harder to handle. This is the goal of the next section, but let us first show sufficient conditions to guarantee the existence of a conflicting component for a pair of paths.

Lemma 11. *Let G be a $[k]$-shortest graph and i, j two different indices in $[k]$, P_i a path of colour i and P_j a path of colour j. If P_i and P_j have three common vertices, then they have a conflicting component.*

Proof. Let x_1, x_2 and x_3 be three vertices in $P_i \cap P_j$. We claim that they belong to the same bi-coloured component. Indeed, consider the subpaths of P_i and P_j between x_1 and x_2. By Proposition 8 they are both paths of colour i and j and belong to the same component S. Without loss of generality, suppose S is a component of $G^+_{i,j}$ and $x_1 \leq_i x_2$. If x_3 belongs to $P_i[x_1, x_2]$ or $P_j[x_1, x_2]$, we have that $x_3 \in S$, which ends the proof of the claim. Assume now x_3 appears after x_2 on P_i, the other case being symmetrical. If it appears after x_2 on P_j, then the same argument shows that $P_i[x_2, x_3]$ is also a path of S.

Suppose now that x_3 appears before x_1 on P_j. In that case we have that $P_j[x_3, x_2]$ and $P_i[x_2, x_3]$ are both shortest path, and thus have the same size. However, this implies that $P_j[x_3, x_1]$ is strictly shorter than $P_i[x_1, x_3]$, which is a contradiction.

Now that we know that x_1, x_2 and x_3 belong to the same bi-coloured component, let us show that this component is a conflicting component for P_i and P_j. Indeed, since x_1, x_2 and x_3 belong to the same component, they either appear in the same order on the paths P_i and P_j if S is a component of $G^+_{i,j}$ or in reverse order if S is a component of $G^-_{i,j}$. In both cases, the vertex in the middle is the same in both paths, and this implies that P_i and P_j are conflicting on this component. □

4 Blind Paths

As explained earlier, if P_i and P_j are two paths of the solution of some k-DSP problem which are in conflict, then Lemma 10 allows us to show that the paths P_i and P_j can be decomposed into a finite set of requests such that each pair of requests of different colours are without any possible intersection, meaning that for any shortest path P solving the first request and P' solving the second request, $P \cap P' = \emptyset$. Moreover, finding these decompositions only requires to guess the conflicting component and the intersection of P_i and P_j. Unfortunately, it is not true that any positive instance of k-DSP has a solution where every pair of paths of different colours are in conflict. For this purpose we need the definition of blind pair of paths.

Let P_i be some (s_i, t_i)-path of colour i and P_j some (s_j, t_j)-path of colour j which are internally vertex-disjoint. We say that P_i *sees* P_j if there exists an internal vertex x of P_i such that there exists a path of colour i from x to t_i which intersects $P_j \setminus \{s_j, t_j\}$. We say that the pair P_i and P_j is *blind* if P_j does not see P_i and P_i does not see P_j.

Note that if $|P_i| = 2$, then P_i does not see, or is not seen, by any other path P_j. Note also that, if P_i and P_j are blind, then it is possible to find a (s_j, t_j)-path of colour j P'_j and a (s_i, t_i)-path of colour i P'_i such that $P'_i \cap P'_j \neq \emptyset$. In that sense, blind paths is a weaker notion than what we could obtain from Lemma 10 outside of the conflicting component.

The following lemma shows how to use Lemma 10 to find blind paths from conflicting paths.

Lemma 12. *Let G be a k-shortest graph, i, j two integers in $[k]$, P_i a path of colour i and P_j a path of colour j which are internally vertex-disjoint. There exists a path partition L_i of P_i and a path partition L_j of P_j, both of size at most 9 with the following properties:*

- *All the paths of L_a are paths of colour a for $a \in \{i, j\}$*
- *For any pair of paths $H_i \in L_i$ and $H_j \in L_j$, then either H_i and H_j are paths of the same bi-coloured component of colour i, j or H_i does not see H_j.*

Proof. Suppose P_i is an (s_i, t_i)-path and P_j is an (s_j, t_j) path. If P_i does not see P_j, then $L_i = \{P_i\}$ and $L_j = \{P_j\}$ satisfy the properties of the lemma. Suppose now P_i sees P_j and let x_1 denote the last vertex of P_i from which there exists a path Q^1_i of colour i to t_i which uses some vertex of $P_j \setminus \{s_j, t_j\}$. Because P_i and P_j are internally vertex disjoint, $x_1 \neq t_i$. Let x'_1 denote the vertex just after x_1 on P_i. Note that $P_i[x'_1, t_1]$ does not see P_j.

Now let x_2 denote the last vertex of $P_i[s_i, x_1]$ from which there exists a path of colour i to x_1 which uses some vertex of $P_j \setminus \{s_j, t_j\}$. Again, if this vertex does not exist, then $L_i = \{P_i[s_i, x_1], (x_1, x'_1), P_i[x'_1, t_i]\}$ and $L_j = \{P_j\}$ satisfy the properties of the lemma. Suppose from now on that x_2 exists and let x'_2 be the vertex just after x_2 on P_i. Since P_i and P_j are internally vertex-disjoint, $x_2 \neq x_1$ and thus $x'_2 \in P_i[s_i, x_1]$. Again, note that $P_i[x'_2, x_1]$ does not see P_j.

Let x_3 denote the last vertex of $P_i[s_i, x_2]$ from which there exists a path Q_i^3 of colour i to x_2 which uses some vertex of $P_j \setminus \{s_j, t_j\}$. Again, we can assume that this vertex exists or $L_i = \{P_i[s_i, x_2], (x_2, x_2'), P_i[x_2', x_1], (x_1, x_1'), P_i[x_1', t_i]\}$ and $L_j = \{P_j\}$ satisfy the properties of the lemma. Let $x_3' \in P_i[s_i, x_2]$ denote the vertex just after x_3 on P_i. Again, note that $P_i[x_3', x_2]$ does not see P_j.

Note that for any internal vertex $x \in Q_i^1$ and $y \in Q_i^2$, $y <_i x$. This implies that the intersection of Q_i^1 and Q_i^2 is equal to x_1, and the same argument applies for $Q_i^3 \cap Q_i^2$ and $Q_i^3 \cap Q_i^1$. This means that the paths P_j and $P_i' = P_i[s_i, x_3]Q_i^3 Q_i^2 Q_i^1$ intersect on at least 3 vertices and thus are conflicting by Lemma 11. Let S denote the conflicting component of P_i' and P_j.

Suppose first that none of the s_i, t_i, s_j, t_j belong to S and denote by e_i the last edge of P_i' without both endpoints in S, e_j the last edge of P_j before S, h_i the first edge of P_i' after S and h_j the first edge of P_j after S.

Claim 12.1. *All the pairs of paths among $P_i'[s_i, t(e_i)]$,e_i, $P_i'[h(e_i), t(h_i)]$, h_i, $P_i'[h(h_i), t_i]$, $P_j[s_j, t(e_j)]$, e_j, $P_j[h(e_j), t(h_j)]$, h_j and $P_j[h(h_j), t_j]$ are blind, except from $P_j[h(e_j), t(h_j)]$ and $P_i'[h(e_i), t(h_i)]$ which belong to the same bi-coloured component.*

Proof. Since e_i and h_i are not edges of S and there exists a path of colour i in this component from $h(e_i)$ to $t(h_i)$, then by Lemma 9 no path of colour i from s_i to $t(e_i)$ or from $h(h_i)$ to t_i can use any vertex of S. However, any path of colour j from $h(e_j)$ to $t(h_j)$ is a path of S, so it cannot intersect any path of colour i from s_i to $t(e_i)$ or from $h(h_i)$ to t_i. This means that $(P_i'[s_i, t(e_i)], P_j[h(e_j), t(h_j)])$ and $(P_i'[h(h_i), t_i], P_j[h(e_j), t(h_j)])$ are blind pairs. By reversing the role of i and j, it also means that $(P_j[s_j, t(e_j)], P_i'[h(e_i), t(h_i)])$ and $(P_j[h(h_j), t_j], P_i'[h(e_i), t(h_i)])$ are blind pairs.

By the definition of conflicting and Lemma 10, we can show that no path of colour i from s_i to $t(e_i)$ or from $h(h_i)$ to t_i can intersect a path of colour j from s_j to $t(e_j)$ or $h(h_j)$ to t_j. Indeed, suppose for example that there exists a path H_i of colour i from s_i to $t(e_i)$ that intersects a path H_j of colour j from s_j to $t(e_j)$. In that case the paths $H_i P_i'[t(e_i), h(h_i)]$ and $H_j P_j[t(e_j), h(h_j)]$ contradict Lemma 10 as S is a conflicting component for these two paths, but they also intersect outside of S. The other cases are symmetrical and thus all pairs of paths among $P_i'[s_i, t(e_i)]$, $P_i'[h(h_i), t_i]$, $P_j[s_j, t(e_j)]$ and $P_j[h(h_j), t_j]$ are blind.

This ends the proof of the claim as the other pairs contain an edge and are blind by definition and $P_j[h(e_j), t(h_j)]$ and $P_i'[h(e_i), t(h_i)]$ are paths of S. □

Let $L_j = \{P_j[s_j, t(e_j)], e_j, P_j[h(e_j), t(h_j)], h_j, P_j[h(h_j), t_j]\}$. Suppose first that $t(e_i)$ appears after x_3 on P_i'. It means that $P_i[s_i, x_3]$ is a subpath of $P_i'[s_i, t(e_i)]$, and in particular $P_i[s_i, x_3]$ does not see P_j. Setting $L_i = \{P_i[s_i, x_3], (x_3, x_3'), P_i[x_3', x_2], (x_2, x_2'), P_i[x_2', x_1], (x_1, x_1'), P_i[x_1', t_i], \}$, we then have that no path of L_i sees P_j and thus any path of L_j.

Suppose now that $t(e_i)$ appears before x_3 on P_i'. Note that $t(h_i)$ has to appear after x_3 or there is no path from x_3 to t_i intersecting P_j, which contradicts the choice of x_3. In that case, setting $L_i = \{P_i[s_i, t(e_i)], e_i, P_i[h(e_i), x_3], (x_3, x_3'), P_i[x_3, x_2], (x_2, x_2')P_i[x_2', x_1], (x_1, x_1'), P_i[x_1, t_i]\}$, we also have that the only path of L_i that sees a path of L_j is $P_i[h(e_i), x_3]$. Moreover, it can only see $P_j[h(e_j), t(h_j)]$, but these paths belong to the same bi-coloured component S.

The cases where some of the s_i, t_i, s_j, t_j belong to S are treated exactly the same, except that some of the e_i, e_j, h_j, h_i might not exist, which means we have fewer paths to consider. □

By applying the previous lemma several times, we obtain the following:

Lemma 13. *There exists a constant C such that if G is a k-shortest graph, i, j two integers in $[k]$, P_i a path of colour i and P_j a path of colour j which are internally vertex-disjoint, then there exists a path partition L_i of P_i and a path partition L_j of P_j, both of size at most C with the following properties:*

- *Each L_a consists of at most C paths of colour a.*
- *For any pair of path $H_i \in L_i$ and $H_j \in L_j$ which are not blind, then H_i and H_j are paths of the same bi-coloured component.*

Proof. Let Q_i, Q_j be the path partitions obtained by applying Lemma 12 to P_i and P_j. We know that for any pair of paths $H_i \in Q_i$ and $H_j \in Q_j$, then either H_i and H_j are paths of the same bi-coloured component of colours i, j, or H_i does not see H_j.

Now as long as there exists a path in $H_i \in Q_i$ such that there exists some path $H_j \in Q_j$, such that H_j sees H_i and is not a path of the same bi-coloured component as H_i, we do the following. Let $H_{j,1}, \dots, H_{j,r}$ denote all the paths of Q_j which see H_i and do not belong to the same bi-coloured component. For any $a \in [r]$, let $Q_{j,a}$ and $Q_{i,a}$ denote the set of path partitions obtained by applying Lemma 12 to $H_{j,a}$ and H_i. Let Q'_i be the intersection of all the partitions $Q_{i,a}$ of P_i. Because every path of Q'_i is a subpath of some $Q_{i,a}$ for any $a \in [r]$, it means that this path is not seen by any path in $Q_{j,a}$ which is not a path of the same bi-coloured component. Let us update Q_i by replacing H_i by Q'_i and update Q_j by replacing each of the $H_{j,a}$ by $Q_{j,a}$. By doing that, the number of paths in Q_i which is seen by some path $H_j \in Q_j$ which is not a path of the same bi-coloured component decreases strictly as none of the paths of Q'_i satisfy these properties. At each step, we multiply the number of paths in Q_j by at most 9 and the number of paths in Q_i by at most $9|Q_j|$. However, we only have to do this 9 steps as initially the sets Q_i and Q_j have size at most 9. Finally, this means that after 9 steps, $|Q_j| \le 9^9$ and $|Q_i| \le 9(9|Q_j|)^9 \le 9^{91}$. This ends the proof for $C = 9^{91}$ □

5 Proof of the main theorem

We are now ready to describe and prove our algorithm. First we will show, using Lemma 12 and induction, that any solution $P_1, \dots, P_l$ of some k-DSP can be decomposed into some path partitions $L_1, \dots, L_l$ where each L_i has size at most $C(k, l)$ for some function C and any pair of paths in the union of the L_i of different colours is blind. The algorithm then consists of guessing the endpoints and colours of the partitions $L_1, \dots, L_l$ (there is at most $n^{O(C(k,l))}$ possible choices) and then solve the k-DSP defined using the fact that all the pairs of paths of different colours are blind. Roughly speaking, each colour class is solved almost independently using Fortune et al. Algorithm.

5.1 Proof of the blind case

Let us first show the existence of the algorithm in the blind case. Note that we also have some list of forbidden components for each path in the partitions $L_1, \dots, L_l$. This is due to some technicalities in the proof of the existence of such decomposition.

Lemma 14. *Let G be a k-shortest graph, $(s_1, t_1), \dots, (s_l, t_l)$ a set of pairs and c a function from $[l]$ into $[k]$. Moreover, suppose that for every i, there is a list F_i of bi-coloured components where one of the colours being $c(i)$. There exists an algorithm running in time $n^{O(l)}$ that either returns a solution $P_1, \dots, P_l$ to the k-DSP defined by G, the (s_i, t_i) and c or shows that no solution is such* *that each P_i does not use any vertex of any component in F_i and moreover, for any indices i and j, either P_j and P_i are blind or P_i is a path of some component of F_j or P_j is a path of some component of F_i.*

To prove this lemma, we will build an auxiliary digraph D such that a solution satisfying the properties of the lemma exists if and only if there exists a directed path in D between two specified vertices.

First note that, by potentially replacing some vertices with an independent set with the same neighbourhood, we can assume that all the s_i and t_i are disjoint.

The vertices of D will correspond to l-tuples $(x_1, \dots, x_l)$ of vertices of G. Intuitively, we are trying to build the paths P_i starting from s_i, and x_i is the last vertex of a prefix of P_i we are considering. For any pair of vertices $(x_1, \dots, x_l)$ and $(y_1, \dots, y_l)$, D contains the arc from $(x_1, \dots, x_l)$ to $(y_1, \dots, y_l)$ if the following are satisfied:

- There exists $i \in [l]$ such that $x_j = y_j$ for all $j \in [l]$, $j \neq i$.
- $x_i y_i$ is an edge of colour $c(i)$ such that there exists a path of colour $c(i)$ from y_i to t_i avoiding the components in F_i.
- For all $j \in [l]$ different from i, $y_i \neq x_j$ and either there is no path of colour $c(j)$ from x_j to t_j that uses the vertex x_i, or x_i is a vertex of a component of F_j.

Let $S = (s_1, \dots, s_l)$ and $T = (t_1, \dots t_l)$. The next two claims finishes the proof of Lemma 14.

Claim 14.1. *If there exists a solution $P_1, \dots, P_l$ to the k-DSP defined by G, (s_i, t_i) and c such that each P_i does not use a vertex of any component in F_i and moreover, for any indices i and j, either P_j and P_i are blind, P_i is a path of some component of F_j or P_j is a path of some component of F_i, then there is a path in D from S to T.*

Proof. Let $P_1, \dots P_l$ denote such a solution in G. Let X be the set of vertices of D corresponding to l-tuples obtained by taking one vertex per path P_i. We can define a natural order on X by considering for each P_i the order induced by the path and taking the lexicographic order. Note that T is the maximal element of X.

Consider now the largest element $A = (x_1, \dots, x_l)$ of X which is reachable in D from S and suppose, in order to reach a contradiction, that this element is not T. Consider some colour c_1 and I the set of indices of i of $[l]$ such that for $c(i) = c_1$ and $x_i \neq t_i$. Because the edges of colour c_1 induce an acyclic digraph, there exists an index $i \in I$ such that for every $j \in I$ with $j \neq i$, there is no path of colour c_1 from x_j to x_i. Now for any $j \in [l]$ such that $c(j) \neq c_1$, then either the path P_i and P_j are blind, in which case there is no path of colour $c(j)$ from x_j to t_j that uses x_i, or P_i (and thus x_i) is in some component of F_j, or P_j is a path of some component of F_i. Note that in the last case, any path from x_j to t_j is a path of some component of F_i, but x_i cannot be a vertex of this component, and thus no such path can use x_i. Therefore, if we note x'_i the vertex just after x_i on P_i and A' the vertex of D obtained from A by only changing x_i into x'_i, then there exists an arc from A to A'. However, this means that A' is reachable from S in D, which contradicts the maximality of A. □

And the opposite direction.

Claim 14.2. *If there exists a path from S to T in D, then there exists a solution $P_1, \dots, P_l$ to the DSP defined by G', the (s_i, t_i) and c such that P_i does not use any vertex of any component in F_i.*

Proof. Suppose there exists a path $P = X_1, \dots, X_r$ from S to T in D. For every $j \in [r]$, note $X_j = (x_1^j, \dots, x_l^j)$. For every i and j, consider the graph P_i^j induced by the vertices x_i^t, for $t \leq j$. By definition of D, P_i^j is a path of colour $c(i)$ from s_i to x_i^j avoiding the components of F_i. We will prove by induction on j, that the paths $P_1^j, \dots, P_l^j$ are such that

- All the P_i^j are internally disjoint.
- For any i and r, there is no path of colour $c(i)$ from x_i^j to t_j avoiding the components in F_i that uses any vertex of P_r^j outside of possibly x_r^j.

Since the path starts at S, all the properties are satisfied when $j = 1$. Suppose now that this is true for some $j \in [r-1]$ and let us show that the properties hold for $j+1$. By definition of the arcs of D, there exists an index i such that $x_i^j x_i^{j+1}$ is an edge of colour $c(i)$ and for every other index s, $x_s^j = x_s^{j+1}$. This means that P_i^{j+1} is the concatenation of P_i^j with x_i^{j+1} and all the P_s^{j+1} are equal to P_s^j for $s \neq i$. Moreover, by definition of D, we know that for any $s \neq i$, x_i^{j+1} is disjoint from all the x_s^j and by induction hypothesis x_i^{j+1} does not belong to any of the P_s^j. This implies that the P_s^j for $s \in [l]$, are disjoint.

Any path of colour $c(i)$ from x_i^{j+1} to t_i avoiding the components of F_i is a subpath of a path of colour $c(i)$ from x_i^j to t_i avoiding the components of F_i. This means that no such path can use any vertex of $P_s^j = P_s^{j+1}$ outside of possibly x_s^j for all $s \in [l]$ different from i.

Finally, for $s \in [l]$ different from i we know that no path of colour $c(s)$ from $x_r^j = x_r^{j+1}$ to t_s avoiding the components in F_s can use any vertex of P_i^j outside of x_s^i by induction hypothesis. Moreover, these paths can also not use x_i^{j+1} by definition of the arcs of D, which ends our induction.

This means that each P_i^r is a path of colour $c(i)$ avoiding the components in F_i from s_i to t_i, and all these paths are disjoints, which ends the proof. □

Therefore, the problem reduces to deciding the existence of a path in D. As $|D| = n^l$, this can be done in $n^{O(l)}$.

5.2 Reducing to the blind case

The next lemma shows how to reduce to the blind case with some forbidden lists.

Lemma 15. *Let G be a k-shortest graph, $(s_1, t_1), \dots (s_l, t_l)$ a set of pairs and c a function from $[l]$ to $[k]$. Let $P_1, \dots, P_l$ be a solution to the k-DSP defined by G, the (s_i, t_i) and c. There exists a constant $C(k, l)$ depending only on k and l, a set of path partitions $L_1, \dots, L_l$, a function a that associates to each path of the L_i a colour in $[k]$ and a function b that associates to each path of the L_i a set of bi-coloured components with the following properties:*

- *For every $i \in [l]$, L_i is a path partition of P_i of at most $C(k, l)$ paths.*
- *If P is a path of some L_i with $a(P) = c_1$, then P is a path of colour c_1 and $b(P)$ consists of a set of at most $C(k, l)$ bi-coloured components where one of the colours is c_1 and such that P does not use any vertex in these components.*
- *For any pair of paths $H_i \in L_i$ and $H_j \in L_j$ such that $a(H_i) \neq a(H_j)$, then either H_i and H_j are blind, H_j is a path contained in one component of $b(H_i)$ or H_i is a path contained in one component of $b(H_j)$.*

Note that if P_i is a path of colour j, then any path partition of P_i consists of paths of colour j. The function a is there to reassign the colour of some paths belonging to bi-coloured components in order to achieve the last property of the lemma.

Proof. Let C be the constant from Lemma 13. We will prove by induction on k that the lemma is true with $C(k,l) \leq (7Cl)^{5^k}$. When $k=1$, there is only one colour and setting $L_i = \{P_i\}$ for all i satisfies the properties, and thus $C(1,l) \leq l$.

Suppose now that $k > 1$, and let I_1 denote the set of indices $i \in [l]$ such that $c(i) = 1$. Note that we can assume I_1 to be non empty, or the induction step is trivial. For every pair of indices $i, j \in [l]$, let $Q_{i,j}, Q_{j,i}$ denote the path partitions of P_i, P_j obtained by applying Lemma 13 to this pair and for every $i \in [l]$, let Q_i denote the intersection of all the $Q_{i,j}$. Note that, since every $Q_{i,j}$ has size C, this implies that the Q_i have size at most Cl. For any $i \in I_1$, let Q'_i denote the set of paths of Q_i, for which there exists some other path among some Q_j with $j \in [k] \setminus I_1$ such that the pair is not blind. By Lemma 13, all these paths belong to some bi-coloured component of colour 1 and some other colour t. Let B_i denote the set of all these bi-coloured components and let $R_i = Q_i \setminus Q'_i$. Let $R = \bigcup_{i \in I_1} R_i$, $B = \bigcup_{i \in I_1} B_i$ and note that $|B| \leq Cl^2$ and $|R| \leq Cl^2$.

For any path $R_t \in R$, we know that the intersections of R_t with any component $C_j \in B$ is a subpath by Lemma 9. Let e_j be the last edge of R_t before C_j and h_j the first edge after. Let a_t and b_t denote the first and last vertex of R_t, and consider $R_{t,j} = \{R_t[a_t, t(e_j)], e_j, R_t[h(e_j), t(h_j)], h_j, R_t[h(h_j), b_t]\}$ a path partition of R_t. Note that, except the two edges e_j and h_j, each path of $R_{t,j}$ is either disjoint from C_j or a path of this component. Let $L(R_t)$ denote the path partition of R_t obtained by taking the intersection of all the $R_{t,j}$. We know that, since $|B| \leq Cl^2$, $L(R_t) \leq 5Cl^2$. Moreover, we know that for any path P' of $L(R_t)$, and any $C_j \in B$, there is a path $r_{t,j} \in R_{t,j}$ such that P' is a subpath of $r_{t,j}$. In particular it means that P' is either an edge, disjoint from C_j, or a path of C_j. Let R_t^2 be the set of paths of $L(R_t)$ which belong to one of the component of B, and $R_t^1 = L(R_t) \setminus R_t^2$. Note that every path in R_t^1 is either an edge or a path disjoint from all the components of B.

Let H_1 denote the set of paths in all the Q'_i for $i \in I_1$ and all the R_t^2 for $R_t \in R$. Note that for every path $H' \in H_1$, H' is path of a bi-coloured component of B. Denote by $c'(H')$ the colour of this component which is not 1. We will now consider H' as a path of colour $c'(H')$ (possibly reversing the endpoints if the component is a component of $G^-_{1,c'(H')}$). Let G_1 be the $(k-1)$-shortest graph obtained from G be removing the partition associated to colour 1 and removing all the edges which are edges of colour 1 only. Consider now the $(k-1)$-DSP problem defined on G_1 by all the endpoints of the paths in Q_i for $i \in [l]$, $i \notin I_1$, considered as path of colour $c(i)$, and all the paths in $H' \in H_1$ considered as path of colour $c'(H')$. Note that the set of paths in Q_i and H_1 is a solution to this problem and moreover, there is at most $7(Cl^2)^2$ requests, as each Q_i is smaller than Cl and H_1 is smaller than $6(Cl^2)^2$.

By induction hypothesis, there exists a path partition of all the paths of the Q_i for $i \in [l]$, $i \notin I_1$ and H_1 as well as two functions a' and b' defined on these paths such that each of these path partitions consists of at most $C(k-1, 7(Cl^2)^2)$ paths, and b' associates to each path at most $C(k-1, 7(Cl^2)^2)$ bi-coloured components. Let us define the path partitions L_i, as well as a and b as follows: For every $i \in I_1$, L_i is the union of all the paths in R_t^1 for some $R_t \in R_i$ as well as all the paths in the path partitions of the paths in R_t^2 and Q'_i obtained by applying induction. For every path P of some R_t^1, let $a(P) = 1$ and $b(P) = B$. For all the other paths of L_i, a and b correspond to the value of a' and b' on these paths. Likewise, for every $i \in [l]$ such that $i \notin I_1$, L_i consists of the union of the path partitions for the paths in Q_i obtained by applying induction, and the function

a and b correspond to the a' and b' on these paths.

Let us now show that the L_i and functions a and b satisfy the required properties. First, it is clear that the L_i thus defined are path partitions, as they are obtained by replacing paths of some path partitions by their own path partition. Moreover, $|L_i| \leq 7(Cl^2)^2 \cdot C(k-1, 7(Cl^2)^2) \leq 7(Cl^2)^2 \cdot (7^2(Cl^2)^2)^{5^{k-1}} \leq (7Cl)^{5^k}$. Likewise, for any paths P in these partitions, $b(P)$ is smaller than $\max\{C(k-1, Cl^2), |B|\} \leq (7Cl)^{5^k}$. Now suppose H_i is a path of L_i and H_j is a path of L_j such that $a(H_i) \neq a(H_j)$. If none of these paths belong to some R_t^1 for $R_t \in R$, then the last property of the lemma is satisfied for H_i and H_j by induction and because a and b correspond to a' and b' on these paths. Suppose now that one of the paths, say H_i belongs to R_t^1 for some $R_t \in R$. Because R_t^1 is a subpath of an element of R, it means that if H_j is not a subpath of some path of Q'_s for $s \in I_1$, then by definition of R, H_i and H_j are blind. However, if H_j is a subpath of some path of Q'_s, then H_j is a path belonging to some component of B. However, $b(H_i) = B$, which ends the proof. □

Finally, we can prove our main result.

Proof of Theorem 4. Suppose there exists a solution $P_1, \ldots, P_l$ to the k-DSP problem defined by G, the (s_i, t_i) and c. Let $L_1, \ldots L_l$, a and b be the path partitions and functions obtained by applying Lemma 15 to $P_1, \ldots, P_l$. For every $i \in [l]$, let $P_{i,1}, \ldots, P_{i,l_i}$ denote the paths of L_i and $(s_1^i, t_1^i), \ldots (s_{l_i}^i, t_{l_i}^i)$ the endpoints of these paths. Remember that by Lemma 15, $|L_i| \leq C(k, l)$ for all $i \in [l]$.

Suppose we guess all the (s_j^i, t_j^i), as well as the functions a and b for each of the $P_{i,j}$, and consider the k-DSP problem defined by all the remaining pairs (s_j^i, t_j^i), then the set of paths $P_{i,j}$ is a solution to this problem such that, for any pair of paths $P_{i,j}$, $P_{i',j'}$ such that $a(P_{i,j}) \neq a(P_{i',j'})$, either $P_{i,j}$ and $P_{i,j}$ are blind, $P_{i,j}$ is a path contained in one component of $b(P_{i',j'})$ or $P_{i',j'}$ is a path contained in one component of $b(P_{i,j})$. This means that we can apply the algorithm of Lemma 14 to find a solution of the k-DSP defined by (s_j^i, t_j^i) in $n^{O(C(k,l))}$. By concatenating for each i all the paths of this solution corresponding to the paths of L_i, we obtain a solution to the initial k-DSP problem.

As there is at most $n^{O(C(k,l))}$ choices for the (s_j^i, t_j^i), $a(P_{i,j})$ and $b(P_{i,j})$ this gives an algorithm running in time $n^{O(C(k,l))}$, which ends the proof. □

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