On the Non-robustness of Essentially Conditional Information Inequalities

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ITW 2012, September 4
Linear information inequalities

**Basic inequality:**

\[
H(a, b) \leq H(a) + H(b) \quad [I(a; b) \geq 0]
\]

\[
H(a, b, c) + H(c) \leq H(a, c) + H(b, c) \quad [I(a; b \mid c) \geq 0]
\]

**Shannon type** inequalities: any positive combination of basic ineq., e.g.,

\[
H(a) \leq H(a \mid b) + H(a \mid c) + I(b; c)
\]

**Non Shannon type** inequalities, e.g., [Z. Zhang, R. W. Yeung, 1998]:

\[
I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) + I(a; b) + I(c; d \mid a) + I(a; c \mid d) + I(a; d \mid c)
\]
Conditional information inequalities

If [some linear constraints for entropies] then [a linear inequality for entropies].

Example 1 (trivial): If \( I(b; c) = 0 \), then \( H(a) \leq H(a \mid b) + H(a \mid c) \).

Explanation:
\[
H(a) \leq H(a \mid b) + H(a \mid c) + I(b; c).
\]

Example 2 (trivial): If \( I(c; d \mid e) = I(c; e \mid d) = I(d; e \mid c) = 0 \), then \( I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) + I(a; b) \).

Explanation:
\[
I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) + I(a; b) + I(c; d \mid e) + I(c; e \mid d) + I(d; e \mid c).
\]

Example 3 (nontrivial) [Zhang–Yeung 1997]: If \( I(a; b) = I(a; b \mid c) = 0 \), then \( I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) \).

Any explanation???
Essentially conditional inequalities

Z. Zhang, R. W. Yeung 97:
if \( I(a; b) = I(a; b \mid c) = 0 \), then \( I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) \).

Theorem [KR 2011] This inequality is essentially conditional, i.e., for all \( \lambda_1, \lambda_2 \) the inequality

\[
I(c; d) \leq I(c; d \mid a) + I(c; d \mid b) + \lambda_1 I(a; b) + \lambda_2 I(a; b \mid c)
\]

does not hold!

Several other examples of essentially conditional inequalities: F. Matúš 99, F. Matúš 07, KR 11.
Non robustness of essentially conditional inequalities

- **An example of a robust conditional inequality. (Ex 2)**
  - If $I(c; d|e) \leq \varepsilon$, $I(c; e|d) \leq \varepsilon$, $I(d; e|c) \leq \varepsilon$, then $I(c; d) \leq I(c; d|a) + I(c; d|b) + I(a; b) + O(\varepsilon)$.

- **An example of a non robust conditional inequality. (ZY97)**
  - Whatever is $\varepsilon > 0$, from $I(a; b) \leq \varepsilon$ and $I(a; b|c) \leq \varepsilon$, it does **not** follow that
    
    $$I(c; d) \leq I(c; d|a) + I(c; d|b) + O(\varepsilon),$$

    moreover, the ratio
    $$\frac{I(c; d)}{I(c; d|a) + I(c; d|b)}$$

    can be arbitrarily large!

- **In this paper:** We classify the known essentially conditional inequalities.

  *Footnote for experts:* Some conditional inequalities are valid only for **entropic points**, the other hold for **almost entropic points** (limits of entropic points).