Fully Optimal Bases
and the Active Bijection
in Graphs, Hyperplane Arrangements,
and Oriented Matroids

Emeric Gioan
CNRS (LIRMM - Université Montpellier 2), France

joint work with Michel Las Vergnas
A FAMOUS CURIOUS PROPERTY

Hyperplane arrangement, represented by its intersection with a central sphere:

two symmetric halvespheres delimited by one the hyperplanes
A FAMOUS CURIOUS PROPERTY

Consider the number of regions that do not touch a given hyperplane (on a given side): bounded regions w.r.t. the hyperplane chosen as hyperplane at infinity
A FAMOUS CURIOUS PROPERTY

This number does not depend on the chosen hyperplane!
A FAMOUS CURIOUS PROPERTY

A bipolar orientation of a graph w.r.t. two adjacent vertices \((s, s')\) is an acyclic orientation with unique source \(s\) and unique sink \(s'\).

The number of bipolar orientations does not depend on the choice of the edge \(e = (s, s')\).
A FAMOUS CURIOUS PROPERTY

The property in graphs is a particular case of the property in hyperplane arrangements

edge $v_i v_j \rightarrow$ hyperplane $x_i - x_j = 0$
spanning trees $\rightarrow$ bases
acyclic orientations $\rightarrow$ regions
bipolar orientations $\rightarrow$ bounded regions
A FAMOUS CURIOUS PROPERTY

More generally the property is true in oriented matroids.

Theorem [Zaslavsky 75, Las Vergnas 77]
The number of bounded regions of an oriented matroid (with no loop nor isthmus), w.r.t. to a given element \( e \), on the positive side on \( e \), does not depend on \( e \) and equals

\[
\beta(M) = t_{1,0}(M)
\]
A FAMOUS CURIOUS PROPERTY

More generally the property is true in oriented matroids.

Theorem [Zaslavsky 75, Las Vergnas 77]
The number of bounded regions of an oriented matroid (with no loop nor isthmus), w.r.t. to a given element e, on the positive side on e, does not depend on e and equals

\[ \beta(M) = t_{1,0}(M) \]

This number is the coefficient of x (or y) in the Tutte polynomial of M:

\[ t(M; x, y) = \sum_{i,j} t_{i,j} x^i y^j \]

Theorem [Tutte 54]
If the ground set of M is linearly ordered, then the number of (i, j)-active bases w.r.t. the linear ordering is an invariant and equals \( t_{i,j} \).
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Let $M$ be an ordered oriented matroid on $E$
(or an ordered hyperplane arrangement $E$, or a graph with an ordered set of edges $E$).

The ground set $E = e_1 < ... < e_n$ is linearly ordered,
and a bounded region (or a bipolar orientation) is always thought of w.r.t. $e_1$.

The *(bounded)* active bijection of $M$ is a bijective mapping
from the set of bounded regions w.r.t. $e_1$
onto the set of $(1,0)$-active bases w.r.t. $(E, <)$.

This bijection has a very simple definition...

...but, first, what is a $(1,0)$-active basis?
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Let $B$ be a basis of $M$.

$C_e = \text{fundamental circuit of } e \notin B \text{ w.r.t. } B = \text{unique circuit in } B \cup e$

$C_b^* = \text{fundamental cocircuit of } b \in B \text{ w.r.t. } B = \text{unique cocircuit in } (E \setminus B) \cup b$

\textit{fundamental tableau of } B = \text{matrix } n \times n \text{ on } \{0, x\} \text{ with}

$C_e$ as non-zero elements of row $e \notin B$,

$C_b^*$ as non-zero elements of column $b \in B$
**Ex.** Fundamental tableau of basis 136

<table>
<thead>
<tr>
<th></th>
<th>$C_1^*$</th>
<th>2</th>
<th>$C_3^*$</th>
<th>4</th>
<th>5</th>
<th>$C_6^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>$C_5$</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Assume the ground set $E$ of $M$ is ordered: $E = e_1 < ... < e_n$

$B$ is $(1,0)$-active if

the smallest non-zero element of a row is non-zero in a previous one

i.e. $\min C_e \in \bigcup_{a < e, \ a \notin B} C_a$

the smallest non-zero element of a column is non-zero in a previous one (except first column)

i.e. $\min C_b^* \in \bigcup_{a < b, \ a \in B} C_a^*$ for $b \neq e_1$
**Ex.** Basis 136 is (1,0)-active

\[
\begin{array}{c|cccccc}
 & C^*_1 & 2 & C^*_3 & 4 & 5 & C^*_6 \\
\hline
1 & x & & & & & \\
C_2 & x & x & x & & & \\
3 & & & & x & & \\
C_4 & x & & x & x & & \\
C_5 & & x & x & x & & \\
6 & & & & & x & \\
\end{array}
\]
THE ACTIVE BIJECTION IN THE BOUNDED CASE

If $M$ is an oriented matroid, or a signed arrangement, or a directed graph,

signed fundamental tableau of $B = \text{matrix } n \times n \text{ on } \{0, +, -\}$ with

$C_e$ as non-zero elements of row $e \notin B$, with its signs in $\{+, -\}$,
and (by convention) $e$ signed $-$

$C_b^*$ as non-zero elements of column $b \in B$, with its signs in $\{+, -\}$,
and (necessarily) $b$ signed $+$
Ex. Signed fundamental tableau of $136$, w.r.t. to the blue region.

<table>
<thead>
<tr>
<th></th>
<th>$C_1^+$</th>
<th>2</th>
<th>$C_3^+$</th>
<th>4</th>
<th>5</th>
<th>$C_6^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>$C_4$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
<td>+</td>
</tr>
<tr>
<td>$C_5$</td>
<td></td>
<td></td>
<td></td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
</tr>
</tbody>
</table>
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Let $M$ be an ordered oriented matroid (or signed arrangement, or directed graph).

A basis $B$ of $M$ is *fully optimal* in $M$ if

- the smallest element of each row is signed $+$
- the smallest element of each column is signed $-$ (except the first).

**Rk.** This implies $M$ is bounded acyclic and $B$ is $(1,0)$-active
**Ex.** The basis 136 is not fully optimal w.r.t. to the blue region.

![Diagram of a graph with labeled vertices 1 to 6 and a triangle with vertices 1, 2, and 3]

<table>
<thead>
<tr>
<th></th>
<th>$C_1^*$</th>
<th>2</th>
<th>$C_3^*$</th>
<th>4</th>
<th>5</th>
<th>$C_6^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
<td>$\oplus$</td>
</tr>
<tr>
<td>$C_5$</td>
<td></td>
<td></td>
<td>+</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>
Ex. The basis 136 is fully optimal w.r.t. to the green region (obtained by reversing 6).

<table>
<thead>
<tr>
<th></th>
<th>$C_1^*$</th>
<th>2</th>
<th>$C_3^*$</th>
<th>4</th>
<th>5</th>
<th>$C_6^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>+</td>
<td></td>
<td>-</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_5$</td>
<td></td>
<td></td>
<td>+</td>
<td>-</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Main Theorem.

A bounded acyclic ordered oriented matroid $M$ has one and only one fully optimal basis, denoted by $\alpha(M)$.

The mapping $\alpha$ is a bijection between bounded regions of $M$ and $(1, 0)$-active bases of $M$.

In particular, a bounded region of an ordered hyperplane arrangement, or a bipolar orientation of an ordered graph, has a unique fully optimal basis.
THE ACTIVE BIJECTION IN THE BOUNDED CASE

Construction of the bijection

From bases to regions: just sign successively the elements one by one the good way!

From regions to bases: linear programming refinements...

Rk. There exist a deletion/contraction construction, and other characterisitic properties.
LINEAR PROGRAMMING REFINEMENTS

In usual linear programming, the vertex intersection of $B \setminus \varepsilon_1$ is optimal

\[ \text{if and only if} \]

in the signed tableau of $B$

- the first column $C^{\varepsilon_1}$ is positive
- the second line $C_{\varepsilon_2}$ is negative (except on $\varepsilon_1$)

Here we take into account the whole fundamental tableau, i.e.

- all lines: multiobjective programming (instead of one objective function)
- all columns: flag optimization (instead of one optimal face)
**Ex.** The region $\alpha^{-1}(136)$ has one fully optimal basis but it has two optimal vertices in usual LP and the same optimal vertex as the region $\alpha^{-1}(135)$.
THE ACTIVE MAPPING IN THE GENERAL CASE

**Th [Tutte 54]**

\[ t(M; x, y) = \sum_{i,j} b_{i,j} x^i y^j \]

where \( b_{i,j} = \# (i, j) \)-active bases

**Th [Las Vergnas 84]**

\[ t(M; x, y) = \sum_{i,j} o_{i,j} \left( \frac{x}{2} \right)^i \left( \frac{y}{2} \right)^j \]

where \( o_{i,j} = \# (i, j) \)-active reorientations

\[ o_{i,j} = 2^{i+j} b_{i,j} \]
THE ACTIVE MAPPING IN THE GENERAL CASE

The *active mapping* $\alpha$ maps an ordered oriented matroid onto one of its bases.

It is defined by

1) $\alpha(M)$ is the fully optimal basis of $M$ if it is bounded acyclic
2) $\alpha(M^*) = E \setminus \alpha(M)$
3) $\alpha(M) = \alpha(M/A) \cup \alpha(M(A))$

where $A$ is the union of all positive circuits of $M$ whose smallest element is the greatest possible minimal element of a positive cocircuit of $M$.

*For a given oriented matroid,*

we get a $2^{i+j} - 1$ *activity preserving correspondence* between all orientations and all bases

and, more specifically,

an activity preserving bijection

between all subsets (related to bases) and all orientations

between no-broken-circuit subsets and acyclic orientations
THE ACTIVE MAPPING IN THE GENERAL CASE

\[ \alpha(M) = \bigcup_{1 \leq k \leq \ell} \alpha(M(F''_k)/F''_{k-1}) \bigcup \bigcup_{1 \leq k \leq \ell} \alpha(M(F'_{k-1})/F'_k) \]

active decomposing sequence of \( M \):

\[ \emptyset = F''_\varepsilon \subset \ldots \subset F'_0 = F''_c = F'_0 \subset \ldots \subset F''_\ell = E \]

Theorem

\[ t(M; x, y) = \sum \left( \prod_{1 \leq k \leq \ell} \beta(M(F'_k)/F'_{k-1}) \right) \left( \prod_{1 \leq k \leq \varepsilon} \beta(M(F''_{k-1})/F''_k) \right) x^\varepsilon y^\ell \]

where the sum is over all active decomposing sequences of bases of \( M \)
125

123456
123456
123456
123456

145+236
145+236

125
### Sum Up

<table>
<thead>
<tr>
<th>structure</th>
<th>active bijection</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>oriented matroids</td>
<td>activity classes of reorientations</td>
<td>bases</td>
</tr>
<tr>
<td></td>
<td>act. cl. of acyclic reorientations</td>
<td>internal bases</td>
</tr>
<tr>
<td></td>
<td>act. cl. of totally cyclic reor.</td>
<td>external bases</td>
</tr>
<tr>
<td></td>
<td>bounded acyclic reorientations</td>
<td>(1,0)-active bases</td>
</tr>
<tr>
<td></td>
<td>reorientations</td>
<td>subsets</td>
</tr>
<tr>
<td></td>
<td>acyclic reorientations</td>
<td>no-broken-circuit subsets</td>
</tr>
<tr>
<td>hyperplane</td>
<td>reorientations = signatures</td>
<td>bases = simplices</td>
</tr>
<tr>
<td>arrangements</td>
<td>acyclic reorientations = regions</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>graphs</td>
<td>reorientations = orientations</td>
<td>bases = spanning trees</td>
</tr>
<tr>
<td></td>
<td>unique sink acyclic orientations</td>
<td>internal spanning trees</td>
</tr>
<tr>
<td></td>
<td>bipolar orientations</td>
<td>(1,0)-active spanning trees</td>
</tr>
<tr>
<td>uniform o.m.</td>
<td>bounded regions</td>
<td>LP optimal vertices</td>
</tr>
<tr>
<td>supersolvable $A_n$</td>
<td>permutations</td>
<td>increasing trees</td>
</tr>
<tr>
<td>supersolvable $B_n$</td>
<td>signed permutations</td>
<td>signed increasing trees</td>
</tr>
</tbody>
</table>