Substitution decomposition of permutations in enumerative combinatorics

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Outline

1. Introduction to substitution decomposition of permutations
   - Context
   - Definition

2. Applications in combinatorics
   - Enumerative combinatorics
     - Enumeration of simple permutations
     - Enumeration of permutation classes
   - Analytic combinatorics
     - Analysis of algorithms: Perfect sorting by reversals

3. Perspectives

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Substitution decomposition

- General framework of [Möhring & Radermacher 84]
- **Modular** decomposition of graphs
- **Substitution** decomposition or **strong interval** decomposition of permutations

**Relies on:**
- a principle for building objects (permutations, graphs) from smaller objects: the **substitution**.
- some “**basic objects**” for this construction: **simple** permutations, **prime** graphs.

**Required properties:**
- every object can be decomposed using only “basic objects”.
- this decomposition is **unique**.
Substitution for permutations

**Substitution** or inflation: \( \sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}] \).

**Example:** Here, \( \pi = 1 3 2 \), and

\[
\begin{align*}
\alpha^{(1)} &= 2 1 = \\
\alpha^{(2)} &= 1 3 2 = \\
\alpha^{(3)} &= 1 =
\end{align*}
\]

Hence \( \sigma = 1 3 2[2 1, 1 3 2, 1] = 2 1 4 6 5 3. \)
Simple permutations

**Interval** (or **block**) = set of elements of \( \sigma \) whose positions **and** values form intervals of integers

**Example:** 5 7 4 6 is an interval of 2 5 7 4 6 1 3

**Simple permutation** = permutation that has no interval, except the trivial intervals: 1, 2, \ldots, \( n \) and \( \sigma \)

**Example:** 3 1 7 4 6 2 5 is simple.

*The smallest simple:* 1 2, 2 1, 2 4 1 3, 3 1 4 2
Substitution decomposition of permutations

**Theorem:** Every $\sigma \neq 1$ is uniquely decomposed as
- $12\ldots k[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
- $k\ldots 21[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable
- $\pi[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where $\pi$ is simple of size $k \geq 4$

**Remarks:**
- $\oplus$-indecomposable: that cannot be written as $12[\alpha^{(1)}, \alpha^{(2)}]$
- First appeared in combinatorics in [Albert & Atkinson 05]
- The $\alpha^{(i)}$ are the maximal strong intervals of $\sigma$
Theorem: Every $\sigma \neq 1$ is uniquely decomposed as
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Decomposing recursively inside the $\alpha(i) \Rightarrow$ decomposition tree

- $\oplus = 12\ldots k$ and $\ominus = k\ldots 21 = \text{linear nodes.}$
- $\pi$ simple of size $\geq 4 = \text{prime node.}$
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Ordered trees.
Computation and examples of application

**Computation:** in **linear** time. [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

**In algorithms:**
- **Computation of modular decomposition trees through factorizing permutations** [Habib, Paul & Viennot 98] [Habib, Montgolfier & Paul 04] [Tedder, Corneil, Habib & Paul 08] [Capelle, Habib & Montgolfier 02] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]
- **Pattern matching** [Bose, Buss & Lubiw 98] [Ibarra 97] [B. & Rossin 06] [B., Rossin & Vialette 07]
- **Algorithms for bio-informatics** [Bérard, Bergeron, Chauve & Paul 07] [Bérard, Chateau, Chauve, Paul & Tannier 08] [B., Chauve, Mishna & Rossin 09]
Computation and examples of application

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**In combinatorics:**

- **Simple permutations** [Albert, Atkinson & Klazar 03]
- **Classes closed by substitution product** [Atkinson & Stitt 02] [Brignall 07] [Atkinson, Ruškuc & Smith 09]
- **Exhibit the structure of classes** [Albert & Atkinson 05] [Brignall, Huczynska & Vatter 08a, 08b] [Brignall, Ruškuc & Vatter 08] [Bassino, B. & Rossin 08] [Bassino, B., Pierrot & Rossin 09, 10]
Substitution Decomposition in Enumerative Combinatorics

- Quick reminder on enumerative combinatorics
- Enumeration of simple permutations
- General results for the enumeration of permutation classes
Enumerative combinatorics: main ideas

$\mathcal{C}$ a family of combinatorial objects (*permutations*)
- Notion of size ($|\sigma| = n$ for $\sigma$ permutation on $\{1, \ldots, n\}$)
- For any $n$, finite number of objects of size $n$
- $c_n =$ number of objects of size $n$ in $\mathcal{C}$

Many ways of providing the enumeration of $\mathcal{C}$:
- Closed formula of $c_n$
- Recurrence satisfied by $c_n$
- Asymptotic equivalent of $c_n$
- Explicit expression of the generating function $C(z) = \sum c_n z^n$
- Equations or properties satisfied by the generating function
Enumeration of simple permutations

[Albert, Atkinson & Klazar 03]

- The enumeration sequence of simple permutations is not
  \( P \)-recursive
- No hope for a closed formula
- Asymptotic equivalent: \( \frac{n!}{e^2} (1 - \frac{4}{n} + O(\frac{1}{n^2})) \)

⇒ Asymptotically, a proportion \( \frac{1}{e^2} \) of decomposition trees are reduced to one prime node.

Rmk: Asymptotically, the proportion of decomposition trees of this shape is 1

[B., Chauve, Mishna & Rossin 09]
**Permutation** Represented by $\sigma(1)\sigma(2) \cdots \sigma(n)$ or on a grid

**Pattern** [Knuth 73] Sub-permutation with normalization

Example: $2134 \preceq 312854796$ since $3279 \equiv 2134$

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Substitution decomposition of permutations in enumerative combinatorics
Permutations, patterns, and permutation classes

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**Pattern** [Knuth 73] Sub-permutation with normalization

Example: $2 1 3 4 \preceq 3 1 2 8 5 4 7 9 6$ since $3 2 7 9 \equiv 2 1 3 4$

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& \bullet & & & & & & & \\
& & & & & & & \bullet & \\
& & & & & & & & \bullet \\
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& & & & & \bullet & & & \\
& & & & & & \bullet & & \\
& & & & & & & \bullet & \\
\end{array}
\]

$\sigma = 3 1 2 8 5 4 7 9 6$
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Permutations, patterns, and permutation classes

**Permutation** Represented by $\sigma(1)\sigma(2)\cdots\sigma(n)$ or on a grid

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Example: $2 1 3 4 \preceq 3 1 2 8 5 4 7 9 6$ since $3 2 7 9 \equiv 2 1 3 4$

**Class** Set downward-closed for $\preceq$. Characterized by a (finite or infinite) basis $B$ of excluded patterns: $\mathcal{C} = Av(B)$

**Some results**
- Enumeration for many specific $B$’s
- General enumerative result: Stanley-Wilf ex-conjecture
  
  [Marcus & Tardos 04] $\exists c$ s.t. $|\mathcal{C} \cap S_n| \leq c^n$
- Since 2005: Finding general properties of permutation classes
Substitution decomposition for general enumerative results

**Theorem** [Albert & Atkinson 05]: If \( C \) contains a finite number of simple permutations, then

- \( C \) has a finite basis
- \( C \) has an algebraic generating function

**Proof**: relies on the substitution decomposition.

**Construction**: compute the gen. fun. from the simples in \( C \)

**Algorithmically**:

- **Semi-decision procedure**

\[ \rightarrow \text{ Find simples of size } 4, 5, 6, \ldots \text{ until } k \text{ and } k + 1 \text{ for which there are } 0 \text{ simples} \quad [\text{Schmerl & Trotter 93}] \]

- **Disastrous complexity** (\( \sim n! \)) for computing the simples in \( C \)
Substitution decomposition for general enumerative results

Theorem [Brignall, Ruškuc & Vatter 08]: It is decidable whether $\mathcal{C}$ given by its finite basis contains a finite number of simples.

Complexity of this procedure: $2\text{ExpTime}$

Improvement of the complexity:
[Bassino, B., Pierrot & Rossin 09,10]

- For substitution-closed classes: $\mathcal{O}(n \log n)$
  with $n = \sum_{\pi \in B} |\pi|$
- In general: $\mathcal{O}(n^{3k})$
  with $n = \max_{\pi \in B} |\pi|$ and $k = |B|$

Proof: relies on the substitution decomposition of pin-permutations
Open questions

- Efficient algorithm for finding the simples in a class instead of the $O(n!)$ procedure
- Compute algorithmically the generating function automatizing the proof of [Albert & Atkinson 05]
- Perform random generation in permutation classes starting with substitution-closed classes
Substitution Decomposition in Analytic Combinatorics

- Analytic combinatorics for analysis of algorithms
- Example of the perfect sorting by reversals
Perfect sorting by reversals

*Perfect reversal* on a signed permutation $\sigma = \sigma(1) \ldots \sigma(n)$

$\sigma = \sigma(1) \ldots \sigma(n)$

= Reverse the orientation and the signs of a contiguous fragment of the permutation, *without breaking any interval*

**Problem:**

- **Input:** A signed permutation $\sigma$
- **Output:** A parsimonious perfect scenario from $\sigma$ to $Id$ or $\overline{Id}$

**Complexity:**

- *NP-hard* problem [Figeac & Varré 04]
- **FPT algorithm** [Bérard, Bergeron, Chauve & Paul 07]: uses the decomposition tree, in time $O(2^p \cdot n^{O(1)})$
- Complexity parametrized by
  $p = \text{number of prime nodes (with a prime parent)}$
Idea of the algorithm on an example

\[ \sigma = 5 \ 6 \ 7 \ 9 \ 4 \ 3 \ 1 \ 2 \ 8 \ 10 \ 17 \ 13 \ 15 \ 12 \ 11 \ 14 \ 18 \ 19 \ 16 \]
Idea of the algorithm on an example

\[ \sigma = 5 \ 6 \ 7 \ 9 \ 4 \ 3 \ 1 \ 2 \ 8 \ 10 \ 17 \ 13 \ 15 \ 12 \ 11 \ 14 \ 18 \ 19 \ 16 \]

Leaves: sign of \( \sigma_1(i) \)
Idea of the algorithm on an example

\[ \sigma = 5 \ 6 \ 7 \ 9 \ 4 \ 3 \ 1 \ 2 \ 8 \ 10 \ 17 \ 13 \ 15 \ 12 \ 11 \ 14 \ 18 \ 19 \ 16 \]

Linear: copy the sign
Idea of the algorithm on an example

\[ \sigma = 5 \, 6 \, 7 \, 9 \, 4 \, 3 \, 1 \, 2 \, 8 \, 10 \, 17 \, 13 \, 15 \, 12 \, 11 \, 14 \, 18 \, 19 \, 16 \]

Prime with linear parent:
sign of the parent

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Idea of the algorithm on an example

\( \sigma = 5 \ 6 \ 7 \ 9 \ 4 \ 3 \ 1 \ 2 \ 8 \ 10 \ 17 \ 13 \ 15 \ 12 \ 11 \ 14 \ 18 \ 19 \ 16 \)

Prime with prime parent: ???

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Substitution decomposition of permutations in enumerative combinatorics
Idea of the algorithm on an example

\[ \sigma = 5 \ 6 \ 7 \ 9 \ 4 \ 3 \ 1 \ 2 \ 8 \ 10 \ 17 \ 13 \ 15 \ 12 \ 11 \ 14 \ 18 \ 19 \ 16 \]
Complexity results

Previous results [Bérard, Bergeron, Chauve & Paul 07]:
- \( O(2^p n \sqrt{n \log n}) \), where \( p = \) number of prime nodes
- polynomial on separable permutations \((p = 0)\)

Complexity analysis [B., Chauve, Mishna & Rossin 09]:
- polynomial on average
  → with lemmas on the number of trees with \( p \) prime nodes
- in a parsimonious scenario for separable permutations
  - average number of reversals \( \sim 1.2n \)
  - average size of a reversal \( \sim 1.02\sqrt{n} \)
  → with bivariate generating functions and analytic combinatorics

Probability distribution: always uniform
Average value of parameters with [Flajolet & Sedgewick 09]

Average number of reversals for separable permutations

\[
= \begin{cases} 
\text{average number of internal nodes (except root)} \\
+ \text{average number of leaves with label different from its parent} \\
= \text{average number of internal nodes } - 1 + n/2
\end{cases}
\]

Bivariate generating function: \( S(x, y) = \sum s_{n,k} x^n y^k \) where \( s_{n,k} \) = number of trees with \( n \) leaves and \( k \) internal nodes

Equation on \( S(x, y) \) giving \( S(x, y) = \frac{(x+1) - \sqrt{(x+1)^2 - 4x(y+1)}}{2(y+1)} \)

Average number of internal nodes \( = \frac{\sum k s_{n,k}}{\sum s_{n,k}} = \frac{[x^n] \frac{\partial S(x,y)}{\partial y} |_{y=1}}{[x^n] S(x,1)} \)

Asymptotic equivalent \( \frac{[x^n] \frac{\partial S(x,y)}{\partial y} |_{y=1}}{[x^n] S(x,1)} \sim \frac{n}{\sqrt{2}} \)

Conclusion Asymptotically \( \frac{1+\sqrt{2}}{2} n \) reversals on average
Open questions

- Not only the average of the parameters but also variance and distribution
- Extend this analysis to decomposition trees containing some (constrained) prime nodes
- Apply similar methods to other algorithms involving decomposition trees (ex: Double-Cut and Join)
Further results in combinatorics of permutation classes with substitution decomposition, making use of the many concepts and results on graph decomposition.

Investigate combinatorial or enumerative questions about graphs by adapting the methods that have been developed for permutations.