On parameterized Multicut

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CIRM

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1 Reducing the treewidth

2 Multicut is FPT
Multicut

Input: graph $G$, set of requests (pairs of vertices of $G$), integer $k$
Output: TRUE if there exist $k$ edges of $G$ which disconnect every request
Multicut

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Output: TRUE if there exist $k$ edges of $G$ which disconnect every request

- NP-complete
- No constant-factor Approx (Under the Unique Games Conj.)
- FPT in $k + \text{number of requests}$
- FPT in $k$ (PolyKernel) on Trees
- 2-F.P.Approx
Question

Is Multicut FPT in the solution size $k$, ie algorithm in $f(k) \times Poly(n)$?
Reducing the treewidth

Joint work with Christophe Paul, Anthony Perez and Stéphan Thomassé.

Result
Multicut can be reduced to graphs of treewidth bounded in $k$. 
Key Idea

When \((z, t)\) is a request \(\forall t \in T \rightarrow\) request \((z, x)\) is *Irrelevant*

\[T \text{ large enough (in } k): \]
\[\exists x \in T \text{ s.t.} \]
\[\text{If } |F| \leq k \text{ cuts } z \text{ from } T - x \]
\[\text{Then } F \text{ cuts } z \text{ from } x.\]
Remark

We can now assume that each vertex has a bounded (in $k$) number of requests.
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Stronger version:

$T$ big enough (in $k, l$):

$\exists x \in T$ s.t.

If $|F| \leq k$ disconnects $z$ from $|S| \geq |T| - l \subseteq T - x$

Then $F$ disconnects $z$ from $x$. 
Gathered Set: For every $k$-cut, only 1 component has 'size' > 1
Gathered Sets

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Theorem (Main Reduction Rule)
Big gathered set of terminals: there exists an irrelevant request
Treewidth reduction

Structure: Large treewidth $\rightarrow$ large grid minor
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Easy: in a true grid, big gathered set:
Clean Subgrid?

Idea: There are either:

- Many jumps $\rightarrow$ a large Clique-Minor
- or a 'Clean' Subgrid
Theorem (Robertson & Seymour)

Grid + many 'jumps' (= non-planar edges) with endpoints:
- Far away from the border
- Far away from each other
- Long jumps

... has a large Clique-Minor.
Without a large Clique-Minor

Treewidth Reduction

An instance of Multicut with large treewidth but no large Clique-Minor has a ’clean’ subgrid with either

- An ’empty’ zone $\rightarrow$ edge contraction
- terminals everywhere $\rightarrow$ large gathered set $\rightarrow$ irrelevant request

Hence instances of Multicut with large treewidth but no large Clique-Minor can be reduced.
Large Clique-Minor

Instance with a large clique minor: transformation into:

- \( k + 1 \) - connected minor
- Small 'vertex cover' of the requests
- Many distance-layers
Large Clique-Minor

Instance with a large clique minor: transformation into:

- $k + 1$ - connected minor
- Small 'vertex cover' of the requests
- Many distance-layers

Once again: Terminals far apart form a gathered set, ie either large empty zone (edge contraction) or a gathered set (irrelevant requests).
Joint work with Nicolas Bousquet and Stéphan Thomassé
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Result

Multicut is FPT in the solution size $k$. 
Key Idea

Multicut FPT on Treewidth 2 graphs?
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Multicut FPT on Treewidth 2 graphs?

A bounded (in $k$) number of petals
Branch on the partition of the solution edges w.r.t the petals
The problem is now 2-SAT, i.e. Polynomial!

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Key Idea

Multicut FPT on Treewidth 2 graphs?

- Bounded (in $k$) number of Petals
- Branch on the partition of the solution edges w.r.t. the Petals
- The problem is now 2-SAT, i.e. Polynomial!
Multicut is FPT

Request $R$ is cut if:
- $x_A \leq 2$ or $x_D \geq 2$
- $x_A \geq 1$ or $x_D \leq 1$
- $x_A \leq 2$ or $x_D \geq 1$
- $x_A \geq 1$ or $x_D \leq 2$

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Outline
Reducing the treewidth
Multicut is FPT

Multicut on a Flower

Request $R$ is cut $\iff$ "$x_A \leq 2$" or "$x_D \leq 2$" and "$x_A \geq 1$" or "$x_D \geq 1$" and "$x_A \leq 2$" or "$x_D \leq 2$" and "$x_A \geq 1$" or "$x_D \geq 1$"
Request $R$ is cut $\iff$:

"$x_A \leq 2$" or "$x_D ??$" and
Request $R$ is cut $\iff$

"$x_A \leq 2$" or "$x_D \geq 2$" and

"$x_A \geq 1$" or "$x_D \geq 2$" and
Request \( R \) is cut \( \iff \):

- "\( x_A \leq 2 \)" or "\( x_D \geq 2 \)"
- "\( x_A \geq 1 \)" or "\( x_D \geq 1 \)"
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Iterative compression

Marx & Razgon (ESA 2009)
Multicut is equivalent to Richer Multicut where:
- The input contains a $(\leq k + 1)$-Vertex-Multicut $Y$
- The solution has to split $Y$
Iterative compression

Marx & Razgon (ESA 2009)

Multicut is equivalent to *Richer Multicut* where:
- The input contains a \((\leq k + 1)\)-Vertex-Multicut \(Y\)
- The solution has to split \(Y\)
Outline of the proof

- Branch to decide the partition of the solution edges in the components
- Break components with 3 or more attachment vertices (technical, branching)
- Bounded number of important cuts in a Cherry
- Reduce (branching) Lemons
Lemonification

$xy$-lemon, connectivity $\lambda$.
Lemonification

$xy$-lemon, connectivity $\lambda$.

$d(x) = d(y) = \lambda$
Lemonification

xy-lemon, connectivity \( \lambda \).

- \( d(x) = d(y) = \lambda \)
- Backbone
Lemonification

xy-lemon, connectivity $\lambda$.

- $d(x) = d(y) = \lambda$
- Backbone
- Slices
Lemonification

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- Lemonize the Lemons & reduce

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Lemonification

$x y$-lemon, connectivity $\lambda$.

- $d(x) = d(y) = \lambda$
- Backbone
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- Lemonize the Lemons & reduce

Cuts can be partitioned into a bounded number of **linearly ordered sets**
Conclusion and perspectives

Result

Multicut is FPT in the solution size.
Conclusion and perspectives

Result

Multicut is FPT in the solution size.

Questions:

- Multiflow? FPT in $k$?
- (No) Polynomial Kernel for Multicut?
Thanks for your attention!