CLASSIFYING GRAPH CLASSES
(AND CLASSES OF FINITE STRUCTURES)

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A JOINT PROJECT
WITH

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EHESS
PARIS

$A, B, \ldots, F, D, \ldots$ FINITE RELATIONAL STRUCTURES WITH GIVEN SIGNATURE τ

(THINK OF GRAPHS)

$A \longrightarrow B$

$A \leq B$

EXISTENCE OF A HOMOMORPHISM

HOMOMORPHISM QUASIORDER $\mathcal{H}$

$\rightsquigarrow$

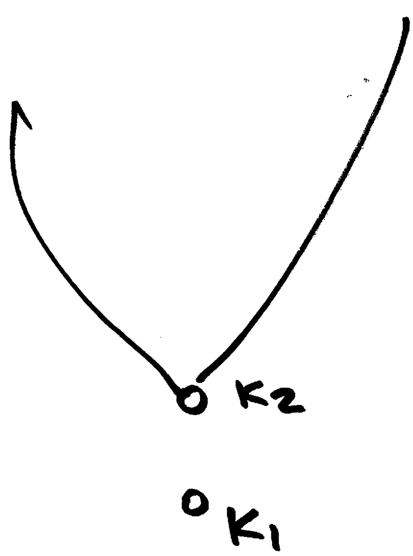
HOMOMORPHISM ORDER
(DEFINED ON CORE-GRAPHS)

SPECTACULAR PROPERTIES OF HOMOMORPHISM ORDER

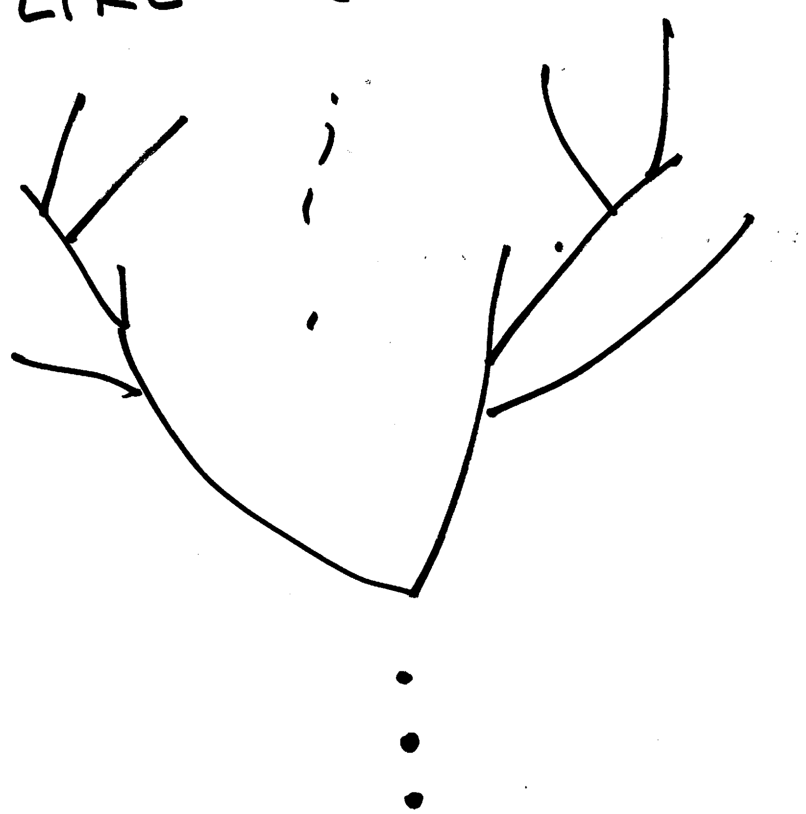
$\mathcal{H}$

- DENSE
- UNIVERSAL
- ∞ CONNECTED (UNDIRECTED)

CACTI-LIKE (DIRECTED)



UNDIRECTED



DIRECTED

$F \not\to \; = \{ A : F \not\to A \}$

FORB(F)

FORB($\mathcal{F}$) FOR $\mathcal{F} = \{ F_1, \dots, F_t \}$.

$\boxed{\text{THM}}$ ANY NP PROBLEM POLYNOMIALLY EQUIVALENT TO MEMBERSHIP FORB($\mathcal{F}'$) FOR A LIFT (EXPANSION τ' OF τ)

(KUN, N.)

$\boxed{\text{THM}}$ ANY HOMOMORPHISM CLOSED FIRST ORDER DEFINABLE CLASS IS OF FORM $\mathcal{F} \longrightarrow$

(ROSSMAN)

$\longrightarrow D = \{A : A \longrightarrow D\}$

CSP (D)

(FEDER, VARDI) – CSP

(N. PULTR) –

$\boxed{\text{FINITE DUALITY}}$ (N. PULTR)

$\mathrm{FORB}(\mathcal{F}) = \mathrm{CSP}(D)$

FINITE DUALITIES CHARACTERIZED

- COMBINATORICS
  ($\mathcal{F}$ A SET OF TREES)
  (D^2 DISMANTABLE)

  KOMÁREK
  N., TARDIF

  LAROSE, LOTTEN, TARDIF

- LOGIC
  (ONLY FO DEFINABLE CSP)

  ATSERIAS
  ROSSMAN

- HOMOMORPHISM POSET

GAPS CUTS BOUNDS

(HEYTING POSETS)

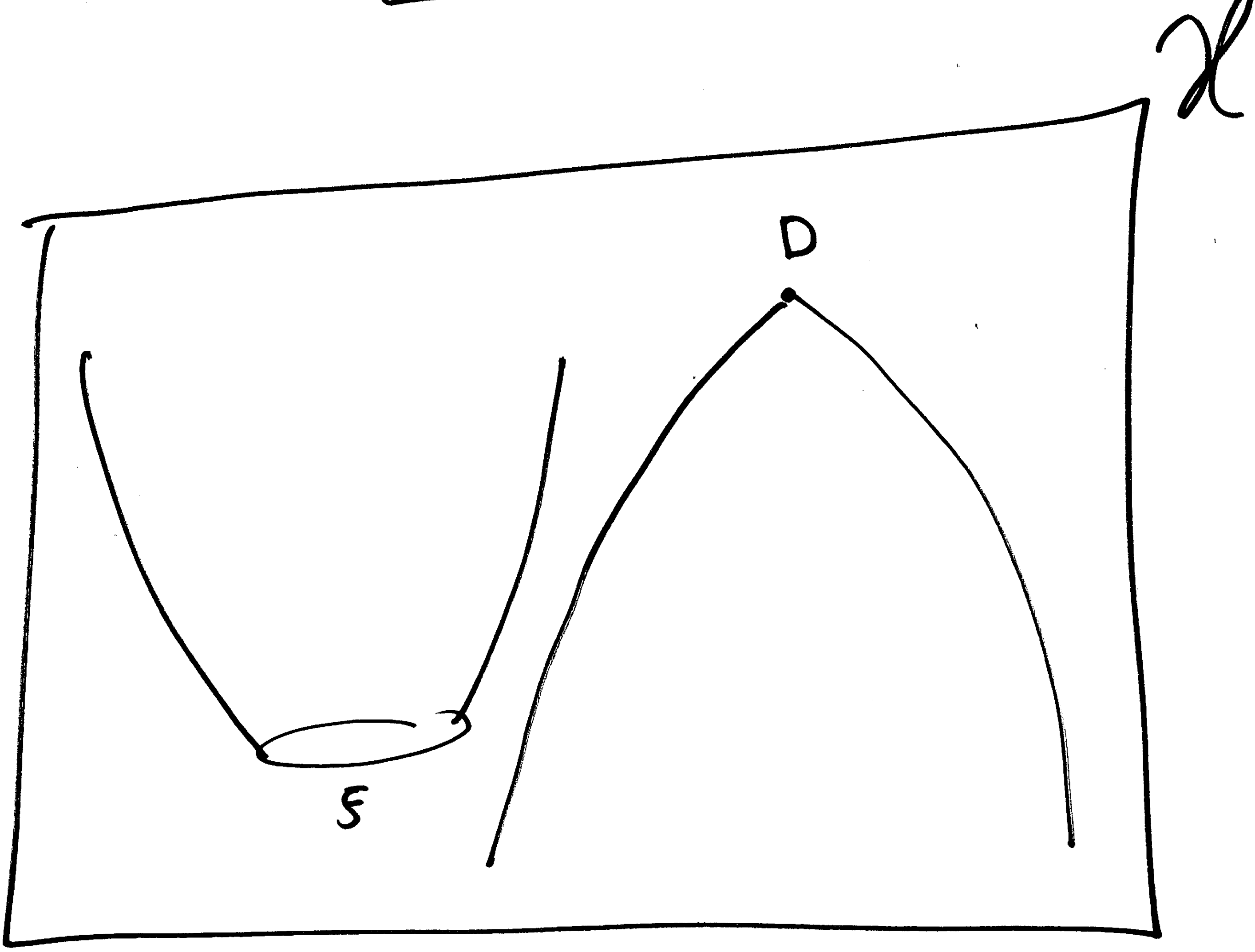
$\mathcal{C}$- $\boxed{\text{RESTRICTED DUALITY}}$

$\mathcal{C}$ A CLASS OF STRUCTURES

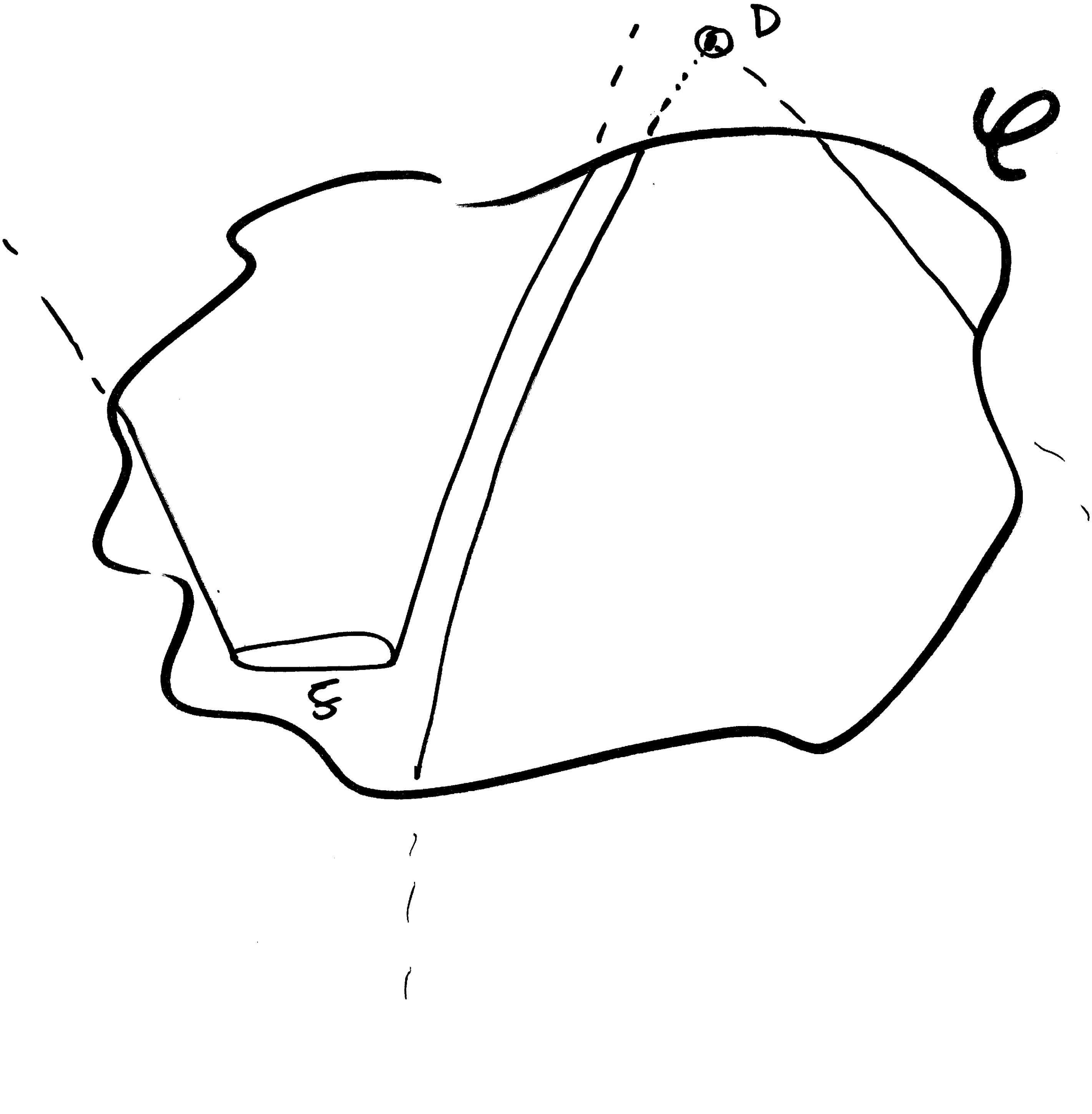
$\mathcal{F} \subseteq \mathcal{C}$

$$\boxed{\mathrm{FORB}(\mathcal{F}) \cap \mathcal{C} = \mathrm{CSP}(D) \cap \mathcal{C}}$$

DUALITY



RESTRICTED DUALITY



$\mathcal{C}$ HAS $\boxed{\text{ALL}}$ RESTRICTED DUALITIES

IFF

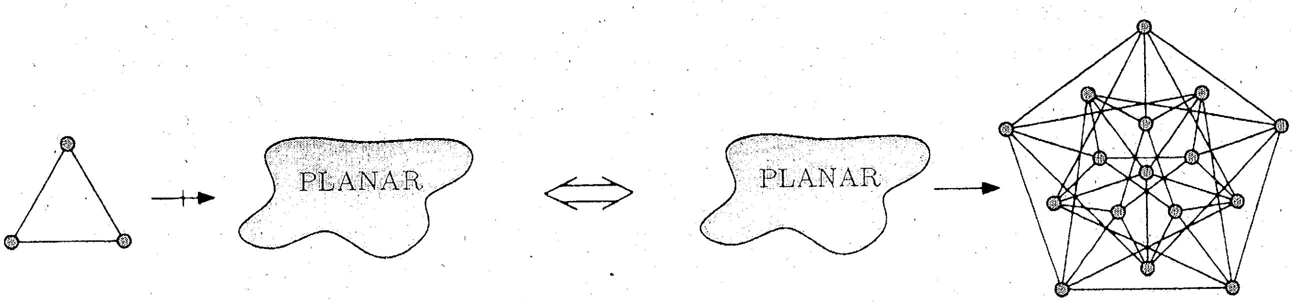
FOR EVERY FINITE SET CONNECTED $\mathcal{F} \subseteq \mathcal{C}$

THERE EXISTS $D_{\mathcal{F}}$ SUCH THAT

$$\mathcal{C} \cap \mathrm{FORB}(\mathcal{F}) = \mathcal{C} \cap \mathrm{CSP}(D_{\mathcal{F}})$$

- BOUNDED DEGREE GRAPHS HAVE ALL RESTRICTED DUALITIES (HAGGKVIST, HELL)

- PLANAR GRAPHS HAVE ARD (N., P. OSSONA DE MENDEZ)



PLANAR – RESTRICTED DUALITY

WHICH CLASSES HAVE ARD ?

CHARACTERIZATION BY

- METRIC PROPERTIES OF HOMOMORPHISM ORDER

- ORIENTED AND ACYCLIC LIFTS

- BY SUBDIVISIONS

- BY FO DEFINABILITY (MODULO $\underbrace{\ }_{\text{WEAK}}$ ERDÖS-HAJNAL CONJECTURE)

$$\mathrm{dist}(A,B) = 2^{-k}$$

$$k = \min\left\{ |C| : \begin{array}{c} C \longrightarrow A \ \& \ C \not\longrightarrow B \\ \text{OR} \\ C \not\longrightarrow A \ \& \ C \longrightarrow B \end{array} \right\}$$

$\varepsilon > 0$

$$\phi^{\varepsilon}(A) = \min\left\{ |B| : \begin{array}{c} A \longrightarrow B \\ \mathrm{dist}(A,B) < \varepsilon \end{array} \right\}$$

$\boxed{\text{THM}}$ (N., POM)

FOR A CLASS $\mathcal{C}$

① $\mathcal{C}$ HAS ALL RESTRICTED DUALITIES

$\Updownarrow$

② $\sup_{A \in \mathcal{C}} \phi^{\varepsilon}(A) < \infty$

(FOR EVERY $\varepsilon > 0$)

dist DEFINES $\overline{\mathcal{H}}$ COMPLETION OF THE HOMOMORPHISM ORDER

DUALITIES IN $\overline{\mathcal{H}}$ CHARACTERIZED

$(\overline{F}, \overline{D})$ DUALITY IN $\overline{\mathcal{H}}$

$\Updownarrow$

EITHER $\overline{F}$ IS A CONNECTED GRAPH

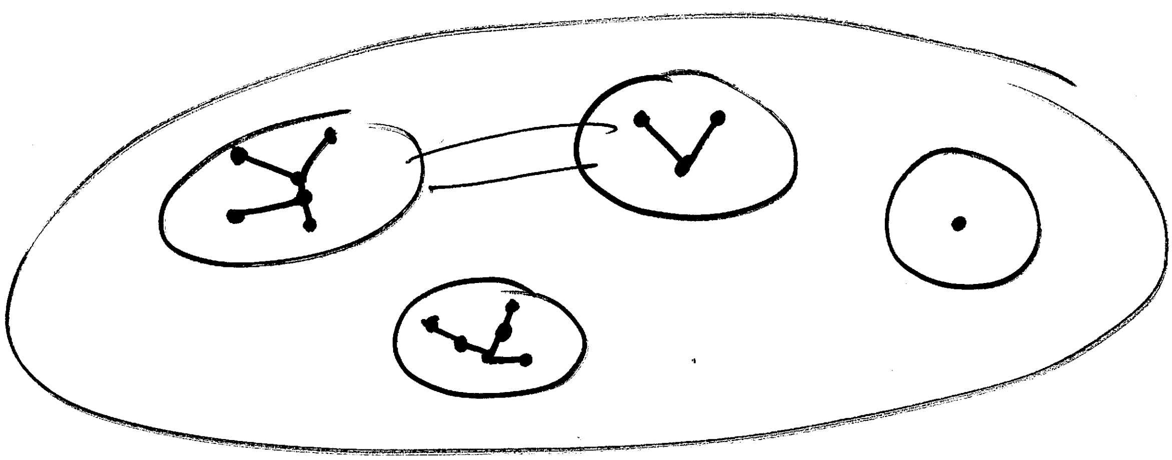
OR $\overline{D}$ IS A MULTIPLICATIVE GRAPH

EXAMPLES OF ARD

CLASSES WITH BOUNDED EXPANSION

$$\mathcal{C} \subseteq \mathcal{C}\nabla 0 \subseteq \mathcal{C}\nabla 1 \subseteq \mathcal{C}\nabla 2 \subseteq \ldots$$

$\mathcal{C}\nabla i \equiv$ ALL GRAPHS WHICH WE OBTAIN FROM GRAPHS IN $\mathcal{C}$ BY CONTRACTING SUBGRAPH WITH RADIUS $\leq i$



$\mathcal{C}$ HAS $\boxed{\text{BOUNDED EXPANSION}}$

IF ALL GRAPHS IN $\mathcal{C} \nabla i$ HAVE

BOUNDED EDGE DENSITY :

$$\sup_{G \in \mathcal{C} \nabla i} \frac{|E(G)|}{|V(G)|} < \infty$$

$\boxed{\text{THM}}$ (N., POM)

ANY $\mathcal{C}$ WITH **BOUNDED** EXPANSION HAS ALL RESTRICTED DUALITIES.

ESSENTIALLY BEST POSSIBLE

CLASS RESOLUTION IN TIME

$$\mathcal{C} \subseteq \mathcal{C} \nabla 0 \subseteq \mathcal{C} \nabla 1 \subseteq \mathcal{C} \nabla 2 \subseteq \dots$$

$$\xrightarrow{\quad\quad\quad\quad}$$

TIME

$\boxed{\text{DEF}}$ (PLOTKIN, RAO, SMITH)

$\mathcal{C} \nabla i$ IS THE CLASS OF ALL i- SHALLOW MINORS (SHALLOW MINORS AT DEPTH i) OF GRAPHS FROM $\mathcal{C}$

DEPTH OF A MINOR $\equiv$ MAX RADIUS OF CONTRACTED SUBGRAPH

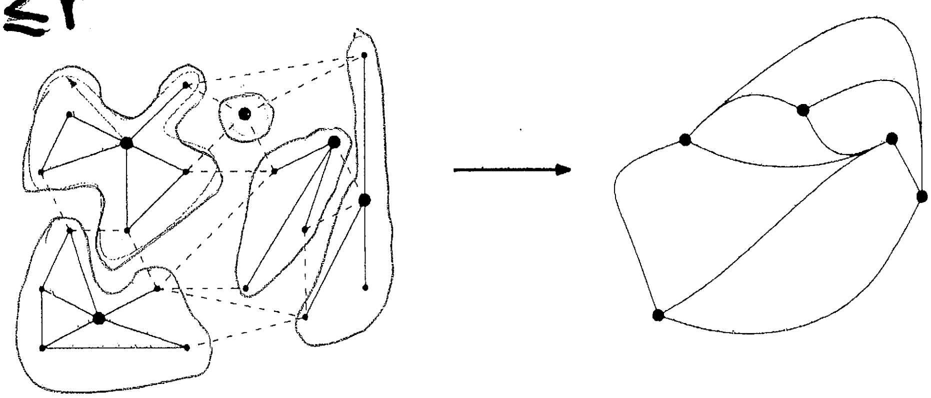
(ALL GRAPHS ARE SIMPLE)

(3')

Class Resolution $(r=1)$

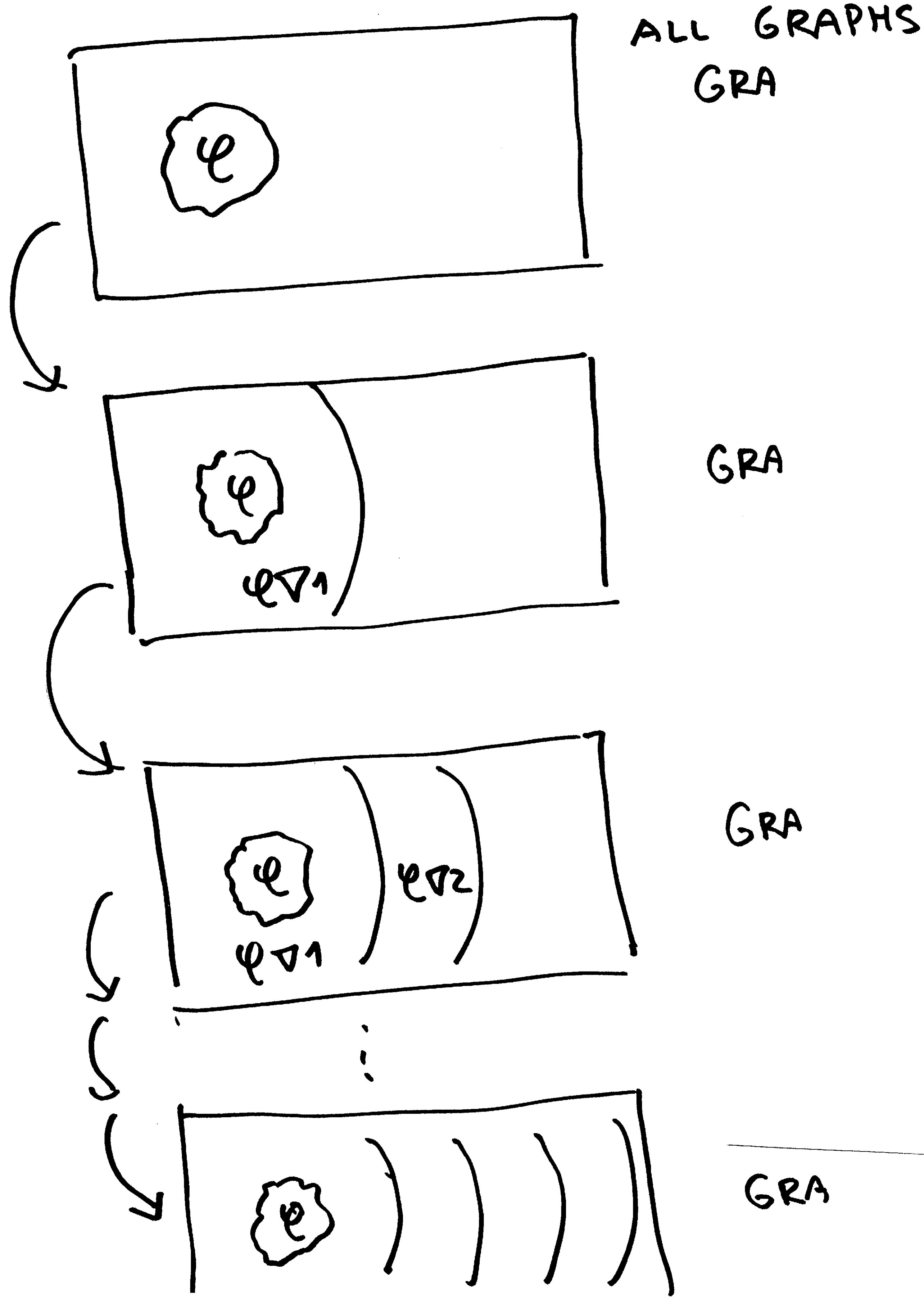
$$\mathcal{C} \subseteq \mathcal{C} \triangledown 0 \subseteq \cdots \subseteq \mathcal{C} \triangledown r \subseteq \ldots$$

$\leq r$



- If there exists r such that $\mathcal{C} \triangledown r$ contains all graphs, then $\mathcal{C}$ is *somewhere dense*,
- Otherwise $\mathcal{C}$ is *nowhere dense*

PARAMETRIZATION OF GRAPHS

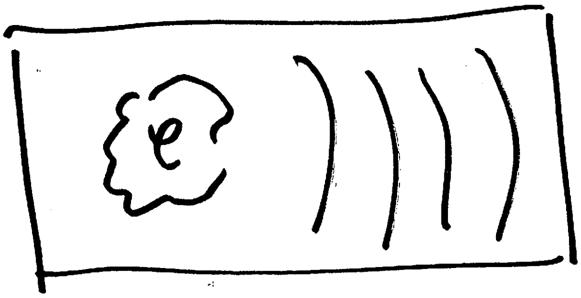


EXAMPLES

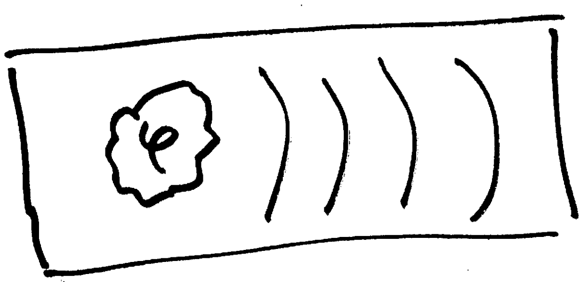
PLANAR

$\mathcal{C} \nabla i = \mathcal{C}$

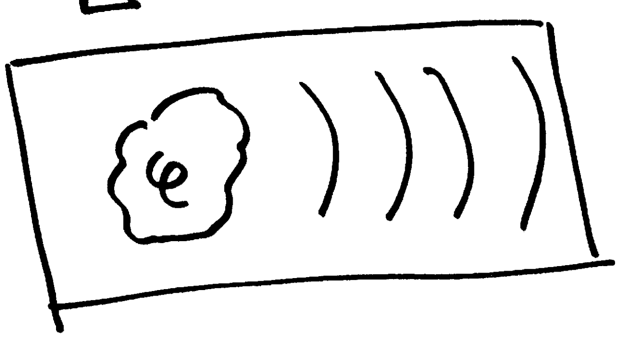
$\Delta \leq d$ BOUNDED DEGREE



FORBIDDEN TOPOLOGICAL MINOR



$\Delta \leq$ GIRTH



SCALLING OF ALL GRAPHS

- CLASS RESOLUTION CAPTURES MANY COMBINATORIAL PROPERTIES

- TIME IS A GOOD PARAMETRIZATION
  LEADS TO EFFECTIVE ALGORITHMS

- LEADS TO DICHOTOMY

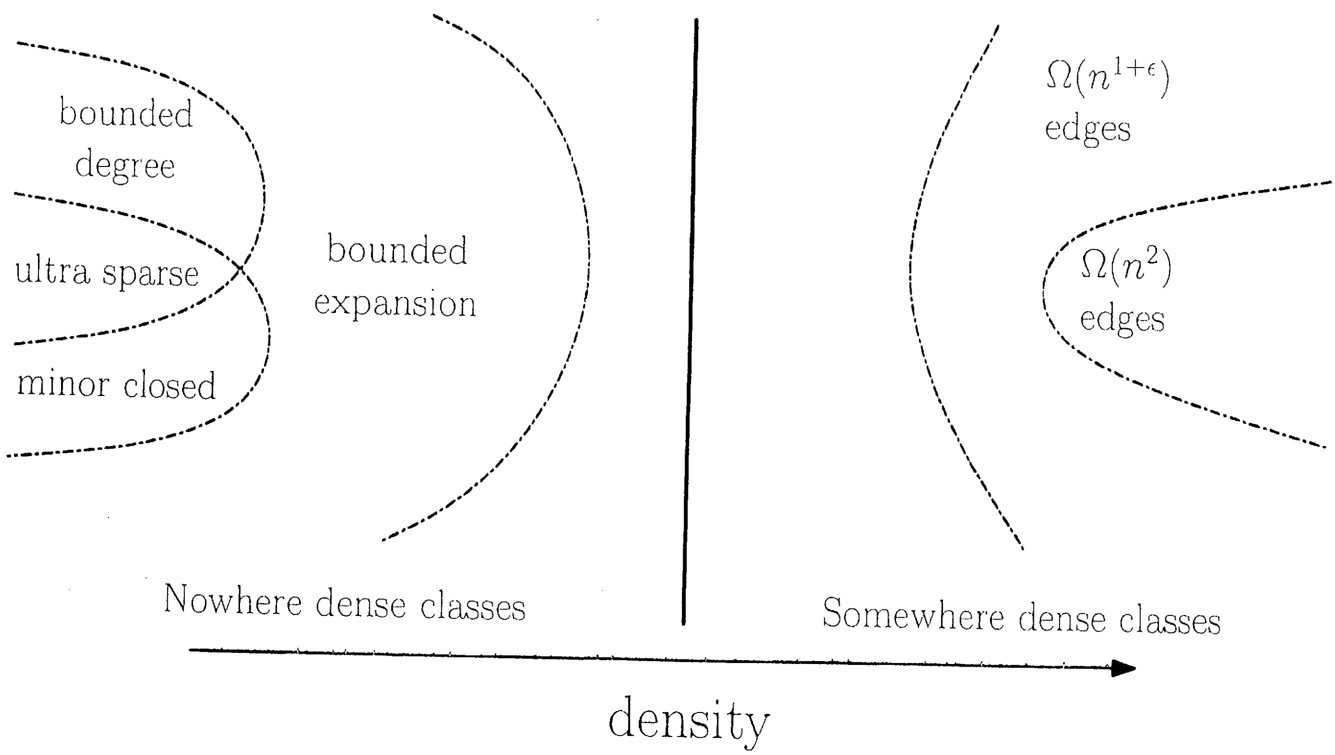
DEF

$\mathcal{C}$ is SOMEWHERE DENSE
IF THERE EXISTS i_0 SUCH THAT
$\mathcal{C} \nabla i_0$ IS THE CLASS OF ALL GRAPHS

$\mathcal{C}$ IS NOWHERE DENSE OTHERWISE

ND VS SD
DICHOTOMY

4T


bounded
degree
ultra sparse
minor closed
bounded
expansion
$\Omega(n^{1+\epsilon})$
edges
$\Omega(n^2)$
edges
Nowhere dense classes
Somewhere dense classes
density

NOT ARBITRARY DEF

POSSIBLE TO DESCRIBE BY MANY (VIRTUALLY ALL) GRAPH PARAMETERS :

α, ω, χ, WCOL

ALSO BY

EDGE DENSITIES OF

$\mathcal{C} \nabla i$ .... SHALLOW MINORS

$\mathcal{C} \tilde{\nabla} i$ .... SHALLOW TOPOLOGICAL MINORS

$\mathcal{C} \overset{\propto}{\nabla} i$ .... SHALLOW IMMERSIONS

ALSO BY COUNTING

CHARACTERISATION OF ND

Let $\mathcal{C}$ be an unbounded size infinite class of graphs, let F be a graph with at least one edge and let q be a positive integer. Then the following conditions are equivalent:

(1) *$\mathcal{C}$ is a class of nowhere dense graphs,*

(2) *for every integer r, $\mathcal{C} \mathbin{\triangledown} r$ is not the class of all finite graphs,*

(3) *for every integer r, $\mathcal{C} \mathbin{\widetilde{\triangledown}} r$ is not the class of all finite graphs,*

(4) *$\mathcal{C}$ is a uniformly quasi-wide class,*

(5) *$\mathrm{H}(\mathcal{C})$ is a quasi-wide class,*

(6) $\displaystyle \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\triangledown r} \frac{\log \|G\|}{\log|G|} = 1,$

(7) $\displaystyle \lim_{r\to\infty} \limsup_{G\in\mathcal{C}\widetilde{\triangledown} r} \frac{\log \|G\|}{\log|G|} = 1,$

(8) $\displaystyle \lim_{r\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{\log|G|} = 0,$

(9) $\displaystyle \lim_{r\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \widetilde{\nabla}_r(G)}{\log|G|} = 0,$

(10) $\displaystyle \lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log|G|} = 0,$

(11) $\displaystyle \lim_{i\to\infty} \limsup_{G\in\mathcal{C}\triangledown i} \frac{\log \chi\ (G)}{\log|G|} = 0,$

(12) $\displaystyle \lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \mathrm{col}_p(G)}{\log|G|} = 0,$

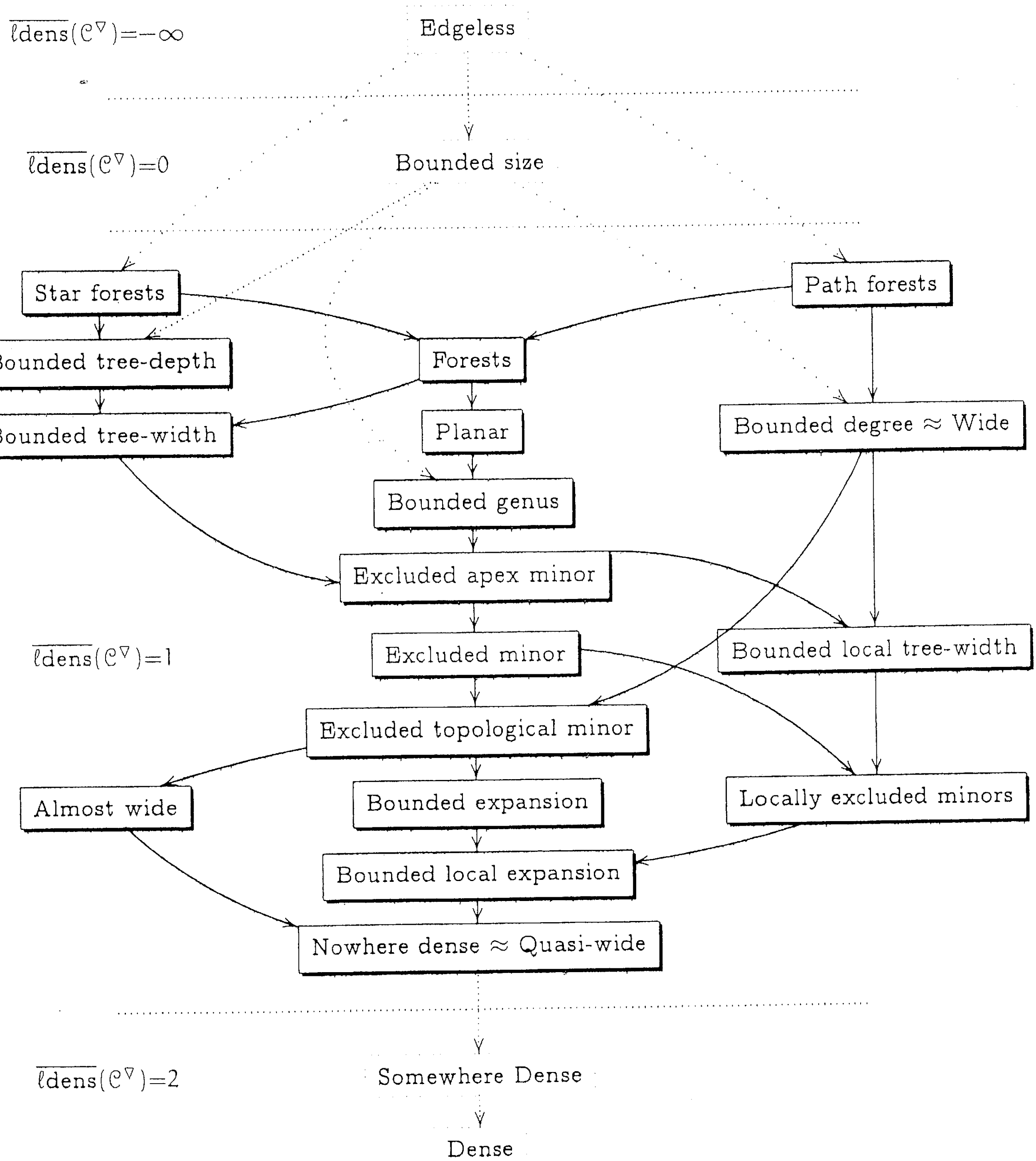
(13) $\displaystyle \lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \mathrm{wcol}_p(G)}{\log|G|} = 0,$

(14) *for every integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ is a class of nowhere dense graphs,*

(15) $\displaystyle \lim_{i\to\infty} \limsup_{G\in\mathcal{C}\triangledown i} \frac{\log(\#F \subseteq G)}{\log|G|} < |F|,$

(16) *for every polynomial P, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \leq P(\nabla_0(G))$ and $G \in \mathcal{C}$ form a class of nowhere dense graphs,*

like quasirandomness! one graph enough!



Inclusion map of some hereditary classes.

EXPLANATION I.

log - DENSITY:

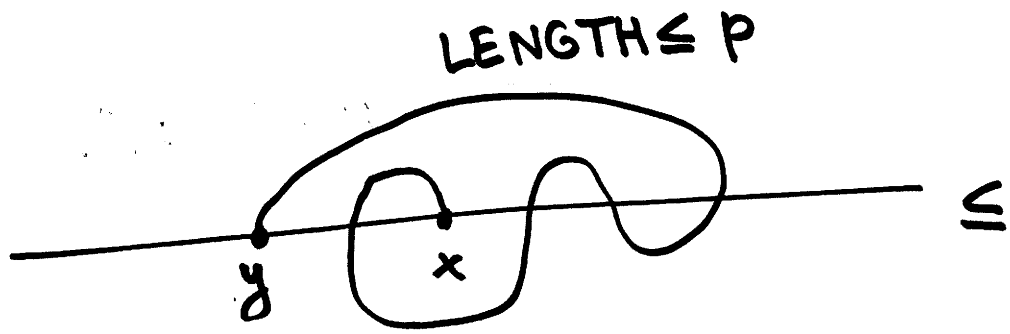
$$\frac{\log \|G\|}{\log |G|} < 1+\varepsilon \quad \dots \quad \|G\| < |G|^{1+\varepsilon}$$

GRAD

$$\nabla_r(G) = \max_{H \in G \nabla r} \frac{\|H\|}{|H|}$$

"r-DEGENERACY"

WCOL$_p$

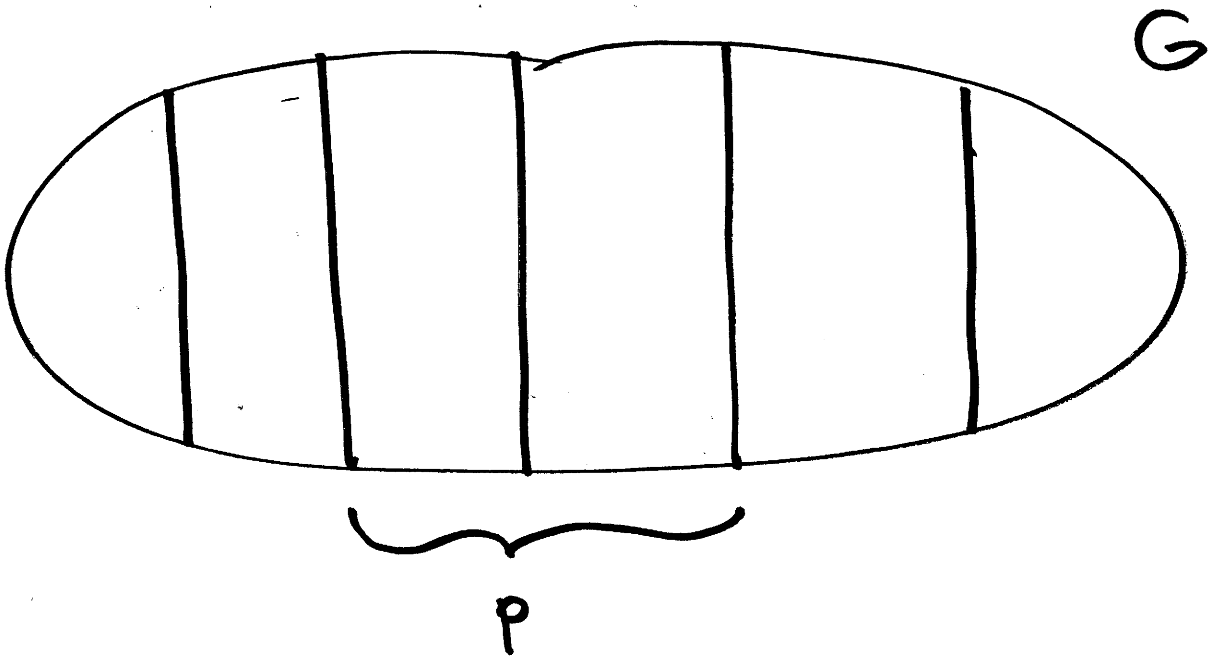


$\# y$ TO THE LEFT WHICH ARE p-REACHABLE

(KIERSTED, YANG)

EXPLANATION II

$\chi_p \equiv$ LOW TREE DEPT DECOMPOSITION



ANY p CLASSES INDUCE SUBGRAPH WITH TREE DEPTH $\leq p$

$td(G) =$ MIN HIGHT OF A ROOTED TREE T SUCH THAT $G \subseteq$ CLOS (T)

"ANCESTOR RELATION"

$\boxed{\text{EXPLANATION}}$ III.

$1\text{-ind} \equiv \text{ind}$

$\boxed{r\text{- INDEPENDENT}}$

$A \subseteq V(G) \quad \mathrm{dist}_G(x,y) > r.$

$\boxed{\text{QUASIWIDE } \mathcal{C}}$

$\forall r \ \exists s(r) \ \forall n \ \exists N :$

$\forall G \in \mathcal{C}$ HOLDS

$|G| > N \Rightarrow \exists S \ \big(|S| \leq s(r) \big)$

$G \setminus S$ HAS r- INDEPENDENT SET A

$|A| \geq n$

$\sim$ SCATTERED SETS

$$\text{ND} \iff \lim_{r\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \nabla_r(G)}{\log |G|} = 0$$

$$\nabla_r(G) = \max_{H \in G \nabla r} \frac{\|H\|}{|H|}$$

$$\Downarrow$$

$$\|H\| = o\left(|H|^{\varepsilon+1}\right)$$

FOR EVERY $\varepsilon > 0$

ALMOST LINEAR ALGORITHMS

$\boxed{\text{SPECIAL CASE}}$

$$f(r) \overset{\text{DEF}}{=} \limsup_{G \in \mathcal{C}} \nabla_r(G) < \infty$$

$\boxed{\text{BOUNDED EXPANSION}}$ $\equiv$ degenerated $\forall r$

EXPANSION FUNCTION

- CONSTANT (PROPER MINOR CLOSED)

- EXPONENTIAL (REGULAR α)

- POLYNOMIAL (GEOMETRIC INTERSECTION)

⋮

- ~~EVERY~~ GROWTH POSSIBLE (ARBITRARY FAST)

SMALL EXPANSION $\Rightarrow$ GOOD PROPERTIES

— SUBEXPONENTIAL GROWTH

$\Downarrow$

SUBLINEAR SEPARATORS

$\Downarrow$ (DVOŘÁK, NORINE)

— EXPANSION $c^{r^{1/3-\varepsilon}}$ YIELDS SMALL CLASS

(# OF LABELLED GRAPHS $\leq n!\,\alpha^n$)

– SUBEXPONENTIAL CLIQUE-GROWTH

$$\Downarrow$$

$\mathcal{C}$ IS HYPERFINITE AND THUS

$\forall k \; \exists c(k)$ $G - c(k)$ vel hom $\leq k$

EVERY MONOTONNE PROPERTY IS TESTABLE IN THE BOUNDED DEGREE MODEL

(BENJAMINI, SCHRAMM, SHAPIRA)

EXTENSION OF

ω - EXPANSION $\quad i \longmapsto \sup_{G \in \mathcal{C} \triangledown i} \omega(G)$

SUB-EXPONENTIAL ω - EXPANSION

$$\limsup_{i \to \infty} \sup_{G \in \mathcal{C} \triangledown i} \frac{\log \omega(G)}{i} = 0$$

(VERY) SMALL EXPANSION

SUBLINEAR SEPARATORS

ALL MONOTONNE PROPERTIES TESTABLE IN THE BOUNDED DEGREE MODEL

SMALL CLASS

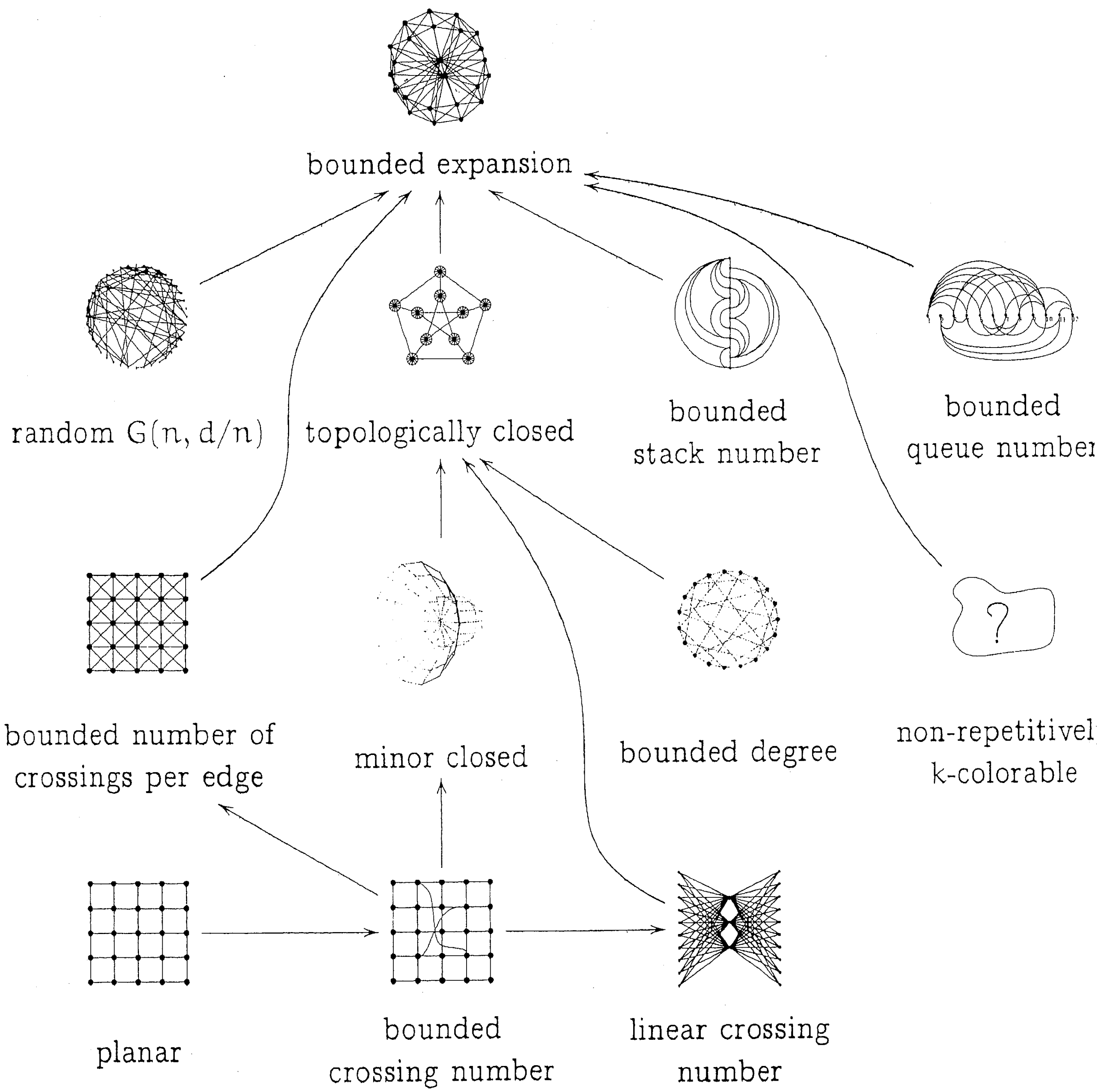
INCLUDES ALL PROPER MINOR CLOSED CLASSES

CHARACTERISATION OF BOUNDED EXPANSION CLASSES

Let $\mathcal{C}$ be a class of graphs. Then the following conditions are equivalent:

(1) *$\mathcal{C}$ has bounded expansion,*
(2) *for every integer r, $\sup_{G\in\mathcal{C}} \nabla_r(G) < \infty$,*
(3) *for every integer r, $\sup_{G\in\mathcal{C}} \widetilde{\nabla}_r(G) < \infty$,*
(4) *for every integer p, $\sup_{G\in\mathcal{C}} \chi_p(G) < \infty$,*
(5) *for every integer p, $\sup_{G\in\mathcal{C}} \mathrm{col}_p(G) < \infty$,*
(6) *for every integer p, $\sup_{G\in\mathcal{C}} \mathrm{wcol}_p(G) < \infty$,*
(7) *for every integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ has bounded expansion,*
(8) *$\mathcal{C}$ has low tree-width colorings,*
(9) *$\mathcal{C}$ has low tree-depth colorings,*
(10) *for every integer p, there exists an integer $X(p)$ such that every graph $G \in \mathcal{C}$ has a p-centered colorings using at most $X(p)$ colors,*
(11) *for every integer k, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \le k$ and $G \in \mathcal{C}$ form a class with bounded expansion,*
(12) *the class $\mathcal{C}$ is a degenerate class of graphs (that is: $\nabla_0(G)$ is bounded on $\mathcal{C}$) and there exists a function F such that every orientation $\vec{G}$ of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \le Q(\Delta^-(\vec{G}), i)$,*
(13) *there exists a function f such that every graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \le f(i)$.*

EXAMPLES OF BOUNDED EXPANSION



(+ D. WOOD)

BOUNDED EXPANSION

$\Downarrow$

LINEAR ALGORITHM FOR SUBGRAPH PROBLEM

LINEAR ALGORITHM FOR COLORING χ_p

$$\chi = \chi_1 \leq \chi_2 \leq \chi_3 \leq \dots \leq \chi_\infty = td$$

STAR COLORING

CENTERED COLORING

HOMOMORPHISM ORDER $\times$ COUNTING

DECISION PROBLEM — STATISTICS TESTING

DENSE GRAPHS (LOVÁSZ & ....)

$$t(F,G) = \frac{\hom(F,G)}{|G|^{|F|}}$$

VERY SPARSE GRAPHS (BENJAMINI, SCHRAMM ...)

$$\mathrm{dens}(F,G) = \frac{\#\; F \subseteq G}{|G|}$$

$$\begin{tikzcd}
F_1 \arrow[dr] & \\
F_2 \arrow[r] & G \\
F_3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1 \arrow[dr] & \\
F_2 \arrow[r] & G \\
F_\infty \arrow[ur] &
\end{tikzcd}$$

LOOKING FOR THE EXPONENT (DEGREE OF FREEDOM)

– HOW IS $\dfrac{\log(\# F \subseteq G)}{\log|G|}$ BOUNDED WHEN G IS RESTRICTED TO A CLASS $\mathcal{C}$?

– HOW MANY VERTICES CAN BE CHOSEN INDEPENDENTLY WHEN LOOKING FOR A COPY OF F?

$\boxed{\text{THM}}$

FOR EVERY INFINITE CLASS OF GRAPHS $\mathcal{C}$

AND EVERY GRAPH F

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log(\# F \subseteq G)}{\log|G|} \in \{-\infty, 0, 1, \dots, \alpha(F), |F|\}$$

WHERE $\alpha(F)$ IS INDEPENDENCE NUMBER

- DEGREES OF FREEDOM $\longrightarrow$ RANDOMNESS
- $\rho_F(G) = \dfrac{\hom(F,G)}{|G|^{\alpha(F)}}$ $\longrightarrow$ LIMITS
- TREE REDUCTIONS $\longrightarrow$ REGULARITY

COROLLARY

$\mathcal{C}$ NOWHERE DENSE

$$\Updownarrow$$

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log(\# F \subseteq G)}{\log|G|}$$

$$\in$$

$$\{-\infty, 0, 1, \dots, \alpha(F)\}$$

FOR EVERY F.

WITH AT LEAST ONE EDGE SO THAT $\alpha(F) < |F|$.

$\boxed{\text{PROOF}}$

IT SUFFICES TO PROVE $\Downarrow$.

PROOF COMBINES ALL CHARACTERIZATIONS

- TREE DEPTH td
- LOW TREE DEPTH DECOMPOSITIONS χ_p
- STEPPING UP LEMMA FOR "(k, F) - SUNFLOWERS"
- COUNTING OF COLORED TREES

PROOF II

FIRST CASE: $F = K_2$

THM (TRICHOTOMY THM)

FOR EVERY INFINITE CLASS $\mathcal{C}$ THE LIMIT

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \nabla i} \frac{\log \|G\|}{\log |G|}$$

$$\in$$

$$\{-\infty, 0, 1, 2\}$$

≤ 1 IFF $\mathcal{C}$ IS NOWHERE DENSE

2 IFF $\mathcal{C}$ IS SOMEWHERE DENSE

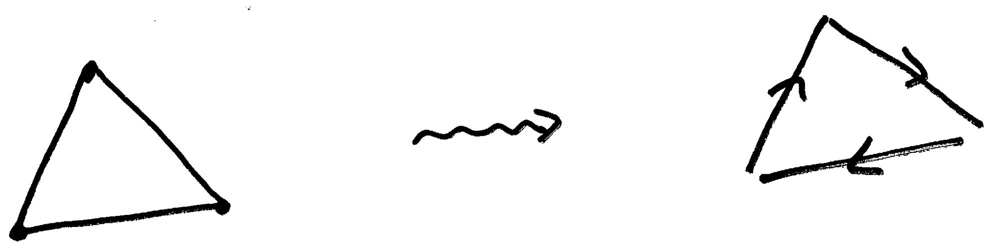
THM (CHARACTERIZATION BY ORIENTATIONS)

LET $\mathcal{C}$ BE A CLASS OF GRAPHS.

① THE CLASS $\vec{\mathcal{C}}$ OF ALL ORIENTATIONS OF ALL GRAPHS IN $\mathcal{C}$ HAS ARD

$\Updownarrow$

② $\mathcal{C}$ HAS BOUNDED EXPANSION.



ORIENTATION

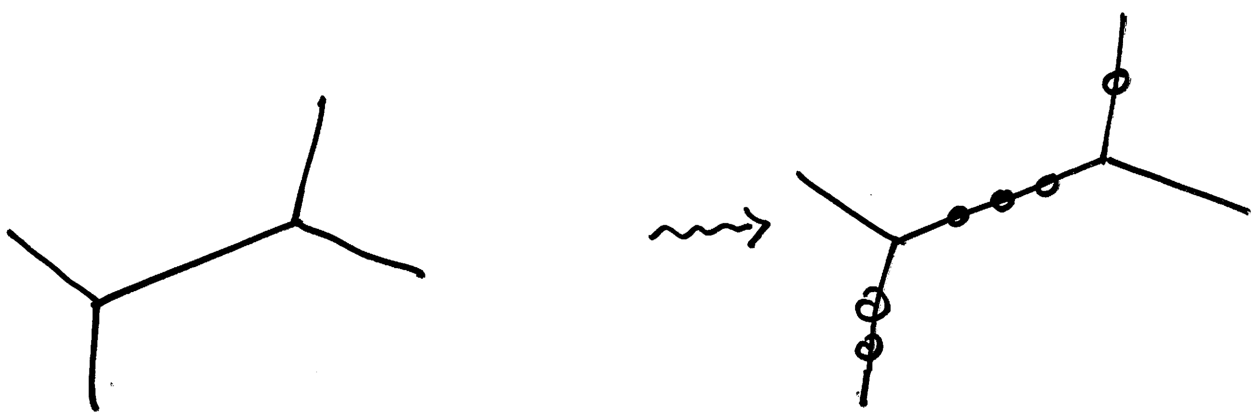
$\boxed{\text{THM}}$ (CHARACTERIZATION BY SUBDIVISIONS)

LET $\mathcal{C}$ BE A TOPOLOGICALLY CLOSED CLASS OF GRAPHS.

① $\mathcal{C}$ HAS ARD

$\Downarrow$

② $\mathcal{C}$ HAS BOUNDED EXPANSION.



SUBDIVISION

ANY SUBDIVISION BELONGS TO $\mathcal{C}$

|||

TOPOLOGICALLY CLOSED

$\boxed{\text{ERDÖS – HAJNAL CONJ.}}$

$\exists\, f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ SUCH THAT

$$\chi(G) \geqslant f(n,\ell) \Rightarrow \exists\, G' \subseteq G$$

$$\chi(G') \geqslant n$$

$$\&$$

$$\text{girth}(G') \geqslant \ell.$$

$\ell = 4$ (RÖDL)

$\boxed{\text{WEAK ODD GIRTH CONJ.}}$

$\exists\, \tilde{f} : \mathbb{N}^3 \longrightarrow \mathbb{N}$ SUCH THAT.

$$\chi(G) \geqslant \tilde{f}(n,\ell,s) \Rightarrow \exists\, G' \subseteq G$$

EITHER $\chi(G') \geqslant n$ & $\text{ODD GIRTH}(G') \geqslant \ell$

OR G' IS $\leq s$ SUBDIVISION OF K_n.

$\boxed{\text{THM}}$ (CHARACTERIZATION BY FO DEFINABILITY)

LET $\mathcal{C}$ BE MONOTONE TOPOLOGICALLY CLOSED.

ASSUME $\boxed{\text{WOGC}}$.

$\Downarrow$

1. $\mathcal{C}$ HAS ARD
2. $\mathcal{C}$ HAS BOUNDED EXPANSION
3. FOR EVERY p THERE EXISTS A GRAPH H_p WITH ODD GIRTH $\geq p$ SUCH THAT $CSP(H_p) \cap \mathcal{C}$ IS EQUIVALENT (ON $\mathcal{C}$) WITH FO FORMULA ϕ_q:

$$\forall G \in \mathcal{C} \left(G \models \phi_g \iff G \to H_g \right)$$

OPEN PROBLEMS (DREAMS)

- IS THERE A NOWHERE DENSE CLASS $\mathcal{C}$ WHICH IS ALGEBRAICALLY UNIVERSAL

- IS THERE A GOOD STATISTICS OF SHALLOW TOPOLOGICAL SUBGRAPHS

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STATISTICS OF POWERS OF MATRICES

"SD ANALYSIS"

P. OSSONA DE MENDEZ, J.N.

GRAPH SPARSITY

(TO APPEAR)

THANK YOU FOR
ATTENTION